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Differential geometry

A remark on the Bismut–Ricci form on 2-step nilmanifolds ☆

Une remarque sur la forme de Bismut–Ricci des espaces homogènes sous l'action d'un groupe nilpotent de classe ≤ 2

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ABSTRACT

In this note, we observe that, on a 2-step nilpotent Lie group equipped with a left-invariant SKT structure, the $(1,1)$-part of the Bismut–Ricci form is seminegative definite. As an application, we give a simplified proof of the non-existence of invariant SKT static metrics on 2-step nilmanifolds and of the existence of a long-time solution to the pluriclosed flow in 2-step nilmanifolds.

RÉSUMÉ

Nous observons que, sur un groupe de Lie nilpotent de classe ≤ 2, équipé d'une structure de Kähler forte avec torsion (SKT), invariante à gauche, la partie $(1,1)$ de la forme de Bismut–Ricci est définie semi-négative. Comme application, nous donnons une démonstration simplifiée de la non-existence d'une métrique statique SKT sur un espace homogène sous l'action d'un groupe nilpotent de classe ≤ 2. Nous montrons également l'existence d'une solution à long terme du flot plurifermé dans ces mêmes espaces.

1. The Bismut–Ricci form on 2-step SKT nilmanifolds

An Hermitian manifold is called SKT if its fundamental form is $\partial\bar\partial$-closed. The SKT condition can be described in terms of the Bismut connection by requiring that the torsion form is closed. Indeed, on any Hermitian manifold (M,g), there is a unique Hermitian connection ∇ such that the tensor $c := g(T(\cdot,\cdot),\cdot)$ is skew-symmetric in its entries [2], where T is the torsion of ∇. The metric g is SKT if and only if $dc = 0$. In this note, we focus on the Ricci form of ∇. In analogy to the Kähler case, the form is defined by

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$$\rho^{\mathrm{B}}(X,Y)=\mathrm{tr}_\omega R^{\mathrm{B}}(X,Y,\cdot,\cdot)\,,$$

ω being the fundamental form of g and R^{B} the curvature tensor of ∇.

We consider as a manifold M a 2-step nilpotent Lie group G equipped with an invariant Hermitian structure (J,g). Under these assumptions, the form ρ^{B} takes the following expression

$$\rho^{\mathrm{B}}(X,Y)=\mathrm{i}\sum_{r=1}^{n} g([X,Y],[Z_r,\bar Z_r])\,,\qquad \text{for every } X,Y\in\mathfrak{g}\,, \tag{1}$$

where $\{Z_r\}$ is an arbitrary g-unitary frame of the Lie algebra $\mathfrak{g}$ of G (see [6,12], taking into account that here we adopt the convection $\omega(\cdot,\cdot)=g(J\cdot,\cdot)$ in contrast to the one adopted in [6]). More generally, if G is just a Lie group with an invariant Hermitian structure, ρ^{B} takes the following expression

$$\rho^{\mathrm{B}}(X,Y)=-\mathrm{i}\sum_{r=1}^{n}\left\{g([[X,Y]^{1,0},Z_r],\bar Z_r)-g([[X,Y]^{0,1},\bar Z_r],Z_r)-g([X,Y],[Z_r,\bar Z_r])\right\} \tag{2}$$

(see [6,12], again). We have the following proposition.

Proposition 1.1. *Let G be a $2n$-dimensional 2-step nilpotent Lie group with a left-invariant SKT structure (J,g). Then*

$$\rho^{\mathrm{B}}(Z,\bar Z)=-\mathrm{i}\sum_{r=1}^{n}\|[Z,\bar Z_r]\|^2$$

for every $Z\in\mathfrak{g}^{1,0}$, where $\{Z_r\}$ is an arbitrary unitary frame. In particular,

$$\rho^{\mathrm{B}}(X,JX)\le 0$$

for every $X\in\mathfrak{g}$.

Proof. Let Z and W be vector fields of type $(1,0)$ on $\mathfrak{g}\otimes\mathbb{C}$, and let ω be the fundamental form of g. Then we directly compute

$$\begin{aligned}
\partial\bar\partial\omega(Z,\bar Z,W,\bar W)=&-\bar\partial\omega([Z,\bar Z],W,\bar W)+\bar\partial\omega([Z,W],\bar Z,\bar W)-\bar\partial\omega([Z,\bar W],\bar Z,W)\\
&-\bar\partial\omega([\bar Z,W],Z,\bar W)+\bar\partial\omega([\bar Z,\bar W],Z,W)-\bar\partial\omega([W,\bar W],Z,\bar Z)\\
=&-\bar\partial\omega([Z,\bar Z]^{0,1},W,\bar W)+\bar\partial\omega([Z,W],\bar Z,\bar W)-\bar\partial\omega([Z,\bar W]^{0,1},\bar Z,W)\\
&-\bar\partial\omega([\bar Z,W]^{0,1},Z,\bar W)+\bar\partial\omega([\bar Z,\bar W],Z,W)-\bar\partial\omega([W,\bar W]^{0,1},Z,\bar Z)\\
=&-\omega([Z,\bar Z]^{0,1},[W,\bar W]^{1,0})+\omega([Z,W],[\bar Z,\bar W])-\omega([Z,\bar W]^{0,1},[\bar Z,W]^{1,0})\\
&-\omega([\bar Z,W]^{0,1},[Z,\bar W]^{1,0})+\omega([\bar Z,\bar W],[Z,W])-\omega([W,\bar W]^{0,1},[Z,\bar Z]^{1,0})\\
=&+\mathrm{i}g([Z,\bar Z]^{0,1},[W,\bar W]^{1,0})+\mathrm{i}g([Z,W],[\bar Z,\bar W])+\mathrm{i}g([Z,\bar W]^{0,1},[\bar Z,W]^{1,0})\\
&+\mathrm{i}g([\bar Z,W]^{0,1},[Z,\bar W]^{1,0})-\mathrm{i}g([\bar Z,\bar W],[Z,W])+\mathrm{i}g([W,\bar W]^{0,1},[Z,\bar Z]^{1,0})\\
=&+\mathrm{i}g([Z,\bar Z],[W,\bar W])+\mathrm{i}g([Z,\bar W],[\bar Z,W]).
\end{aligned}$$

The SKT assumption $\partial\bar\partial\omega=0$ implies

$$g([Z,\bar Z],[W,\bar W])=-g([Z,\bar W],[\bar Z,W])\,.$$

Therefore, in view of (1), we get

$$\rho^{\mathrm{B}}(Z,\bar Z)=\mathrm{i}\sum_{r=1}^{n}g([Z,\bar Z],[Z_r,\bar Z_r])=-\mathrm{i}\sum_{r=1}^{n}g([Z,\bar Z_r],[\bar Z,Z_r])\,,$$

being $\{Z_r\}$ an arbitrary unitary frame, and the claim follows. □

Remark 1.2. Another description of the Bismut–Ricci form on 2-step nilmanifolds can be found in [1].

Next we observe that in general the form ρ^{B} is not seminegative definite if we drop the assumption on G to be nilpotent or on the metric to be SKT.

Example 1.3. Let $\mathfrak{g}$ be the solvable unimodular Lie algebra with structure equations

$$de^1 = 0, \quad de^2 = -e^{13}, \quad de^3 = e^{12}, \quad de^4 = -e^{23},$$

equipped with the complex structure $Je_1 = e_4$ and $Je_2 = e_3$ and the SKT metric

$$g = \sum_{r=1}^{4} e^r \otimes e^r + \frac{1}{2}(e^1 \otimes e^3 + e^3 \otimes e^1) - \frac{1}{2}(e^2 \otimes e^4 + e^4 \otimes e^2).$$

By using (2) with respect to a unitary frame $\{Z_r\}$, we easily get

$$\rho^B = \frac{2}{3}e^{12} - \frac{2}{3}e^{13} + \frac{4}{3}e^{23}.$$

In particular,

$$\rho^B(e_2, Je_2) = \frac{4}{3} \quad \text{and} \quad \rho^B(4e_1 + e_2, J(4e_1 + e_2)) = -\frac{4}{3}$$

which implies that ρ^B is not seminegative definite as $(1,1)$-form.

Example 1.4. Let $(\mathfrak{g}, J)$ be the 2-step nilpotent Lie algebra with structure equations

$$de^1 = de^2 = de^3 = 0, \quad de^4 = e^{12}, \quad de^5 = -e^{23}, \quad de^6 = e^{13},$$

and equipped with the complex structure $Je_1 = e_2$, $Je_3 = e_4$ and $Je_5 = e_6$ and the non-SKT metric

$$g = \sum_{r=1}^{6} e^r \otimes e^r + \frac{1}{2}(e^3 \otimes e^6 + e^6 \otimes e^3) - \frac{1}{2}(e^4 \otimes e^5 + e^5 \otimes e^4).$$

Again by using (2) with respect to a unitary frame $\{Z_r\}$, we easily get

$$\rho^B = -e^{12} - \frac{1}{2}e^{23},$$

which implies that ρ^B is not seminegative definite as $(1,1)$-form.

2. Non-existence of invariant SKT metrics satisfying $(\rho^B)^{1,1} = \lambda\omega$

In this section, we observe that our Proposition 1.1 easily implies that, on a 2-step nilpotent Lie group, there are no SKT invariant metrics such that

$$(\rho^B)^{1,1} = \lambda\,\omega$$

for some constant λ. This result is already known: the case $\lambda = 0$ was studied in [4], while the case $\lambda \neq 0$ follows from [5].

Indeed, in the setting of Proposition 1.1, if we assume $(\rho^B)^{1,1} = \lambda\omega$, then, taking into account that the center of G in not trivial, formula (1) implies $\lambda = 0$ and, from Proposition 1.1, it follows $[\mathfrak{g}^{1,0}, \mathfrak{g}^{0,1}] = 0$. Therefore, if $\{\zeta^k\}$ is a unitary co-frame in $\mathfrak{g}$, we have

$$\bar\partial \zeta^k = 0$$

and we can write

$$\partial \zeta^k = c^k_{rs} \zeta^r \wedge \zeta^s,$$

for some c^k_{rs} in $\mathbb{C}$. Then

$$\partial\bar\partial\omega = \mathrm{i}\,\partial\bar\partial\left(\sum_{k=1}^{n} \zeta^k \wedge \bar\zeta^k\right) = -\mathrm{i}\,\partial\left(\sum_{k=1}^{n} \bar c^k_{rs} \zeta^k \wedge \bar\zeta^r \wedge \bar\zeta^s\right) = -\mathrm{i}\sum_{k=1}^{n} c^k_{ab}\bar c^k_{rs} \zeta^a \wedge \zeta^b \wedge \bar\zeta^r \wedge \bar\zeta^s$$

and the SKT assumption implies that all the c^k_{rs}'s vanish in contrast to the assumption on G to be not abelian.

3. Long-time existence of the pluriclosed flow on 2-step nilmanifolds

The pluriclosed flow (PCF) is a parabolic flow of Hermitian metrics that preserves the SKT condition. The flow is defined on an SKT manifold (M,ω) as

$$\partial_t\omega_t = -(\rho^{\mathrm{B}}_{\omega_t})^{1,1}\,,\quad \omega_{|t=0}=\omega\,,$$

where $\rho^{\mathrm{B}}_{\omega_t}$ is computed with respect to ω_t and the superscript "1, 1" is the $(1,1)$-component with respect to J. The flow was introduced in [8] and then investigated in [3,8–11], and it is a powerful tool in SKT geometry.

In [6], it is proved that on a 2-step nilpotent Lie group, the flow has always a long-time solution for any initial invariant datum. The proof makes use of the bracket flow device introduce by Lauret in [7].

In our setting, let G be a 2-step nilpotent Lie group with a left-invariant complex structure J and consider the PCF equation starting form an invariant SKT form ω. The solution ω_t holds invariant for every t and, therefore, the flow can be regarded as on ODE on $\Lambda^2\mathfrak{g}^*\otimes\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. The bracket flow device consists in evolving the Lie bracket on $\mathfrak{g}$ instead of the form ω. For this purpose, one considers the bracket variety $\mathcal{A}$ consisting of the elements $\lambda\in\Lambda^2\mathfrak{g}^*\otimes\mathfrak{g}$ such that

$$\lambda(\lambda(X,Y),V))=0\,, \tag{3}$$

$$\lambda(JX,JY)-J\lambda(JX,Y)-J\lambda(X,JY)-\lambda(X,Y)=0\,, \tag{4}$$

$$\partial_\lambda\bar\partial_\lambda\omega=0\,, \tag{5}$$

for every $X,Y,V\in\mathfrak{g}$, where the operators ∂_λ and $\bar\partial_\lambda$ are computed by using the bracket λ. Any $\lambda\in\mathcal{A}$ gives a structure of 2-step nilpotent Lie algebra to $\mathfrak{g}$ such that (J,ω) is an SKT structure. It turns out that the PCF is equivalent to a bracket flow-type equation, i.e. an ODE in $\mathcal{A}$. The equivalence between the two equations is obtained by evolving the initial bracket μ of $\mathfrak{g}$ as

$$\mu_t(X,Y)=h_t\mu(h_t^{-1}X,h_t^{-1}Y)\,,\quad X,Y\in\mathfrak{g}\,,$$

being h_t the curve in $\mathrm{End}(\mathfrak{g})$ solving

$$\frac{\mathrm{d}}{\mathrm{d}t}h_t=-\frac{1}{2}h_tP_{\omega_t}\,,\quad h_{|t=0}=\mathrm{I}$$

and $P_{\omega_t}\in\mathrm{End}(\mathfrak{g})$ is defined by

$$\omega_t(P_{\omega_t}X,Y)=\frac{1}{2}\left(\rho^{\mathrm{B}}_{\omega_t}(X,Y)+\rho^{\mathrm{B}}_{\omega_t}(JX,JY)\right)\,.$$

The form ω_t reads in terms of h_t as

$$\omega_t(X,Y)=\omega(h_tX,h_tY)\,.$$

Now, in view of formula (1),

$$\rho^{\mathrm{B}}_{\omega_t}(X,\cdot)=0 \text{ for every } X\in\xi$$

and then $\omega_t(X,\cdot)=\omega(X,\cdot)$ for every $X\in\xi$, where ξ is the center of μ. Let $\xi^\perp$ be the g-orthogonal complement of ξ in $\mathfrak{g}$ and let g_t be the Hermitian metric corresponding to the solution to the PCF equation starting from ω. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}g_t(X,\cdot)=0 \text{ for every } X\in\xi$$

and g_t preserves the splitting $\mathfrak{g}=\xi\oplus\xi^\perp$, and the flow evolves only the component of g in $\xi^\perp\times\xi^\perp$. It follows that h_t preserves the splitting $\mathfrak{g}=\xi\oplus\xi^\perp$ and

$$h_{t|\xi}=\mathrm{I}_\xi\,.$$

Since $(\mathfrak{g},\mu)$ is 2-step nilpotent, then $\mu(X,Y)\in\xi$ for every $X,Y\in\mathfrak{g}$ and

$$\mu_t(X,Y)=\mu(h_t^{-1}X,h_t^{-1}Y)\,.$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_t(X,Y)=-\mu(h_t^{-1}\dot h_th_t^{-1}X,h_t^{-1}Y)-\mu(h_t^{-1}X,h_t^{-1}\dot h_th_t^{-1}Y)$$

$$=-\mu_t(\dot h_th_t^{-1}X,Y)-\mu_t(X,\dot h_th_t^{-1}Y)=\frac{1}{2}\mu_t(P_{\mu_t}X,Y)+\frac{1}{2}\mu_t(X,P_{\mu_t}Y)\,,$$

where for any $\lambda \in \mathcal{A}$ we set

$$\omega(P_\lambda X, Y) = \mathrm{i}\frac{1}{2}\sum_{r=1}^{n}\big(g(\lambda(X,Y),\lambda(Z_r,\bar{Z}_r)) + g(\lambda(JX,JY),\lambda(Z_r,\bar{Z}_r))\big)$$

being $\{Z_r\}$ an arbitrary g-unitary frame and in the last step we have used

$$h_t P_{\omega_t} = P_{\mu_t} h_t\,.$$

Hence the bracket flow equations writes as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_t(X,Y) = \frac{1}{2}\mu_t(P_{\mu_t}X,Y) + \frac{1}{2}\mu_t(X,P_{\mu_t}Y)\,, \quad \mu_{|t=0} = \mu \tag{6}$$

and its solution satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} g(\mu_t,\mu_t) = 2g(\dot{\mu}_t,\mu_t) = 4\sum_{r,s=1}^{2n} g(\mu_t(P_{\mu_t}e_r,e_s),\mu_t(e_r,e_s))$$

being $\{e_r\}$ an arbitrary g-orthonormal frame. In view of Proposition 1.1, all the eigenvalues of any P_{μ_t} are nonpositive. Fixing t and taking as $\{e_r\}$ an orthonormal basis of eigenvectors of P_{μ_t}, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} g(\mu_t,\mu_t) = 4\sum_{r,s=1}^{2n} a_r g(\mu_t(e_r,e_s),\mu_t(e_r,e_s)) \leq 0\,.$$

It follows that solution μ_t to (6) will stay forever in a compact subset, which implies that μ_t is defined for every $t \in [0,\infty)$, and the claim follows.

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References

[1] R. Arroyo, R. Lafuente, The long-time behavior of the homogeneous pluriclosed flow, arXiv:1712.02075.
[2] J.-M. Bismut, A local index theorem for non-Kähler manifolds, Math. Ann. 284 (4) (1989) 681–699.
[3] J. Boling, Homogeneous solutions of pluriclosed flow on closed complex surfaces, J. Geom. Anal. 26 (3) (2016) 2130–2154.
[4] N. Enrietti, Static SKT metrics on Lie groups, Manuscr. Math. 140 (3–4) (2013) 557–571.
[5] N. Enrietti, A. Fino, L. Vezzoni, Tamed symplectic forms and strong Kähler with torsion metrics, J. Symplectic Geom. 10 (2) (2012) 203–223.
[6] N. Enrietti, A. Fino, L. Vezzoni, The pluriclosed flow on nilmanifolds and Tamed symplectic forms, J. Geom. Anal. 25 (2) (2015) 883–909.
[7] J. Lauret, The Ricci flow for simply connected nilmanifolds, Commun. Anal. Geom. 19 (5) (2011) 831–854.
[8] J. Streets, G. Tian, A parabolic flow of pluriclosed metrics, Int. Math. Res. Not. (2010) 3101–3133.
[9] J. Streets, G. Tian, Regularity results for pluriclosed flow, Geom. Topol. 17 (4) (2013) 2389–2429.
[10] J. Streets, Pluriclosed flow, Born–Infeld geometry, and rigidity results for generalized Kähler manifolds, Commun. Partial Differ. Equ. 41 (2) (2016) 318–374.
[11] J. Streets, Pluriclosed flow on manifolds with globally generated bundles, Complex Manifolds 3 (2016) 222–230.
[12] L. Vezzoni, A note on canonical Ricci forms on 2-step nilmanifolds, Proc. Amer. Math. Soc. 141 (1) (2013) 325–333.