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On an inequality of Brendle in the hyperbolic space

Sur une inégalité de Brendle dans l'espace hyperbolique

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ABSTRACT

We give a spinorial proof of a Heintze–Karcher-type inequality in the hyperbolic space proved by Brendle [4]. The proof relies on a generalized Reilly formula on spinors recently obtained in [7].

RÉSUMÉ

On donne une nouvelle démonstration d'une inégalité de type Heintze–Karcher dans l'espace hyperbolique prouvée par Brendle [4]. Cette preuve repose sur une formule de Reilly généralisée pour l'opérateur de Dirac, que nous avons récemment obtenue dans [7].

1. Introduction

The classical Alexandrov theorem [1] asserts that, if Σ is a closed embedded hypersurface in $\mathbb{R}^{n+1}$ with constant mean curvature, then Σ is a round sphere. There are different proofs and generalizations of this theorem. Here we are interested in that of [12] (inspired by Reilly's proof [11]), which relies on the following Heintze–Karcher-type inequality

$$n\int_\Sigma \frac{1}{H}\,\mathrm{d}\Sigma \geq -\int_\Sigma \langle \xi, N\rangle\,\mathrm{d}\Sigma \tag{1}$$

which holds for all closed embedded mean convex hypersurfaces Σ in $\mathbb{R}^{n+1}$. Here ξ, N and H denote respectively the position vector field, the unit inner vector field normal to Σ, and the mean curvature of Σ (with our conventions, the unit sphere in $\mathbb{R}^{n+1}$ satisfies $H=n$). Moreover, equality holds if and only if Σ is a round sphere. Now assuming the constancy of the mean curvature H, which has to be positive since Σ is compact, the classical Minkowski formula

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$$n\int_\Sigma \mathrm{d}\Sigma = -\int_\Sigma H\langle \xi, N\rangle\,\mathrm{d}\Sigma$$

implies that equality is achieved in (1) and thus the Alexandrov theorem is proved.

More recently, Brendle [4] generalized this inequality for domains in certain warped product manifolds, which, in particular, include the classical space forms. This formula is then used to prove some unicity theorems for constant mean curvature hypersurfaces in such manifolds generalizing results from the second author [9]. In the hyperbolic space, this inequality reads as:

Theorem 1. *Let $\Omega \subset \mathbb{H}^{n+1}$ be a compact $(n+1)$-dimensional domain with smooth boundary Σ and let $V(x) = \cosh \mathrm{dist}_{\mathbb{H}^{n+1}}(x, b)$ for some fixed point $b \in \Omega$. If Σ is mean convex, then*

$$n\int_\Sigma \frac{V}{H}\,\mathrm{d}\Sigma \ge (n+1)\int_\Omega V\,\mathrm{d}\Omega. \tag{2}$$

The equality in (2) holds if and only if Ω is isometric to a geodesic ball.

The proof of this result relies on the fact that the quantity $\int_\Sigma \frac{V}{H}\,\mathrm{d}\Sigma$ is monotone non-increasing along a geometric flow. It is worth noticing that this approach has recently been successfully adapted in several works. For example, in [13], the authors obtain an analogous formula for codimension-two submanifolds in some warped products spacetime that admit a conformal Killing–Yano 2-form. Then as an application, they prove generalizations of the Alexandrov theorem for adapted curvature conditions in this context. Note that in our work [7], we weaken their assumptions when the spacetime is the Minkowski space. In a same manner, Li and Xia [8] (see also [10]) were able to prove a Heintze–Karcher-type inequality for sub-static manifolds. Their proof relies on a generalization of the classical Reilly formula and fits more with the approach developed by Reilly and Ros.

In this note, we prove that, in the case of the hyperbolic space, inequality (2) is a simple consequence of a generalized Reilly-type inequality on spinors. It follows the spinorial proof of the Heintze–Karcher inequality in $\mathbb{R}^{n+1}$ by Desmonts [5,6].

2. An integral inequality from [7]

Here we specialize an integral inequality proved in our work [7] for certain boundaries of spacelike domains in spacetimes satisfying the Einstein equation and the dominant energy condition in the case of the Minkowski spacetime $\mathbb{R}^{n+1,1}$. For more details, we refer the reader to [7].

Let Ω be a compact $(n+1)$-dimensional domain with smooth boundary Σ of the hyperbolic space $\mathbb{H}^{n+1}$ realized as the spacelike hypersurface of the Minkowski spacetime $\mathbb{R}^{n+1,1}$ defined by

$$\mathbb{H}^{n+1} = \{(x_0, x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+2} \,/\, -x_0^2 + \sum_{i=1}^{n+1} x_i^2 = -1\}.$$

Since $\Omega \subset \mathbb{H}^{n+1}$, the position vector ξ in $\mathbb{R}^{n+1,1}$ is a future-directed timelike vector field normal to Ω in $\mathbb{R}^{n+1,1}$. We will also denote by N the inner unit spacelike vector field normal to Σ in Ω. In this frame, the second fundamental form of Σ^n in $\mathbb{R}^{n+1,1}$ is given by

$$\mathrm{II}(X, Y) = \langle AX, Y\rangle N + \langle X, Y\rangle \xi$$

for all $X, Y \in \Gamma(T\Sigma)$ and where $AX := -\nabla^\Omega_X N$ denotes the shape operator of Σ in Ω. Here ∇^Ω denotes the Levi-Civita connection of the induced Riemannian metric $\langle\ ,\ \rangle$ on Ω. The mean curvature vector field $\mathcal{H}$ of Σ in $\mathbb{R}^{n+1,1}$, defined by $\mathcal{H} = \mathrm{tr}\,\mathrm{II}$, can be expressed as:

$$\mathcal{H} = HN + n\xi$$

where $H = \mathrm{tr}\,A$ is the mean curvature of Σ in Ω.

On $\mathbb{R}^{n+1,1}$, we define the bundle of complex spinors as being the trivial vector bundle $\mathbb{S}\mathbb{R}^{n+1,1} = \mathbb{R}^{n+2} \times \mathbb{C}^m$ where $m = 2^{[\frac{n+2}{2}]}$. The natural action of $\omega \in \mathbb{C}l(\mathbb{R}^{n+1,1})$, an element of the complex Clifford bundle over $\mathbb{R}^{n+1,1}$, on a spinor field $\psi \in \Gamma\big(\mathbb{S}\mathbb{R}^{n+1,1}\big)$ will be denoted by $\widetilde{\gamma}(\omega)\psi$. The existence of a unit timelike vector ξ normal to $\mathbb{H}^{n+1}$ (hence to Ω) allows us to define the restricted spinor bundle over Ω by $\not{\mathbb{S}}\Omega = \mathbb{S}\mathbb{R}^{n+1,1}{}_{|\Omega}$. According to [2], this spinor bundle carries a positive-definite inner product denoted by $\langle\ ,\ \rangle$ and such that

$$\langle \widetilde{\gamma}(X)\varphi, \psi\rangle = -\langle \varphi, \widetilde{\gamma}(X)\psi\rangle \qquad \text{and} \qquad \langle \widetilde{\gamma}(\xi)\varphi, \psi\rangle = \langle \varphi, \widetilde{\gamma}(\xi)\psi\rangle \tag{3}$$

for all $X \in \Gamma(T\Omega)$ and $\varphi, \psi \in \Gamma\big(\not{\mathbb{S}}\Omega\big)$. Moreover, they also satisfy the compatibility relation

$$\widetilde{\nabla}_X(\widetilde{\gamma}(Y)\psi) = \widetilde{\gamma}(\widetilde{\nabla}_X Y)\psi + \widetilde{\gamma}(Y)\widetilde{\nabla}_X\psi \tag{4}$$

for $X, Y \in \Gamma(T\Omega)$ and $\psi \in \Gamma(\$\Omega)$.

From a Lorentzian point of view, the Gauß formula gives a relation between the Levi-Civita connection $\widetilde{\nabla}$ of $\mathbb{R}^{n+1,1}$ and the one induced on $T\Omega$. Namely, we have

$$\widetilde{\nabla}_X Y = \nabla^{\Omega}_X Y + \langle X, Y\rangle \xi$$

for all $X, Y \in \Gamma(T\Omega)$. The spin Gauß formula gives the corresponding formula in the spinorial setting:

$$\widetilde{\nabla}_X\psi = \nabla\!\!\!\!/^{\Omega}_X\psi + \frac{1}{2}\widetilde{\gamma}\big(X\big)\widetilde{\gamma}(\xi)\psi \tag{5}$$

for all $X \in \Gamma(T\Omega)$, $\psi \in \Gamma\big(\$\Omega\big)$ and where $\widetilde{\nabla}$ and $\nabla\!\!\!\!/^{\Omega}$ correspond to the spin Levi-Civita connections obtained by lifting to the spinor bundle $\$\Omega$ the Lorentzian and Riemannian connections $\widetilde{\nabla}$ and ∇^{Ω}. Moreover, it is also simple to check that the following compatibility relation

$$X\langle\psi, \varphi\rangle = \langle\nabla\!\!\!\!/^{\Omega}_X\psi, \varphi\rangle + \langle\psi, \nabla\!\!\!\!/^{\Omega}_X\varphi\rangle \tag{6}$$

holds for all $X \in \Gamma(T\Omega)$ and $\psi, \varphi \in \Gamma\big(\$\Omega\big)$.

The orientation of Ω induces an orientation on Σ that provides the existence of a unit vector field $N \in \Gamma(T\Omega_{|\Sigma})$ normal to Σ and pointing inward Ω. The existence of this vector field allows us to induce on Σ a spin structure from that on Ω. It follows that the bundle $\$\Sigma := \$\Omega_{|\Sigma}$ is well defined and is endowed with a spinorial Levi-Civita connection $\nabla\!\!\!\!/^{\Sigma}$. The latter is induced by the Riemannian connection on Σ and satisfies

$$\nabla\!\!\!\!/^{\Omega}_X\psi = \nabla\!\!\!\!/^{\Sigma}_X\psi + \frac{1}{2}\widetilde{\gamma}\big(AX\big)\widetilde{\gamma}(N)\psi. \tag{7}$$

It is also important in our result to note that, since the map

$$\mathcal{I} : \psi \in \$\Sigma \mapsto i\widetilde{\gamma}(N)\psi \in \$\Sigma$$

is a linear involution, the spinor bundle $\$\Sigma$ splits into $\$\Sigma = \$^+\Sigma \oplus \$^-\Sigma$, where $\$^{\pm}\Sigma$ denote the vector bundles over Σ whose fiber is the eigenspaces associated with the eigenvalues ± 1 of $\mathcal{I}$. The projections onto these subbundles are denoted by $P_{\pm} = (1/2)(Id \pm i\,\widetilde{\gamma}(N))$. Finally, we define the extrinsic Dirac operator of Σ as the first-order elliptic linear differential operator acting on $\$\Sigma$, whose local expression is given by

$$D\!\!\!\!/^{\Sigma}\psi = \sum_{j=1}^{n} \widetilde{\gamma}(e_j)\widetilde{\gamma}(N)\nabla\!\!\!\!/^{\Sigma}_{e_j}\psi$$

for all $\psi \in \Gamma\big(\$\Sigma\big)$ and where $\{e_1, \cdots, e_n\}$ is a local orthonormal frame of $T\Sigma$. It is then straightforward to check that $D\!\!\!\!/^{\Sigma}$ is formally self-adjoint for the L^2-scalar product on $\$\Sigma$.

We are now in a position to state the result proved in our work [7], and from which Inequality (2) follows. Note that the statement here is much less general than the original in view of our purpose. In fact, we have:

Theorem 2. *Let Σ^n be a mean convex closed hypersurface bounding a compact domain Ω^{n+1} in $\mathbb{H}^{n+1}$. If $\mathcal{D}$ is the first order differential operator defined, for all $\psi \in \Gamma(\$\Sigma)$, by*

$$\mathcal{D}\psi := D\!\!\!\!/^{\Sigma}\psi + \frac{n}{2}\widetilde{\gamma}(N)\widetilde{\gamma}(\xi)\psi \tag{8}$$

then,

$$\int_{\Sigma} \Big(\frac{1}{H}|\mathcal{D}\psi|^2 - \frac{H}{4}|\psi|^2\Big)\,\mathrm{d}\Sigma \geq 0. \tag{9}$$

Moreover, equality occurs if and only if there exist two spinor fields $\Phi_1, \Phi_2 \in \Gamma(\$\Omega)$ such that $\widetilde{\nabla}_X\Phi_1 = \widetilde{\nabla}_X\Phi_2 = 0$ for all $X \in \Gamma(T\Omega)$ and with $P_+\Phi_1 = P_+\psi$ and $P_-\Phi_2 = P_-\psi$.

Remark 1. We apply here Proposition 5 in [7]. Let us check that this can be done in our situation. First, the Minkowski spacetime obviously satisfies the Einstein equation with the dominant energy condition. Moreover, the connection 1-form $\alpha_N(X) := \langle\widetilde{\nabla}_X\xi, N\rangle = \langle X, N\rangle$ is clearly zero for all $X \in \Gamma(T\Sigma)$. The only thing that is not satisfied is the fact that Σ is an outer untrapped submanifold that is $H \geq n$. However, it is a simple verification to see that Proposition 5 in [7] also holds under the weaker assumption of mean convexity of Σ in Ω.

3. Proof of Theorem 1

First note that from the definition (8) of the Dirac-type operator $\not{\mathcal{D}}$ and the spin Gauß formula (5) and (7), the formula

$$\not{\mathcal{D}}\psi = \frac{1}{2}H\psi + \sum_{j=1}^{n}\widetilde{\gamma}(e_j)\widetilde{\gamma}(N)\widetilde{\nabla}_{e_j}\psi \tag{10}$$

holds for all $\psi \in \Gamma(\not{\mathbb{S}}\Sigma)$. Now take $\phi \in \Gamma(\mathbb{S}\mathbb{R}^{n+1,1})$ a parallel spinor on $\mathbb{R}^{n+1,1}$, which is such that $\widetilde{\nabla}_X\phi = 0$ for all $X \in \Gamma(T\mathbb{R}^{n+1,1})$, and consider the restriction to Σ of the field $\Phi := \widetilde{\gamma}(\xi)\phi$. Applying formula (10) to Φ, we first compute that

$$\not{\mathcal{D}}\Phi = \frac{1}{2}H\Phi + n\widetilde{\gamma}(N)\phi \tag{11}$$

using (4) and so

$$|\not{\mathcal{D}}\Phi|^2 = \frac{1}{4}H^2|\Phi|^2 + n^2|\phi|^2 + nH\, Re\langle\widetilde{\gamma}(\xi)\widetilde{\gamma}(N)\phi,\phi\rangle$$

using (3). Now from the fact that ϕ is parallel, the Gauß formula (5) and the compatibility relation (6), we deduce that

$$Re\langle\widetilde{\gamma}(\xi)\widetilde{\gamma}(N)\phi,\phi\rangle = 2\,Re\langle\not{\nabla}^{\Omega}_N\phi,\phi\rangle = \frac{\partial|\phi|^2}{\partial N}$$

and so, if we let $V = |\phi|^2$, we get

$$\frac{1}{H}|\not{\mathcal{D}}\Phi|^2 = \frac{1}{4}H|\Phi|^2 + n^2\frac{V}{H} + n\frac{\partial V}{\partial N}.$$

Integrating this identity on Σ and using Inequality (9) in Theorem 2 yields

$$0 \le \int_\Sigma \Big(\frac{1}{H}|\not{\mathcal{D}}\Phi|^2 - \frac{1}{4}H|\Phi|^2\Big)\mathrm{d}\Sigma = n\int_\Sigma \Big(n\frac{V}{H} + \frac{\partial V}{\partial N}\Big)\mathrm{d}\Sigma$$

that is

$$n\int_\Sigma \frac{V}{H}\,\mathrm{d}\Sigma \ge -\int_\Sigma \frac{\partial V}{\partial N}\,\mathrm{d}\Sigma = -\int_\Omega \Delta V\,\mathrm{d}\Omega = (n+1)\int_\Omega V\,\mathrm{d}\Omega$$

where the last equality comes from the fact that V satisfies $\Delta V = -(n+1)V$. Moreover it is not difficult to see (see [3]) that there exists a (unique) point $b \in \mathbb{H}^{n+1}$ such that $V(x) = \cosh \mathrm{dist}_{\mathbb{H}^{n+1}}(x,b)$ for all $x \in \mathbb{H}^{n+1}$. This concludes the proof of Inequality (2).

Assume now that equality is achieved and then equality is also achieved in (9), so that there exist two spinor fields $\Phi_1,\Phi_2 \in \Gamma(\not{\mathbb{S}}\Omega)$ such that

$$\widetilde{\nabla}_X\Phi_1 = \widetilde{\nabla}_X\Phi_2 = 0 \quad\text{and}\quad P_+\Phi_1 = P_+\Phi,\ P_-\Phi_2 = P_-\Phi \tag{12}$$

for all $X \in \Gamma(T\Omega)$. A simple calculation shows that $\not{\mathcal{D}}P_\pm = P_\mp\not{\mathcal{D}}$ and then it follows from (10) and (12) that

$$\not{\mathcal{D}}\Phi = \not{\mathcal{D}}(P_+\Phi_1 + P_-\Phi_2) = P_-\not{\mathcal{D}}\Phi_1 + P_+\not{\mathcal{D}}\Phi_2 = \frac{H}{2}(P_-\Phi_1 + P_+\Phi_2).$$

This equality leads to $\frac{2}{H}\not{\mathcal{D}}\Phi + \Phi = \Phi_3$, where we let $\Phi_3 := \Phi_1 + \Phi_2 \in \Gamma(\not{\mathbb{S}}\Omega)$. Note that Φ_3 satisfies $\widetilde{\nabla}_X\Phi_3 = 0$ because of (12). Moreover using (11), we compute

$$\Phi_3 = 2\widetilde{\gamma}(\xi + \frac{n}{H}N)\phi$$

and then, for all $X \in \Gamma(T\Sigma)$, we have

$$0 = \widetilde{\nabla}_X\Phi_3 = 2\Big(\widetilde{\gamma}\big(X + \frac{n}{H}\widetilde{\nabla}_X N\big)\phi - n\frac{X(H)}{H^2}\widetilde{\gamma}(N)\phi\Big).$$

However, since $\alpha_N(X) = 0$ for all $X \in \Gamma(T\Sigma)$ (see Remark 1), we deduce that $\widetilde{\nabla}_X N = -A(X)$ and so

$$\widetilde{\gamma}\big(X - \frac{n}{H}A(X)\big)\phi - n\frac{X(H)}{H^2}\widetilde{\gamma}(N)\phi = 0.$$

Taking the scalar product with $\widetilde{\gamma}\big(X - \frac{n}{H}A(X)\big)\phi$ in this equality finally gives

$$\underbrace{\left|X - \frac{n}{H}A(X)\right|^2 |\phi|^2}_{\in\mathbb{R}} = \underbrace{n\frac{X(H)}{H^2}\langle\widetilde{\gamma}\big(X - \frac{n}{H}A(X)\big)\phi, \widetilde{\gamma}(N)\phi\rangle}_{\in i\mathbb{R}}$$

from which we obtain that $A(X) = \frac{H}{n}X$, since ϕ is non-zero everywhere on Σ (see [7] for more details). We conclude that Σ is a totally umbilical hypersurface of the hyperbolic space and so it is a geodesic sphere and Ω is a geodesic ball. The converse is obvious and follows from the fact that the restriction of any parallel spinor $\phi \in \Gamma(\mathbb{SR}^{n+1,1})$ satisfies the equation $\not{\mathcal{D}}\phi = \frac{1}{2}H\phi$ (by (10)).

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