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Complex analysis

# Logarithmic potentials on $\mathbb{P}^n$

## *Potentiels logarithmiques sur $\mathbb{P}^n$*

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A B S T R A C T

We study the projective logarithmic potential $\mathbb{G}_\mu$ of a probability measure $\mu$ on the complex projective space $\mathbb{P}^n$. We prove that the range of the operator $\mu \longrightarrow \mathbb{G}_\mu$ is contained in the (local) domain of definition of the complex Monge–Ampère operator acting on the class of quasi-plurisubharmonic functions on $\mathbb{P}^n$ with respect to the Fubini–Study metric. Moreover, when the measure $\mu$ has no atom, we show that the complex Monge–Ampère measure of its logarithmic potential is an absolutely continuous measure with respect to the Fubini–Study volume form on $\mathbb{P}^n$.



R É S U M É

On étudie le potentiel logarithmique projectif $\mathbb{G}_\mu$ d'une mesure de probabilité $\mu$ sur l'espace projectif complexe $\mathbb{P}^n$. On établit que l'image de l'opérateur $\mu \longrightarrow \mathbb{G}_\mu$ est contenue dans le domaine de définition (local) de l'opérateur de Monge–Ampère complexe agissant sur les fonctions quasi-plurisousharmoniques dans $\mathbb{P}^n$ par rapport à la métrique de Fubini–Study. Si $\mu$ n'a pas d'atomes, on montre que la mesure de Monge–Ampère complexe du potentiel logarithmique de $\mu$ est absolument continue par rapport à la forme volume de Fubini–Study de $\mathbb{P}^n$.



## 1. Introduction and statement of the results

Logarithmic potentials of Borel measures in the complex plane play a fundamental role in Logarithmic Potential Theory. This is due to the fact that this theory is associated with the Laplace operator which is a linear elliptic partial differential operator of second order. It is well known that in higher dimension plurisubharmonic functions are rather connected to the complex Monge–Ampère operator, which is a fully non-linear second-order partial differential operator. Therefore, Pluripotential Theory cannot be described by logarithmic potentials. However, the class of logarithmic potentials gives a nice class of plurisubharmonic functions that turns out to be in the local domain of definition of the complex Monge–Ampère operator.

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This study was carried out by Carlehed [5] in the case of a compactly supported measure on $\mathbb{C}^n$ or a bounded hyperconvex domain in $\mathbb{C}^n$.

Our main goal is to extend this study to the complex projective space motivated by the fact that the complex Monge–Ampère operator plays an important role in Kähler geometry (see [13]). A large class of singular potentials on which the complex Monge–Ampère is well defined was introduced (see [12], [8], [4]). However, the global domain of definition of the complex Monge–Ampère operator on compact Kähler manifolds is not yet well understood. Using the characterization of the local domain of definition given by Cegrell and Blocki (see [2], [3], [7]), we show that the class of projective logarithmic potentials $\mathbb{P}^n$ is contained in the local domain of definition of the complex Monge–Ampère operator on the complex projective space $(\mathbb{P}^n, \omega)$ equipped with the Fubini–Study metric $\omega = \omega_{FS}$.

Let $\mu$ be a probability measure on $\mathbb{P}^n$. Then its projective logarithmic potential is defined on $\mathbb{P}^n$ as follows: for $\zeta \in \mathbb{P}^n$,

$$\mathbb{G}_\mu(\zeta) := \int_{\mathbb{P}^n} G(\zeta, \eta)\, \mathrm{d}\mu(\eta) \quad \text{where} \quad G(\zeta, \eta) := \log \frac{|\zeta \wedge \eta|}{|\zeta||\eta|}.$$

**Theorem 1.1.** *Let $\mu$ be a probability measure on $\mathbb{P}^n$. Then the following properties hold.*
*1. The potential $\mathbb{G}_\mu$ is a negative $\omega$-plurisubharmonic function on $\mathbb{P}^n$ normalized by the following condition*

$$\int_{\mathbb{P}^n} \mathbb{G}_\mu\, \omega^n = -\alpha_n,$$

*where $\alpha_n$ is a numerical constant.*
*2. $\mathbb{G}_\mu \in W^{1,p}(\mathbb{P}^n)$ for any $0 < p < 2n$.*
*3. $\mathbb{G}_\mu \in DMA_{\mathrm{loc}}(\mathbb{P}^n, \omega)$.*

We also show a regularizing property of the operator $\mu \to \mathbb{G}_\mu$ acting on probability measures on $\mathbb{P}^n$.

**Theorem 1.2.** *Let $\mu$ be a probability measure on $\mathbb{P}^n$ with no atoms ($n \geq 2$). Then the Monge–Ampère measure $(\omega + \mathrm{dd^c}\mathbb{G}_\mu)^n$ is absolutely continuous with respect to the Fubini–Study volume form on $\mathbb{P}^n$.*

## 2. The logarithmic potential and proof of Theorem 1.1

The complex projective space can be covered by a finite number of charts given by $\mathcal{U}_k := \{[\zeta_0, \zeta_1, \cdots, \zeta_n] \in \mathbb{P}^n \ : \ \zeta_k \neq 0\}$ $(0 \leq k \leq n)$ and the corresponding coordinate chart is defined on $\mathcal{U}_k$ by the formula

$$z^k(\zeta) = z^k := (z^k_j)_{0 \leq j \leq n, j \neq k} \quad \text{where} \quad z^k_j := \frac{\zeta_j}{\zeta_k} \quad \text{for} \quad j \neq k.$$

The Fubini–Study metric $\omega = \omega_{FS}$ is given on $\mathcal{U}_k$ by $\omega|_{\mathcal{U}_k} = \frac{1}{2}\mathrm{dd^c}\log(1 + |z^k|^2)$. The projective logarithmic kernel on $\mathbb{P}^n \times \mathbb{P}^n$ is naturally defined by the following formula

$$G(\zeta, \eta) := \log \frac{|\zeta \wedge \eta|}{|\zeta||\eta|} = \log \sin \frac{d(\zeta, \eta)}{\sqrt{2}} \text{ where } |\zeta \wedge \eta|^2 = \sum_{0 \leq i < j \leq n} |\zeta_i \eta_j - \zeta_j \eta_i|^2,$$

where $d$ is the geodesic distance associated with the Fubini–Study metric (see [15], [6]).

We recall some definitions and give a useful characterization of the local domain of definition of the complex Monge–Ampère operator given by Z. Błocki (see [2], [3]).

**Definition 2.1.** Let $\Omega \subset \mathbb{C}^n$ be a domain. By definition, the set $DMA_{\mathrm{loc}}(\Omega)$ denotes the set of plurisubharmonic functions $\phi$ on $\Omega$ for which there a positive Borel measure $\sigma$ on $\Omega$ such that for all open $U \subset\subset X$ and $\forall\ (\phi_j) \in PSH(U) \cap C^\infty(U) \searrow \phi$ in $U$, the sequence of measures $(\mathrm{dd^c}\phi_j)^n$ converges weakly to $\sigma$ in $U$. In this case, we put $(\mathrm{dd^c}\phi)^n = \sigma$.

The following result of Blocki gives a useful characterization of the local domain of definition of the complex Monge–Ampère operator.

**Theorem 2.2** *(Z.Błocki [2], [3]). 1. If $\Omega \subset \mathbb{C}^2$ is an open set then $DMA_{\mathrm{loc}}(\Omega) = PSH(\Omega) \cap W^{1,2}_{\mathrm{loc}}(\Omega)$.*

*2. If $n \geq 3$, a plurisubharmonic function $\phi$ on a open set $\Omega \subset \mathbb{C}^n$ belongs to $DMA_{\mathrm{loc}}(\Omega)$ if and only if for any $a \in \Omega$ there exists a neighborhood $U_a$ of $a$ in $\Omega$ and a sequence $(\phi_j) \subset PSH(U_z) \cap C^\infty(U_a) \searrow \phi$ in $U_a$ such that the sequences*

$$|\phi_j|^{n-p-2}\mathrm{d}\phi_j \wedge \mathrm{d^c}\phi_j \wedge (\mathrm{dd^c}\phi_j)^p \wedge (\mathrm{dd^c}|z|^2)^{n-p-1}, \quad p = 0, 1, \cdots, n-2$$

*are locally weakly bounded in $U_a$.*

Observe that, by Bedford and Taylor [1], the class of locally bounded plurisubharmonic functions in $\Omega$ is contained in $DMA_{\rm loc}(\Omega)$. By the work of J.-P. Demailly [9], any plurisubharmonic function in $\Omega$ bounded near the boundary $\partial\Omega$ is contained in $DMA_{\rm loc}(\Omega)$. Let $(X,\omega)$ be a Kähler manifold of dimension $n$. We denote by $PSH(X,\omega)$ the set of $\omega$-plurisubharmonic functions in $X$. Then it is possible to define in the same way the local domain of definition $DMA_{\rm loc}(X,\omega)$ of the complex Monge–Ampère operator on $(X,\omega)$. A function $\varphi \in PSH(X,\omega)$ belongs to $DMA_{\rm loc}(X,\omega)$ if for any local chart $(U,z)$, the function $\phi := \varphi + \rho$ belongs to $DMA_{\rm loc}(U)$ where $\rho$ is a Kähler potential of $\omega$. Then the previous theorem extends trivially to this general case. Let $(\chi_j)_{0\leq j\leq n}$ be a fixed partition of unity subordinated to the covering $(\mathcal{U}_j)_{0\leq j\leq n}$. We define $m_j = \int \chi_j \,{\rm d}\mu$ and $J = \{j \in \{0,1,\cdots,n\} \ : \ m_j \neq 0\}$. Then $J \neq \emptyset$ and for $j \in J$, the measure $\mu_j := \frac{1}{m_j}\chi_j\mu$ is a probability measure on $\mathbb{P}^n$ supported in $\mathcal{U}_j$, and we have the following convex decomposition of $\mu$

$$\mu = \sum_{j\in J} m_j \mu_j.$$

Therefore, the potential $\mathbb{G}_\mu$ can be written as a convex combination

$$\mathbb{G}_\mu = \sum_{j\in J} m_j \mathbb{G}_{\mu_j}.$$

To show that $\mathbb{G}_\mu \in DMA_{\rm loc}(\mathbb{P}^n,\omega)$, it suffices to consider the case of a compact measure supported in an affine chart. Without loss of generality, we may always assume that $\mu$ is compactly supported in $\mathcal{U}_0$ and we are reduced to the study of the potential $\mathbb{G}_\mu$ on the open set $\mathcal{U}_0$. The restriction of $G(\zeta,\eta)$ to $\mathcal{U}_0 \times \mathcal{U}_0$ can be expressed in affine coordinates as

$$G(\zeta,\eta) = N(z,w) - \frac{1}{2}\log(1+|z|^2)$$

where

$$N(z,w) := \frac{1}{2}\log\frac{|z-w|^2 + |z\wedge w|^2}{1+|w|^2},$$

will be called the projective logarithmic kernel on $\mathbb{C}^n$.

**Lemma 2.3.** *1. The kernel $N$ is upper semicontinuous in $\mathbb{C}^n \times \mathbb{C}^n$ and smooth off the diagonal of $\mathbb{C}^n \times \mathbb{C}^n$.*
*2. For any fixed $w \in \mathbb{C}^n$, the function $N(.,w) \ : \ z \to N(z,w)$ is plurisubharmonic in $\mathbb{C}^n$ and satisfies the following inequality*

$$\frac{1}{2}\log\frac{|z-w|^2}{1+|w|^2} \leq N(z,w) \leq \frac{1}{2}\log(1+|z|^2), \quad \forall\, (z,w) \in \mathbb{C}^n \times \mathbb{C}^n.$$

From Lemma 2.3, we have the following properties of the projective logarithmic kernel $G$ on $\mathbb{P}^n \times \mathbb{P}^n$.

**Corollary 2.4.** *1. The kernel $G$ is a non-positive upper semi-continuous function on $\mathbb{P}^n \times \mathbb{P}^n$ and smooth off the diagonal of $\mathbb{P}^n \times \mathbb{P}^n$.*
*2. For any fixed $\eta \in \mathbb{P}^n$, the function $G(.,\eta) \ : \ \zeta \to G(\zeta,\eta)$ is a non positive $\omega$-plurisubharmonic function in $\mathbb{P}^n$ and smooth in $\mathbb{P}^n \setminus \{\eta\}$, hence $G(.,\eta) \in DMA_{\rm loc}(\mathbb{P}^n,\omega)$. Moreover, $(\omega + {\rm dd^c}G(\cdot,\eta))^n = \delta_\eta$.*

For a probability measure $\nu$ on $\mathbb{C}^n$, we define the projective logarithmic potential of $\nu$ as follows: for $z \in \mathbb{C}$

$$\mathbb{V}_\nu(z) := \frac{1}{2}\int_{\mathbb{C}^n} \log\frac{|z-w|^2 + |z\wedge w|^2}{1+|w|^2}\,{\rm d}\nu(w).$$

**Proposition 2.5.** *Let $\nu$ be a probability measure $\nu$ on $\mathbb{C}^n$. Then the function $\mathbb{V}_\nu(z)$ is plurisubharmonic in $\mathbb{C}^n$ and for all $z \in \mathbb{C}^n$*

$$\mathbb{V}_\nu(z) \leq \frac{1}{2}\log(1+|z|^2).$$

*Also $\mathbb{V}_\nu \in DMA_{\rm loc}(\mathbb{C}^n)$ and*

$$({\rm dd^c}\mathbb{V}_\nu)^n = \int_{\mathbb{C}^n\times\cdots\times\mathbb{C}^n} {\rm dd}^{\rm c}_z N(.,w_1)\wedge\cdots\wedge {\rm dd}^{\rm c}_z N(.,w_w)\,{\rm d}\nu(w_1)\cdots{\rm d}\nu(w_n).$$

**Proof of Theorem 1.1.** As we have seen, we have

$$\mathbb{G}_\mu = \sum_{j\in J} m_j \mathbb{G}_{\mu_j},$$

where $\mu_j$ is compactly supported in the affine chart $\mathcal{U}_j$.

Observe that for a fixed $k$ one can write on $\mathcal{U}_k$

$$\mathbb{G}_{\mu_k}(\zeta) + \frac{1}{2}\log(1+|z|^2) = \mathbb{V}_{\mu_k}(z)$$

where $z := z^k(\zeta) \in \mathbb{C}^n$, which is plurisubharmonic in $\mathbb{C}^n$. Hence $\mathbb{G}_\mu$ is $\omega$-plurisubharmonic in $\mathbb{P}^n$.

2. By the co-area formula (see [10])

$$\int_{\mathbb{P}^n} \mathbb{G}_\mu(\zeta)\,\mathrm{d}V(\zeta) = \int_0^{\pi/\sqrt{2}} \log\sin\frac{r}{\sqrt{2}} A(r)\,\mathrm{d}r = -\frac{c_n}{\sqrt{2}n^2},$$

where $A(r) := c_n \sin^{2n-2}(r/\sqrt{2})\sin(\sqrt{2}r)$ is the area of the sphere about $\eta$ and radius $r$ on $\mathbb{P}^n$ and $c_n$ is a numerical constant (see [14], page 168, or [11], lemma 5.6).

Let $p \geq 1$. Since $|\nabla d|_\omega = 1$, also by the co-area formula

$$\int_{\mathbb{P}^n} |\nabla \mathbb{G}_\mu(\zeta)|^p \mathrm{d}V(\zeta) \leq \int_{\mathbb{P}^n} \cot^p\Big(\frac{d(\zeta,\eta)}{\sqrt{2}}\Big)\,\mathrm{d}\mu(\eta)\,\mathrm{d}V(\zeta) \leq 2\sqrt{2}c_n \int_0^{\pi/2} \sin^{2n-1-p} t\,\mathrm{d}t,$$

which is finite if and only if $p < 2n$. Hence, for all $p \in ]0, 2n[\,:\ \mathbb{G}_\mu \in W^{1,p}(\mathbb{P}^n)$ (by concavity of $x^p$).

3. When $n = 2$, we can apply the previous result to conclude that $\mathbb{G}_\mu \in DMA_{\mathrm{loc}}(\mathbb{P}^2)$. When $n \geq 3$, we apply Blocki's characterization stated above to show that $\mathbb{G}_{\mu_k} \in DMA_{\mathrm{loc}}(\mathcal{U}_k)$. We consider the following approximating sequence

$$\mathbb{V}^\epsilon_\mu(z) = \frac{1}{2}\int_{\mathbb{C}^n} \log\Big(\frac{|z-w|^2 + |z\wedge w|^2}{1+|w|^2} + \epsilon^2\Big)\mathrm{d}\mu_k(w) \searrow \mathbb{V}_\mu(z),$$

and use the next classical lemma on Riesz potentials to show a uniform estimate on their weighted gradients, as required in Blocki's theorem.

**Lemma 2.6.** *Let $\mu$ be a probability measure on $\mathbb{C}^n$. For $0 < \alpha < 2n$, define the Riesz potential of $\mu$ by*

$$J_{\mu,\alpha}(z) := \int_{\mathbb{C}^n} \frac{\mathrm{d}\mu(w)}{|z-w|^\alpha}.$$

*If $0 < p < 2n/\alpha$ then $J_{\mu,\alpha} \in L^p_{\mathrm{loc}}(\mathbb{C}^n)$.* $\square$

## 3. Regularizing property and proof of Theorem 1.2

We prove a regularizing property of the operator $\mu \to \mathbb{G}_\mu$. By the localization process explained above, the proof of Theorem 1.2 follows from the following theorem, which generalizes and improves a result of Carlehed (see [5]).

**Theorem 3.1.** *Let $\mu$ be a probability measure on $\mathbb{C}^n$ ($n \geq 2$) with no atoms, and let $\psi \in \mathcal{L}(\mathbb{C}^n)$. Assume that $\psi$ is smooth in some open subset $U \subset \mathbb{C}^n$. Then for any $0 \leq m \leq n$, the Monge–Ampère measure $(\mathrm{dd^c}\mathbb{V}_\mu)^m \wedge (\mathrm{dd^c}\psi)^{n-m}$ is absolutely continuous with respect to the Lebesgue measure on $U$.*

The proof is based on the following elementary lemma.

**Lemma 3.2.** *Assume $n \geq 2$ and let $(w_1, \cdots, w_n) \in (\mathbb{C}^n)^n$ fixed such that $w_1 \neq w_2$. Let $\psi \in \mathcal{L}(\mathbb{C}^n)$. Assume that $\psi$ is smooth in some open subset $U \subset \mathbb{C}^n$. Then for any integer $0 \leq m \leq n$, the measure*

$$\bigwedge_{1\leq j\leq m} \mathrm{dd^c}\log(|\cdot - w_j|^2 + |\cdot \wedge w_j|^2) \wedge (\mathrm{dd^c}\psi)^{n-m}$$

*is absolutely continuous with respect to the Lebesgue measure on $U$.*

**Proof of Theorem 3.1.** We first assume that $m = n$. Let $K \subset \mathbb{C}^n$ be a compact set such that $(dd^c|z|^2)^n(K) = 0$. Set $\Delta = \{(w, w, \cdots, w) \ : \ w \in \mathbb{C}^n\}$. Since $\mu$ puts no mass at any point, it follows by Fubini's theorem that $\mu^{\otimes n}(\Delta) = 0$. By Proposition 2.5

$$\int_K (dd^c \mathbb{V}_\mu)^n = \int_{(\mathbb{C}^n)^n \setminus \Delta} f(w_1, \cdots, w_n)\, d\mu^{\otimes n}(w_1, \cdots, w_n),$$

where

$$f(w_1, \cdots, w_n) = \int_K dd^c \log(|z - w_1|^2 + |z \wedge w_1|^2) \wedge \cdots \wedge dd^c \log(|z - w_n|^2 + |z \wedge w_n|^2).$$

By Lemma 3.2, for any $(w_1, \cdots, w_n) \notin \Delta$, $f(w_1, \cdots, w_n) = 0$, hence $(dd^c \mathbb{V}_\mu)^n(K) = 0$. The case $1 \le m < n$ follows from Lemma 3.2 in the same way. The proof is complete.

**Proof of Theorem 1.2.** As we have seen in the proof of Theorem 1.1, one can write on each coordinate chart $\mathcal{U}_k$,

$$\mathbb{G}_\mu(\zeta) = m_k \mathbb{G}_{\mu_k} + \psi_k(z),$$

where $\psi_k \in \mathcal{L}(\mathbb{C}^n)$ is a smooth function in $\mathbb{C}^n$. Using Theorem 3.1 again we conclude that $\mathbb{G}_\mu \in DMA_{\mathrm{loc}}(\mathcal{U}_k)$. Therefore $\mathbb{G}_\mu \in DMA_{\mathrm{loc}}(\mathbb{P}^n)$.

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## References

[1] E. Bedford, B.A. Taylor, The Dirichlet problem for the complex Monge–Ampère equation, Invent. Math. 37 (1) (1976) 1–44.
[2] Z. Blocki, On the definition of the Monge–Ampère operator in $\mathbb{C}^2$, Math. Ann. 328 (3) (2004) 415–423.
[3] Z. Blocki, The domain of definition of the complex Monge–Ampère operator, Amer. J. Math. 128 (2) (2006) 519–530.
[4] S. Boucksom, P. Essidieux, V. Guedj, A. Zeriahi, Monge–Ampère equations in big cohomology classes, Acta Math. 205 (2) (2010) 199–262.
[5] M. Carlehed, Potentials in pluripotential theory, Ann. Fac. Sci. Toulouse Math. (6) 8 (3) (1999) 439–469.
[6] E. Cartan, Leçons sur la géometrie projective complexe, Gauthier-Villars, 1950.
[7] U. Cegrell, The general definition of the complex Monge–Ampère operator, Ann. Inst. Fourier (Grenoble) 54 (1) (2004) 159–179.
[8] D. Coman, V. Guedj, A. Zeriahi, Domains of definition of Monge–Ampère operator on compact Kähler manifolds, Math. Z. 259 (2) (2008) 393–418.
[9] J.-P. Demailly, Monge–Ampère operators, Lelong numbers and intersection theory, in: V. Ancona, A. Silva (Eds.), Complex Analysis and Geometry, in: The University Series in Mathematics, Plenum Press, New York, 1993.
[10] H. Federer, Geometric Measure Theory, Springer Verlag, 1996.
[11] S. Helgason, The Radon transform on Euclidean spaces, two point homogeneous spaces and Grassmann manifolds, Acta Math. 113 (1965) 153–180.
[12] V. Guedj, A. Zeriahi, The weighted Monge–Ampère energy of quasiplurisubharmonic functions, J. Funct. Anal. 250 (2) (2007) 442–482.
[13] V. Guedj, A. Zeriahi, Degenerate Complex Monge–Ampère Equations, EMS Tracts Math, vol. 26, 2017.
[14] D.L. Ragozin, Constructive polynomial approximation on spheres and projective spaces, Trans. Amer. Math. Soc. 162 (1971) 157–170.
[15] E. Study, Kürzeste Wege im komplexen Gebiet, Math. Ann. 60 (1905) 321–378.