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Ordinary differential equations

On the orbital Hausdorff dependence of differential equations with non-instantaneous impulses ☆

Sur la dépendance orbitale de Hausdorff des équations différentielles avec impulsions non instantanées

Dan Yang [a], JinRong Wang [a,*], Donal O'Regan [b]

[a] *Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, China*
[b] *School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland*

A R T I C L E   I N F O

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A B S T R A C T

In this article, we investigate the orbital Hausdorff continuous dependence of the solutions to integer order and fractional nonlinear non-instantaneous differential equations. The concept of orbital Hausdorff continuous dependence is used to characterize the relations of solutions corresponding to the impulsive points and junction points in the sense of the Hausdorff distance. Then, we establish sufficient conditions to guarantee this specific continuous dependence on their respective trajectories. Finally, two examples are given to illustrate our theoretical results.

R É S U M É

Nous étudions ici la dépendance orbitale de Hausdorff continue des solutions des équations différentielles d'ordre entier ou fractionnaire, non linéaires avec impulsion non instantanée. Le concept de dépendance orbitale de Hausdorff continue est utilisé pour évaluer la distance de Hausdorff entre les solutions correspondant aux points d'impulsion et de jonction. Nous montrons ensuite des conditions suffisantes garantissant cette dépendance continue spécifique sur leurs trajectoires respectives. Finalement, nous donnons deux exemples qui illustrent nos résultats théoriques.

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* Corresponding author.
E-mail addresses: wjr9668@126.com, jrwang@gzu.edu.cn (J. Wang).

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1. Introduction

Impulsive differential equations (IDEs) can be divided into two classes depending on the length of the impulsive action:

–instantaneous impulsive differential equations (IIDEs, i.e. the duration of the impulsive perturbation is relatively short compared to the whole evolution process);

–non-instantaneous impulsive differential equations (NIDEs, i.e. the impulsive action starts at fixed points, and remains active on a period of time that may be related to the previous state).

IIDEs is studied extensively in the literature; for more details on the qualitative theory of IIDEs, we refer the reader to the monographs [7,8,12,24,26] and the papers [2,5,11,14,19,20,27,36,37].

NIDEs was introduced in [16] and the existence, the stability, and the control theory for differential equations of this class were studied in [1,3,4,6,9,10,15,17,21–23,28–31,33,34] (these equations describe the dynamics of the evolution processes arising in pharmacotherapy, economy and aquaculture).

The concept of orbital Hausdorff dependence of the solutions to integer order instantaneous impulsive differential equations was introduced in the monograph [13], where the measure between their respective trajectories is given in the whole domain using the Hausdorff distance. In [35], we study asymptotic properties of solutions, continuous dependence and stability, of integer order and fractional order NIDEs. Sufficient conditions are presented to guarantee that the solutions to both the original and the perturbed problems are close to each other in sense of the uniform metric.

In this paper, we extend the ideas in [13,35] to investigate the orbital Hausdorff dependence of the solutions to the integer order NIDEs:

$$\begin{cases} \chi'(\tau) = f(\tau, \chi(\tau)), \ \ \tau \in (\varsigma_i, \tau_{i+1}], \ \tau_{i+1} = \varsigma_i + d, \ i \in \Lambda := \{0, 1, 2, \cdots\}, \\ \chi(\tau_i^+) = g_i(\tau_i, \chi(\tau_i^-)), \ \ i \in \Lambda\backslash\{0\}, \\ \chi(\tau) = g_i(\tau, \chi(\tau_i^-)), \ \ \tau \in (\tau_i, \varsigma_i], \ \varsigma_i = \tau_i + d, \ i \in \Lambda\backslash\{0\}, \\ \chi(0) = \chi_0, \end{cases} \tag{1}$$

and to the fractional order NIDEs:

$$\begin{cases} {}^c\mathbf{D}^{\alpha}_{\varsigma_i,\tau}\chi(\tau) = f(\tau, \chi(\tau)), \ \ \tau \in (\varsigma_i, \tau_{i+1}], \ \tau_{i+1} = \varsigma_i + d, \ i \in \Lambda, \ \alpha \in (0, 1), \\ \chi(\tau_i^+) = g_i(\tau_i, \chi(\tau_i^-)), \ \ i \in \Lambda\backslash\{0\}, \\ \chi(\tau) = g_i(\tau, \chi(\tau_i^-)), \ \ \tau \in (\tau_i, \varsigma_i], \ \varsigma_i = \tau_i + d, \ i \in \Lambda\backslash\{0\}, \\ \chi(0) = \chi_0, \end{cases} \tag{2}$$

where ${}^c\mathbf{D}^{\alpha}_{\varsigma_i,t}$ denotes the classical Caputo fractional derivative of order α by changing the lower limit ς_i [18], τ_i acts as an impulsive point, and ς_i acts as a junction point satisfying $\varsigma_i < \tau_{i+1} \to \infty$ with $\tau_0 = \varsigma_0 = 0$; the constant $d > 0$ is the difference between the impulsive points and the junction points. Now $\chi(\tau_i^+) = \lim_{\varepsilon\to 0^+}\chi(\tau_i + \varepsilon)$ and $\chi(\tau_i^-) = \lim_{\varepsilon\to 0^+}\chi(\tau_i - \varepsilon) := \chi(\tau_i)$. The function $f \in C([0,\infty) \times D, R^n)$, $\emptyset \neq D \subset R^n$ and $g_i \in C([\tau_i, \varsigma_i] \times D, R^n)$, $i \in \Lambda\backslash\{0\}$.

Consider the corresponding perturbation problems of the form:

$$\begin{cases} \widetilde{\chi}'(\tau) = f(\tau, \widetilde{\chi}(\tau)), \ \ \tau \in (\widetilde{\varsigma}_i, \widetilde{\tau}_{i+1}], \ \widetilde{\tau}_{i+1} = \widetilde{\varsigma}_i + \widetilde{d}_{\tau_{i+1}}, \ i \in \Lambda, \\ \widetilde{\chi}((\widetilde{\tau}_i)^+) = g_i(\widetilde{\tau}_i, \widetilde{\chi}((\widetilde{\tau}_i)^-)), \ \ i \in \Lambda\backslash\{0\}, \\ \widetilde{\chi}(\tau) = g_i(\tau, \widetilde{\chi}((\widetilde{\tau}_i)^-)), \ \ \tau \in (\widetilde{\tau}_i, \widetilde{\varsigma}_i], \ \widetilde{\varsigma}_i = \widetilde{\tau}_i + \widetilde{d}_{\varsigma_i}, \ i \in \Lambda\backslash\{0\}, \\ \widetilde{\chi}(0) = \widetilde{\chi}_0, \end{cases} \tag{3}$$

and

$$\begin{cases} {}^c\mathbf{D}^{\alpha}_{\widetilde{\varsigma}_i,\tau}\widetilde{\chi}(\tau) = f(\tau, \widetilde{\chi}(\tau)), \ \ \tau \in (\widetilde{\varsigma}_i, \widetilde{\tau}_{i+1}], \ \widetilde{\tau}_{i+1} = \widetilde{\varsigma}_i + \widetilde{d}_{\tau_{i+1}}, \ i \in \Lambda, \ \alpha \in (0, 1), \\ \widetilde{\chi}((\widetilde{\tau}_i)^+) = g_i(\widetilde{\tau}_i, \widetilde{\chi}((\widetilde{\tau}_i)^-)), \ \ i \in \Lambda\backslash\{0\}, \\ \widetilde{\chi}(\tau) = g_i(\tau, \widetilde{\chi}((\widetilde{\tau}_i)^-)), \ \ \tau \in (\widetilde{\tau}_i, \widetilde{\varsigma}_i], \ \widetilde{\varsigma}_i = \widetilde{\tau}_i + \widetilde{d}_{\varsigma_i}, \ i \in \Lambda\backslash\{0\}, \\ \widetilde{\chi}(0) = \widetilde{\chi}_0, \end{cases} \tag{4}$$

where $\widetilde{\tau}_0 = \widetilde{\varsigma}_0 = 0$, $\widetilde{\varsigma}_i < \widetilde{\tau}_{i+1} \to \infty$, the constants $\widetilde{d}_{\tau_{i+1}}, \widetilde{d}_{\varsigma_i} > 0$ denote the differences between the impulsive points and the junction points.

The representation of piecewise continuous solutions to problems (1) and (3), which we denote respectively by $\chi(\cdot; 0, \chi_0) \in PC([0,\infty), R^n)$ and $\widetilde{\chi}(\cdot; 0, \widetilde{\chi}_0) \in PC([0,\infty), R^n)$, is as follows:

$$\chi(\tau; 0, \chi_0) = \begin{cases} \chi_0 + (\mathbf{I}^1_{0,\tau} f)(\tau, \chi), \ \ \tau \in [0, \tau_1], \\ g_i(\tau, \chi(\tau_i^-)), \ \ \tau \in (\tau_i, \varsigma_i], \ i \in \Lambda\backslash\{0\}, \\ g_i(\varsigma_i, \chi(\tau_i^-)) + (\mathbf{I}^1_{\varsigma_i,\tau} f)(\tau, \chi), \ \ \tau \in (\varsigma_i, \tau_{i+1}], \ i \in \Lambda\backslash\{0\}, \end{cases} \tag{5}$$

and

$$\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)=\begin{cases}\widetilde{\chi}_0+(\mathbf{I}^1_{0,\tau}f)(\tau,\widetilde{\chi}), & \tau\in[0,\widetilde{\tau}_1],\\ g_i(\tau,\widetilde{\chi}((\widetilde{\tau}_i)^-)), & \tau\in(\widetilde{\tau}_i,\widetilde{\varsigma}_i],\ i\in\Lambda\backslash\{0\},\\ g_i(\widetilde{\varsigma}_i,\widetilde{\chi}((\widetilde{\tau}_i)^-))+(\mathbf{I}^1_{\widetilde{\varsigma}_i,\tau}f)(\tau,\widetilde{\chi}), & t\in(\widetilde{\varsigma}_i,\widetilde{\tau}_{i+1}],\ i\in\Lambda\backslash\{0\},\end{cases}\tag{6}$$

where

$$(\mathbf{I}^p_{a,\tau}f)(\tau,\chi):=\frac{1}{\Gamma(p)}\int_a^\tau\frac{f(\sigma,\chi(\sigma))}{(\tau-\sigma)^{1-p}}\mathrm{d}\sigma,\ p>0.$$

Similarly, we get the solutions to problems (2) and (4), namely:

$$\chi(\tau;0,\chi_0)=\begin{cases}\chi_0+(\mathbf{I}^\alpha_{0,\tau}f)(\tau,\chi), & \tau\in[0,\tau_1],\\ g_i(\tau,\chi(\tau_i^-)), & \tau\in(\tau_i,\varsigma_i],\ i\in\Lambda\backslash\{0\},\\ g_i(\varsigma_i,\chi(\tau_i^-))+(\mathbf{I}^\alpha_{\varsigma_i,\tau}f)(\tau,\chi) & \tau\in(\varsigma_i,\tau_{i+1}],\ i\in\Lambda\backslash\{0\},\end{cases}\tag{7}$$

and

$$\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)=\begin{cases}\widetilde{\chi}_0+(\mathbf{I}^\alpha_{0,\tau}f)(\tau,\widetilde{\chi}), & \tau\in[0,\widetilde{\tau}_1],\\ g_i(\tau,\widetilde{\chi}((\widetilde{\tau}_i)^-)), & \tau\in(\widetilde{\tau}_i,\widetilde{\varsigma}_i],\ i\in\Lambda\backslash\{0\},\\ g_i(\widetilde{\varsigma}_i,\widetilde{\chi}((\widetilde{\tau}_i)^-))+(\mathbf{I}^\alpha_{\widetilde{\varsigma}_i,\tau}f)(\tau,\widetilde{\chi}), & \tau\in(\widetilde{\varsigma}_i,\widetilde{\tau}_{i+1}],\ i\in\Lambda\backslash\{0\}.\end{cases}\tag{8}$$

The rest of this paper is organized as follows. In Section 2, we introduce the definition of the orbital Hausdorff continuous dependence of solutions for our problems. In Section 3, we establish sufficient conditions to guarantee the Hausdorff continuous dependence of solutions. Two examples are given in the final section to illustrate our results.

2. Preliminaries

Let $J=[0,\infty)$. Consider the piecewise continuous function space $PC(J,R^n):=\{v:J\to R^n: v\in C((t_k,t_{k+1}],R^n),\ k=0,1,\cdots$ and $\exists\ v(t_k^+),\ v(t_k^-), k=1,2,\cdots$ with $v(t_k^-)=v(t_k)\}$ with the norm $\|v\|_{PC}:=\sup_{t\in J}\|v(t)\|$, where $C(J,R^n)=\{v: J\to R^n$ is continuous$\}$.

Next we recall some concepts from [25].

With the points $x=(x_1,x_2,\cdots,x_n)$, $y=(y_1,y_2,\cdots,y_n)\in R^n$, the Euclidean distance and Euclidean norm are defined as: $\rho(x,y)=\sqrt{\Sigma_{j=1}^n(x_j-y_j)^2}$ and $\|x\|=\sqrt{\Sigma_{j=1}^n x_j^2}$. Clearly, $\|x-y\|=\rho(x,y)$.

If $\emptyset\neq X,Y\subset R^n$, the Euclidean and Hausdorff distance between them are introduced as:

$$E(X,Y)=\inf\left\{\inf\{\rho(x,y),x\in X\},y\in Y\right\},$$

and

$$H(X,Y)=\max\left\{\sup\left\{\inf\{\rho(x,y),x\in X\},y\in Y\right\},\sup\left\{\inf\{\rho(x,y),y\in Y\},x\in X\right\}\right\}.$$

When $X=\emptyset$ or $Y=\emptyset$, we suppose that $E(X,Y)=0$ and $H(X,Y)=0$.

Theorem 2.1. *If the sets $X_1,X_2,\cdots,X_k,Y_1,Y_2,\cdots,Y_k\subset R^n$ are bounded and $X=\bigcup_{i=1}^k X_i$, $Y=\bigcup_{i=1}^k Y_i$, then*

$$H(X,Y)=H(X_1\cup X_2\cup\cdots\cup X_k,Y_1\cup Y_2\cup\cdots\cup Y_k)\leq\max\{H(X_1,Y_1),H(X_2,Y_2),\cdots,H(X_k,Y_k)\}.$$

Set the functions $h,\widetilde{h}\in C(R^+,R^n)$ and the constants $t_1,t_2,\widetilde{t}_1,\widetilde{t}_2\in R^+$. We define the notation of the parametric curves:

$$r[t_1,t_2]=\begin{cases}\{h(t);t_1\leq t\leq t_2\},\ t_1\leq t_2;\\ \emptyset;\ t_1>t_2,\end{cases}$$

and

$$\widetilde{r}[\widetilde{t}_1,\widetilde{t}_2]=\begin{cases}\{\widetilde{h}(t);\widetilde{t}_1\leq t\leq\widetilde{t}_2\},\ \widetilde{t}_1\leq\widetilde{t}_2;\\ \emptyset;\ \widetilde{t}_1>\widetilde{t}_2.\end{cases}$$

Similarly, we can also define the parametric curves in half-open and open intervals.

Now we give the Hausdorff distance between continuous parametric curves.

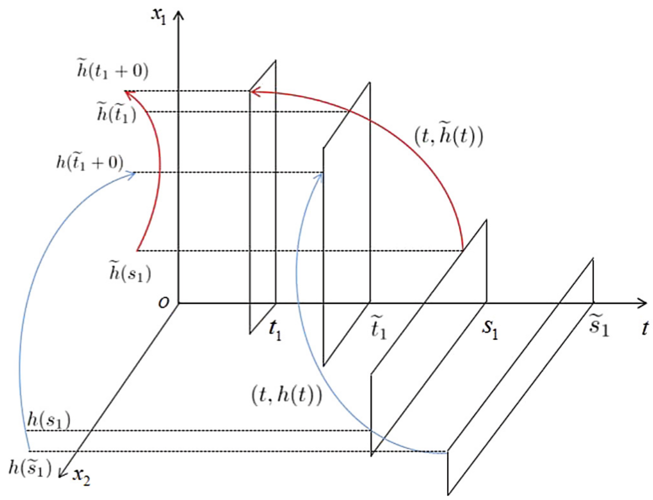

Fig. 1. The blue line denotes the orbital of the solution to the original problem, and the red line denotes the orbital of the solution to the perturbation problem.

Remark 2.2 *([13, Remark 1.4])*. Let $0 \le t_1 \le t_2$, $0 \le \widetilde{t}_1 \le \widetilde{t}_2$. We give the definition concerning the Euclidean, Hausdorff, and uniform distances between the curves $r[t_1,t_2]$ and $\widetilde{r}[\widetilde{t}_1,\widetilde{t}_2]$, respectively:

$$E(\widetilde{r}[\widetilde{t}_1,\widetilde{t}_2],r[t_1,t_2]) = \inf\left\{\inf\left\{\rho(\widetilde{h}(\widetilde{t}),h(t)),\widetilde{t}_1 \le \widetilde{t} \le \widetilde{t}_2\right\},t_1 \le t \le t_2\right\};$$

and

$$\begin{aligned} &H(\widetilde{r}[\widetilde{t}_1,\widetilde{t}_2],r[t_1,t_2]) \\ &= \max\left\{\sup\left\{\inf\left\{\rho(\widetilde{h}(\widetilde{t}),h(t)),\widetilde{t}_1 \le \widetilde{t} \le \widetilde{t}_2\right\},t_1 \le t \le t_2\right\},\right. \\ &\qquad \left.\sup\left\{\inf\left\{\rho(\widetilde{h}(\widetilde{t}),h(t)),\right\},t_1 \le t \le t_2\right\},\widetilde{t}_1 \le \widetilde{t} \le \widetilde{t}_2\right\}; \end{aligned}$$

and

$$R(\widetilde{r}[t_1,t_2],r[t_1,t_2]) = \sup\left\{\rho(\widetilde{h}(t),h(t)),t_1 \le t \le t_2\right\}.$$

For brevity, we set $t_i^{\min} = \min\{\widetilde{t}_i,t_i\}$, $t_i^{\max} = \max\{\widetilde{t}_i,t_i\}$, $s_i^{\min} = \min\{\widetilde{s}_i,s_i\}$ and $s_i^{\max} = \max\{\widetilde{s}_i,s_i\}$, $i = 1,2,\cdots$.

We consider the following hypothesis.

$[H_1]$ The function $f: J \times D \to R^n$ is continuous and $g_i \in C([\tau_i,\varsigma_i] \times D, R^n)$, $i \in \Lambda\backslash\{0\}$.

$[H_2]$ There exists a positive constant L_f such that $\|f(\tau,\chi) - f(\tau,\psi)\| \le L_f\|\chi - \psi\|$, for each $\tau \in [\varsigma_i,\tau_{i+1}]$, $i \in \Lambda$, for all χ, $\psi \in R^n$.

$[H_3]$ There exists a positive constant L_{g_i}, $i \in \Lambda\backslash\{0\}$ such that $\|g_i(\tau_1,\chi) - g_i(\tau_2,\psi)\| \le L_{g_i}(|\tau_1 - \tau_2| + \|\chi - \psi\|)$, for $\tau_1,\tau_2 \in [\tau_i,\varsigma_i]$, $i \in \Lambda\backslash\{0\}$, for all $\chi,\psi \in R^n$.

As in [29, Theorem 4.1], the following theorem is a direct consequence of conditions $[H_1]$, $[H_2]$ and $[H_3]$.

Lemma 2.3. *Assume $[H_1]$, $[H_2]$, and $[H_3]$ are satisfied. Then (1); (2) has a unique solution in $PC(J,R^n)$.*

As in [13, Theorem 2.2], we have the following lemma.

Lemma 2.4. *Suppose that the functions h, $\widetilde{h}: R^+ \to R^n$ are continuous on the left-hand side in R^+, if $i = 1$ and $t_1^{\max} \le s_1^{\min}$ (see Fig. 1). Then*

$$\begin{aligned} &H\left(\widetilde{r}(\widetilde{t}_1,\widetilde{s}_1],r(t_1,s_1]\right) \\ &\le \max\left\{R\left(\widetilde{r}(t_1^{\max},s_1^{\min}],r(t_1^{\max},s_1^{\min}]\right),H\left(h(t_1+0),\widetilde{r}(\widetilde{t}_1,t_1]\right),\right. \\ &\qquad \left.H\left(\widetilde{h}(\widetilde{t}_1+0),r(t_1,\widetilde{t}_1]\right),H\left(h(s_1),\widetilde{r}(s_1,\widetilde{s}_1]\right),H\left(\widetilde{h}(\widetilde{s}_1),r(\widetilde{s}_1,s_1]\right)\right\}. \end{aligned}$$

Motivated from [13, Definition 2.1], we introduce the following definition.

Definition 2.5. The solution to (1); (2) is orbital Hausdorff dependent on the initial condition and the differences between the impulsive points and the junction points, if
$\forall\ (0,\chi_0)\in[0,T]\times D,\ \forall\ d>0,\ \forall\ \epsilon>0,\ \forall\ T>0,\ \exists\ \delta=\delta(\chi_0,d,\epsilon,T)>0$, for $\forall\ \widetilde{\chi}_0\in D,\ \|\widetilde{\chi}_0-\chi_0\|<\delta, \forall\ \widetilde{d}_{\tau_i}>0,\ \widetilde{d}_{\varsigma_i}>0,$ $|\widetilde{d}_{\tau_i}-d|<\delta,\ |\widetilde{d}_{\varsigma_i}-d|<\delta,\ i=1,2,\cdots$, then

$$H(\widetilde{r}[0,T],r[0,T])<\epsilon.$$

3. Main results

In this section, we investigate the orbital Hausdorff continuous dependence of the solutions to our problems.
We need the following condition:
$[H_4]$ There exists a positive constant M such that $\|f(\tau,\chi)\|\le M$, for any $(\tau,\chi)\in J\times R^n$.

Remark 3.1. In fact, $[H_4]$ could be changed to $\sup_{t\in J}\|f(t,0)\|<\infty$. Then, one can apply the impulsive Gronwall inequality [24, Lemma 1]; [32, Lemma 2.8] to derive a prior estimate of solutions to (1); (2) under $[H_2]$. Here we keep $[H_4]$ so that the proofs are more straightforward.

Theorem 3.2. *Suppose $[H_1]$–$[H_4]$ are satisfied. Then, the solution to the problem (1) is orbital Hausdorff dependent on the initial condition and the difference between the impulsive points τ_i and the junction points ς_i, $i=1,2\cdots$.*

Proof. Consider the possible location of the distribution of the impulsive points τ_i, $\widetilde{\tau}_i$ and of the junction points ς_i, $\widetilde{\varsigma}_i$, so we divide our proof into several cases.

Case 1. Let $\tau_i^{\min}=\tau_i,\ \tau_i^{\max}=\widetilde{\tau}_i,\ \varsigma_i^{\min}=\varsigma_i,\ \varsigma_i^{\max}=\widetilde{\varsigma}_i,\ i=1,2,\cdots$; (the case $\tau_i^{\min}=\widetilde{\tau}_i,\ \tau_i^{\max}=\tau_i,\ \varsigma_i^{\min}=\widetilde{\varsigma}_i,\ \varsigma_i^{\max}=\varsigma_i,$ $i=1,2,\cdots$ can be considered similarly).

For the point $(0,\chi_0)\in[0,\infty)\times R^n$, let ϵ and T be positive constants. Since $\tau_i\to\infty\ (i\to\infty)$, then $\exists\ k\in\Lambda\backslash\{0\}$ such that $2kd=\varsigma_k<T<\tau_{k+1}=(2k+1)d$. Therefore, we can select a constant $\delta_T=\delta_T(d,T)>0$, which is sufficiently small, and then $\forall\ \widetilde{d}_{\tau_i},\ \widetilde{d}_{\varsigma_i}>0,\ |\widetilde{d}_{\tau_i}-d|<\delta_T, |\widetilde{d}_{\varsigma_i}-d|<\delta_T$ and $T<\varsigma_{k+1}^{\min},\ \varsigma_{i-1}^{\max}<\tau_i^{\min},\ \tau_i^{\max}<\varsigma_i^{\min},\ i=1,2\cdots,k+1$.

Furthermore,

$\widetilde{\tau}_1<\varsigma_1\Leftrightarrow\widetilde{d}_{\tau_1}<2d\Rightarrow\delta_T<d;$

$\widetilde{\varsigma}_1<\tau_2\Leftrightarrow\widetilde{d}_{\tau_1}+\widetilde{d}_{\varsigma_1}<3d\Rightarrow\delta_T<\frac{d}{2};$

$\widetilde{\tau}_2<\varsigma_2\Leftrightarrow\widetilde{d}_{\tau_1}+\widetilde{d}_{\varsigma_1}+\widetilde{d}_{\tau_2}<4d\Rightarrow\delta_T<\frac{d}{3};$

$\widetilde{\varsigma}_2<\tau_3\Leftrightarrow\widetilde{d}_{\tau_1}+\widetilde{d}_{\varsigma_1}+\widetilde{d}_{\tau_2}+\widetilde{d}_{\varsigma_2}<5d\Rightarrow\delta_T<\frac{d}{4};$

$\vdots$

$\widetilde{\tau}_k<\varsigma_k\Leftrightarrow\widetilde{d}_{\tau_1}+\widetilde{d}_{\varsigma_1}+\cdots+\widetilde{d}_{\varsigma_{k-1}}+\widetilde{d}_{\tau_k}<2kd\Rightarrow\delta_T<\frac{d}{2k-1};$

$\widetilde{\varsigma}_k<T\Leftrightarrow\widetilde{d}_{\tau_1}+\widetilde{d}_{\varsigma_1}+\cdots+\widetilde{d}_{\tau_k}+\widetilde{d}_{\varsigma_k}<T\Rightarrow\delta_T<\frac{T-\varsigma_k}{2k};$

$T<\tau_{k+1}\Leftrightarrow T<(2k+1)d\Rightarrow\delta_T<\frac{\widetilde{\tau}_{k+1}-T}{2k+1}.$

From the inequalities, we suppose that $0<\delta_T<\min\{\frac{d}{2k},\frac{T-\varsigma_k}{2k},\frac{\widetilde{\tau}_{k+1}-T}{2k+1}\}$. Now we consider the Hausdorff distance between the trajectories on the corresponding subintervals.

The Hausdorff distance of the trajectories on the intervals $[0,\widetilde{\tau}_1]$ and $[0,\tau_1]$, according to the property $H(\overline{X},\overline{Y})=H(X,Y)$ and Lemma 2.4, is

$$\begin{aligned}&H\Big(\widetilde{r}[0,\widetilde{\tau}_1],r[0,\tau_1]\Big)=H\Big(\widetilde{r}(0,\widetilde{\tau}_1],r(0,\tau_1]\Big)\\&\le\max\Big\{R\Big(\widetilde{r}(0,\tau_1^{\min}],r(0,\tau_1^{\min}]\Big),H\Big(\chi(\tau_1;0,\chi_0),\widetilde{r}(\tau_1,\widetilde{\tau}_1]\Big),H\Big(\widetilde{\chi}(\widetilde{\tau}_1;0,\widetilde{\chi}_0),r(\widetilde{\tau}_1,\tau_1]\Big)\Big\}.\end{aligned}\tag{9}$$

Since $(\widetilde{\tau}_1,\tau_1]=\emptyset$,

$$H(\widetilde{\chi}(\widetilde{\tau}_1;0,\widetilde{\chi}_0),r(\widetilde{\tau}_1,\tau_1])=0.$$

(See Fig. 2.)

We need to evaluate the other two terms in (9).

Let $0<\eta_{01}<\epsilon$, and we infer that
$\exists\ \delta_0>0,\ \delta_0<\min\{\delta_T,\frac{\eta_{01}}{2M}\},\ \forall\ \widetilde{\chi}_0\in D,\ \|\widetilde{\chi}_0-\chi_0\|<\delta_0,\ \forall\ \widetilde{d}_{\tau_1}>0,\ |\widetilde{d}_{\tau_1}-d|<\delta_0$, then $\|\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)-\chi(\tau;0,\chi_0)\|<\frac{1}{2}\eta_{01}$, $0<\tau\le\tau_1^{\min}$, that is $R(\widetilde{r}(0,\tau_1^{\min}],r(0,\tau_1^{\min}])<\frac{1}{2}\eta_{01}<\epsilon$.

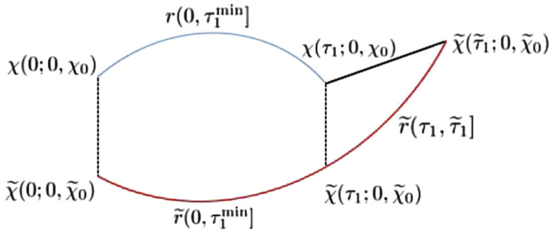

Fig. 2. The blue line denotes the orbital of the solution to (1) in the interval $(0, \tau_1]$, and the red line denotes the orbital of the solution to (3) in the interval $(0, \widetilde{\tau}_1]$.

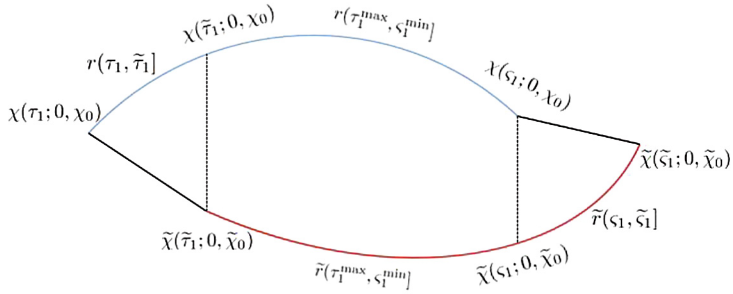

Fig. 3. The blue line denotes the orbital of the solution to (1) in the interval $(\tau_1, \varsigma_1]$, and the red line denotes the orbital of the solution to (3) in the interval $(\widetilde{\tau}_1, \widetilde{\varsigma}_1]$.

Note that $|\widetilde{\tau}_1 - \tau_1| = |\widetilde{d}_{\tau_1} - d| < \delta_0 < \frac{\eta_{01}}{2M}$. For $\tau_1 < \tau \le \widetilde{\tau}_1$,

$$\begin{aligned}\|\chi(\tau_1; 0, \chi_0) - \widetilde{\chi}(\tau; 0, \widetilde{\chi}_0)\| &\le \|\chi(\tau_1; 0, \chi_0) - \widetilde{\chi}(\tau_1; 0, \widetilde{\chi}_0)\| + \|\widetilde{\chi}(\tau; 0, \widetilde{\chi}_0) - \widetilde{\chi}(\tau_1; 0, \widetilde{\chi}_0)\| \\ &< \frac{1}{2}\eta_{01} + \|(\mathbf{I}^1_{\tau_1,\tau} f)(\tau, \widetilde{\chi})\| \\ &\le \frac{1}{2}\eta_{01} + M|\widetilde{\tau}_1 - \tau_1| \\ &< \frac{1}{2}\eta_{01} + M\delta_0 < \eta_{01} < \epsilon,\end{aligned}$$

i.e. $H(\chi(\tau_1; 0, \chi_0), \widetilde{r}(\tau_1, \widetilde{\tau}_1]) < \epsilon$.

Let $\delta_{\tau_1} > 0$ be an arbitrary constant, assume that $\eta_{01} < 2M\delta_{\tau_1}$, and then $|\widetilde{\tau}_1 - \tau_1| < \delta_{\tau_1}$.
Therefore, $\forall\ \delta_{\tau_1} > 0,\ \exists\ \delta_0 > 0,\ \forall\ \widetilde{\chi}_0 \in D,\ \|\widetilde{\chi}_0 - \chi_0\| < \delta_0,\ \forall\ \widetilde{d}_{\tau_1} > 0,\ |\widetilde{d}_{\tau_1} - d| < \delta_0$, then $H(\widetilde{r}[0, \widetilde{\tau}_1], r[0, \tau_1]) < \epsilon,\ |\widetilde{\tau}_1 - \tau_1| < \delta_{\tau_1}$.

For the trajectories $\widetilde{r}(\widetilde{\tau}_1, \widetilde{\varsigma}_1]$ and $r(\tau_1, \varsigma_1]$ (see Fig. 3), the Hausdorff distance is

$$\begin{aligned}&H\Big(\widetilde{r}(\widetilde{\tau}_1, \widetilde{\varsigma}_1], r(\tau_1, \varsigma_1]\Big) \\ &\le \max\Big\{R\Big(\widetilde{r}(\tau_1^{\max}, \varsigma_1^{\min}], r(\tau_1^{\max}, \varsigma_1^{\min}]\Big), H\Big(\chi(\tau_1 + 0; 0, \chi_0), \widetilde{r}(\widetilde{\tau}_1, \tau_1]\Big), \\ &\quad H\Big(\widetilde{\chi}(\widetilde{\tau}_1 + 0; 0, \widetilde{\chi}_0), r(\tau_1, \widetilde{\tau}_1]\Big), H\Big(\chi(\varsigma_1; 0, \chi_0), \widetilde{r}(\varsigma_1, \widetilde{\varsigma}_1]\Big), H\Big(\widetilde{\chi}(\widetilde{\varsigma}_1; 0, \widetilde{\chi}_0), r(\widetilde{\varsigma}_1, \varsigma_1]\Big)\Big\}. \end{aligned} \tag{10}$$

Since $(\widetilde{\tau}_1, \tau_1] = \emptyset$ and $(\widetilde{\varsigma}_1, \varsigma_1] = \emptyset$, $H(\chi(\tau_1 + 0; 0, \chi_0), \widetilde{r}(\widetilde{\tau}_1, \tau_1]) = 0$, $H(\widetilde{\chi}(\widetilde{\varsigma}_1; 0, \widetilde{\chi}_0), r(\widetilde{\varsigma}_1, \varsigma_1]) = 0$.

Set $0 < \eta_{11} < \epsilon$, and we have that
$\exists\ \delta_{\tau_1} > 0,\ \delta_{\tau_1} < \frac{\eta_{11}}{4L_{g_1}},\ \forall\ \widetilde{\chi}_0 \in D,\ \|\widetilde{\chi}_0 - \chi_0\| < \delta_0,\ \forall\ \widetilde{d}_{\varsigma_1} > 0,\ |\widetilde{d}_{\varsigma_1} - d| < \delta_{\tau_1}$, then $\|\widetilde{\chi}(\tau; 0, \widetilde{\chi}_0) - \chi(\tau; 0, \chi_0)\| < \frac{1}{2}\eta_{11},\ \tau_1^{\max} < \tau \le \varsigma_1^{\min}$, that is $R(\widetilde{r}(\tau_1^{\max}, \varsigma_1^{\min}], r(\tau_1^{\max}, \varsigma_1^{\min}]) < \frac{1}{2}\eta_{11} < \epsilon$.

For $\tau_1 < \tau \le \widetilde{\tau}_1$,

$$\begin{aligned}\|\widetilde{\chi}(\widetilde{\tau}_1 + 0; 0, \widetilde{\chi}_0) - \chi(\tau; 0, \chi_0)\| &\le \|\widetilde{\chi}(\widetilde{\tau}_1; 0, \widetilde{\chi}_0) - \chi(\widetilde{\tau}_1; 0, \chi_0)\| + \|\chi(\widetilde{\tau}_1; 0, \chi_0) - \chi(\tau; 0, \chi_0)\| \\ &< \frac{1}{2}\eta_{11} + \|g_1(\widetilde{\tau}_1, \chi(\tau_1^-)) - g_1(\tau, \chi(\tau_1^-))\|\end{aligned}$$

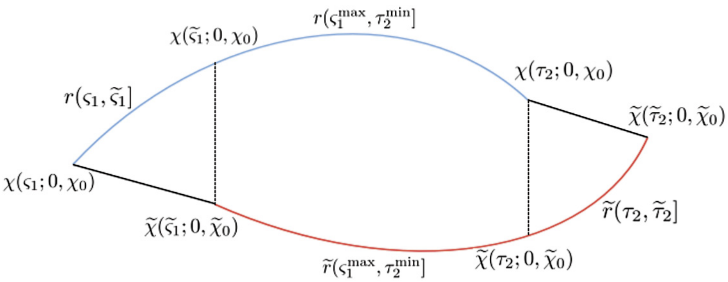

Fig. 4. The blue line denotes the orbital of the solution to (1) in the interval $(\varsigma_1, \tau_2]$, and the red line denotes the orbital of the solution to (3) in the interval $(\widetilde{\varsigma}_1, \widetilde{\tau}_2]$.

$$
\begin{aligned}
&\le \frac{1}{2}\eta_{11} + L_{g_1}|\widetilde{\tau}_1 - \tau_1| \\
&< \frac{1}{2}\eta_{11} + L_{g_1}\delta_{\tau_1} < \eta_{11} < \epsilon,
\end{aligned}
\tag{11}
$$

i.e. $H(\widetilde{\chi}(\widetilde{\tau}_1 + 0; 0, \widetilde{\chi}_0), r(\tau_1, \widetilde{\tau}_1]) < \epsilon$.

Note that $|\widetilde{\varsigma}_1 - \varsigma_1| \le |\widetilde{\tau}_1 - \tau_1| + |\widetilde{d}_{\varsigma_1} - d| < 2\delta_{\tau_1}$. For $\varsigma_1 < \tau \le \widetilde{\varsigma}_1$,

$$
\begin{aligned}
\|\chi(\varsigma_1; 0, \chi_0) - \widetilde{\chi}(\tau; 0, \widetilde{\chi}_0)\| &\le \|\chi(\varsigma_1; 0, \chi_0) - \widetilde{\chi}(\varsigma_1; 0, \widetilde{\chi}_0)\| + \|\widetilde{\chi}(\tau; 0, \widetilde{\chi}_0) - \widetilde{\chi}(\varsigma_1; 0, \widetilde{\chi}_0)\| \\
&< \frac{1}{2}\eta_{11} + \|g_1(\tau, \widetilde{\chi}(\widetilde{\tau}_1^-)) - g_1(\varsigma_1, \widetilde{\chi}(\widetilde{\tau}_1^-))\| \\
&\le \frac{1}{2}\eta_{11} + L_{g_1}|\widetilde{\varsigma}_1 - \varsigma_1| \\
&< \frac{1}{2}\eta_{11} + 2L_{g_1}\delta_{\tau_1} < \eta_{11} < \epsilon,
\end{aligned}
$$

i.e. $H(\chi(\varsigma_1; 0, \chi_0), \widetilde{r}(\varsigma_1, \widetilde{\varsigma}_1]) < \epsilon$.

Set $\eta_{11} < 2L_{g_1}\delta_{\varsigma_1}$, and then $|\widetilde{\varsigma}_1 - \varsigma_1| < \delta_{\varsigma_1}$, where δ_{ς_1} is an arbitrary positive constant.
Hence, $\forall\ \delta_{\varsigma_1} > 0,\ \exists\ \delta_{\tau_1} > 0,\ \forall\ \widetilde{\chi}_0 \in D,\ \|\widetilde{\chi}_0 - \chi_0\| < \delta_0,\ \forall\ \widetilde{d}_{\varsigma_1} > 0,\ |\widetilde{d}_{\varsigma_1} - d| < \delta_{\tau_1}$, then $H(\widetilde{r}(\widetilde{\tau}_1, \widetilde{\varsigma}_1], r(\tau_1, \varsigma_1]) < \epsilon,\ |\widetilde{\varsigma}_1 - \varsigma_1| < \delta_{\varsigma_1}$.

For the Hausdorff distance between the trajectories $\widetilde{r}(\widetilde{\varsigma}_1, \widetilde{\tau}_2]$ and $r(\varsigma_1, \tau_2]$ (see Fig. 4),

$$
\begin{aligned}
&H\Big(\widetilde{r}(\widetilde{\varsigma}_1, \widetilde{\tau}_2], r(\varsigma_1, \tau_2]\Big) \\
&\le \max\Big\{R\Big(\widetilde{r}(\varsigma_1^{\max}, \tau_2^{\min}], r(\varsigma_1^{\max}, \tau_2^{\min}]\Big), H\Big(\chi(\varsigma_1 + 0; 0, \chi_0), \widetilde{r}(\widetilde{\varsigma}_1, \varsigma_1]\Big), \\
&\quad H\Big(\widetilde{\chi}(\widetilde{\varsigma}_1 + 0; 0, \widetilde{\chi}_0), r(\varsigma_1, \widetilde{\varsigma}_1]\Big), H\Big(\chi(\tau_2; 0, \chi_0), \widetilde{r}(\tau_2, \widetilde{\tau}_2]\Big), H\Big(\widetilde{\chi}(\widetilde{\tau}_2; 0, \widetilde{\chi}_0), r(\widetilde{\tau}_2, \tau_2]\Big)\Big\}.
\end{aligned}
\tag{12}
$$

Since $(\widetilde{\varsigma}_1, \varsigma_1] = \emptyset$, $(\widetilde{\tau}_2, \tau_2] = \emptyset$, then $H(\chi(\varsigma_1 + 0; 0, \chi_0), \widetilde{r}(\widetilde{\varsigma}_1, \varsigma_1]) = 0$, $H(\widetilde{\chi}(\widetilde{\tau}_2; 0, \widetilde{\chi}_0), r(\widetilde{\tau}_2, \tau_2]) = 0$.

Let $0 < \eta_{12} < \epsilon$, and we have that
$\exists\ \delta_{\varsigma_1} > 0,\ \delta_{\varsigma_1} < \frac{\eta_{12}}{4M},\ \forall\ \widetilde{\varsigma}_1 \in R^+,\ |\widetilde{\varsigma}_1 - \varsigma_1| < \delta_{\varsigma_1},\ \forall\ \widetilde{d}_{\tau_2} > 0,\ |\widetilde{d}_{\tau_2} - d| < \delta_{\varsigma_1}$, then $\|\widetilde{\chi}(\tau; 0, \widetilde{\chi}_0) - \chi(\tau; 0, \chi_0)\| < \frac{1}{2}\eta_{12}$, $\varsigma_1^{\max} < \tau \le \tau_2^{\min}$, that is $R(\widetilde{r}(\varsigma_1^{\max}, \tau_2^{\min}], r(\varsigma_1^{\max}, \tau_2^{\min}]) < \frac{1}{2}\eta_{12} < \epsilon$.

For $\varsigma_1 < \tau \le \widetilde{\varsigma}_1$,

$$
\begin{aligned}
\|\widetilde{\chi}(\widetilde{\varsigma}_1 + 0; 0, \widetilde{\chi}_0) - \chi(\tau; 0, \chi_0)\| &\le \|\widetilde{\chi}(\widetilde{\varsigma}_1; 0, \widetilde{\chi}_0) - \chi(\widetilde{\varsigma}_1; 0, \chi_0)\| + \|\chi(\widetilde{\varsigma}_1; 0, \chi_0) - \chi(\tau; 0, \chi_0)\| \\
&< \frac{1}{2}\eta_{12} + \|(\mathbf{I}^1_{\tau, \widetilde{\varsigma}_1} f)(\widetilde{\varsigma}_1, \chi)\| \\
&\le \frac{1}{2}\eta_{12} + M|\widetilde{\varsigma}_1 - \varsigma_1| \\
&< \frac{1}{2}\eta_{12} + M\delta_{\varsigma_1} < \eta_{12} < \epsilon,
\end{aligned}
$$

i.e. $H(\widetilde{\chi}(\widetilde{\varsigma}_1 + 0; 0, \widetilde{\chi}_0), r(\varsigma_1, \widetilde{\varsigma}_1]) < \epsilon$.

Note that $|\widetilde{\tau}_2 - \tau_2| \le |\widetilde{\varsigma}_1 - \varsigma_1| + |\widetilde{d}_{\tau_2} - d| < 2\delta_{\varsigma_1}$. For $\tau_2 < \tau \le \widetilde{\tau}_2$,

$$\begin{aligned}\|\chi(\tau_2;0,\chi_0) - \widetilde{\chi}(\tau;0,\widetilde{\chi}_0)\| &\le \|\chi(\tau_2;0,\chi_0) - \widetilde{\chi}(\tau_2;0,\widetilde{\chi}_0)\| + \|\widetilde{\chi}(\tau;0,\widetilde{\chi}_0) - \widetilde{\chi}(\tau_2;0,\widetilde{\chi}_0)\| \\ &< \frac{1}{2}\eta_{12} + \|(\mathbf{I}^1_{\tau_2,\tau} f)(\tau,\widetilde{\chi})\| \\ &\le \frac{1}{2}\eta_{12} + M|\widetilde{\tau}_2 - \tau_2| \\ &< \frac{1}{2}\eta_{12} + 2M\delta_{\varsigma_1} < \eta_{12} < \epsilon, \end{aligned} \tag{13}$$

i.e. $H(\chi(\tau_2;0,\chi_0), \widetilde{r}(\tau_2,\widetilde{\tau}_2]) < \epsilon$.

Let δ_{τ_2} denote an arbitrary positive constant, and we presume that $\eta_{12} < 2M\delta_{\tau_2}$, and then $|\widetilde{\tau}_2 - \tau_2| < \delta_{\tau_2}$.

Therefore, $\forall\ \delta_{\tau_2} > 0,\ \exists\ \delta_{\varsigma_1} > 0,\ \forall\ \widetilde{\varsigma}_1 \in R^+,\ |\widetilde{\varsigma}_1 - \varsigma_1| < \delta_{\varsigma_1},\ \forall\ \widetilde{d}_{\tau_2} > 0,\ |\widetilde{d}_{\tau_2} - d| < \delta_{\varsigma_1}$, then $H(\widetilde{r}(\widetilde{\varsigma}_1,\widetilde{\tau}_2], r(\varsigma_1,\tau_2]) < \epsilon$, $|\widetilde{\tau}_2 - \tau_2| < \delta_{\tau_2}$.

Consider the trajectories $\widetilde{r}(\widetilde{\tau}_2,\widetilde{\varsigma}_2]$ and $r(\tau_2,\varsigma_2]$ and note

$$\begin{aligned} &H\Big(\widetilde{r}(\widetilde{\tau}_2,\widetilde{\varsigma}_2], r(\tau_2,\varsigma_2]\Big) \\ &\le \max\Big\{R\Big(\widetilde{r}(\tau_2^{\max},\varsigma_2^{\min}], r(\tau_2^{\max},\varsigma_2^{\min}]\Big), H\Big(\chi(\tau_2+0;0,\chi_0), \widetilde{r}(\widetilde{\tau}_2,\tau_2]\Big), \\ &\quad H\Big(\widetilde{\chi}(\widetilde{\tau}_2+0;0,\widetilde{\chi}_0), r(\tau_2,\widetilde{\tau}_2]\Big), H\Big(\chi(\varsigma_2;0,\chi_0), \widetilde{r}(\varsigma_2,\widetilde{\varsigma}_2]\Big), H\Big(\widetilde{\chi}(\widetilde{\varsigma}_2;0,\widetilde{\chi}_0), r(\widetilde{\varsigma}_2,\varsigma_2]\Big)\Big\}. \end{aligned} \tag{14}$$

Since $(\widetilde{\tau}_2,\tau_2] = \emptyset$, $(\widetilde{\varsigma}_2,\varsigma_2] = \emptyset$, then $H(\chi(\tau_2+0;0,\chi_0), \widetilde{r}(\widetilde{\tau}_2,\tau_2]) = 0$, $H(\widetilde{\chi}(\widetilde{\varsigma}_2;0,\widetilde{\chi}_0), r(\widetilde{\varsigma}_2,\varsigma_2]) = 0$.

Let $0 < \eta_{22} < \epsilon$, and we have that
$\exists\ \delta_{\tau_2} > 0,\ \delta_{\tau_2} < \frac{\eta_{22}}{4L_{g_2}},\ \forall\ \widetilde{d}_{\varsigma_2} > 0,\ |\widetilde{d}_{\varsigma_2} - d| < \delta_{\tau_2}$, then $\|\widetilde{\chi}(\tau;0,\widetilde{\chi}_0) - \chi(\tau;0,\chi_0)\| < \frac{1}{2}\eta_{22}$, $\tau_2^{\max} < \tau \le \varsigma_2^{\min}$, that is $R(\widetilde{r}(\tau_2^{\max},\varsigma_2^{\min}], r(\tau_2^{\max},\varsigma_2^{\min}]) < \frac{1}{2}\eta_{22} < \epsilon$.

For $\tau_2 < \tau \le \widetilde{\tau}_2$,

$$\begin{aligned}\|\widetilde{\chi}(\widetilde{\tau}_2+0;0,\widetilde{\chi}_0) - \chi(\tau;0,\chi_0)\| &\le \|\widetilde{\chi}(\widetilde{\tau}_2;0,\widetilde{\chi}_0) - \chi(\widetilde{\tau}_2;0,\chi_0)\| + \|\chi(\widetilde{\tau}_2;0,\chi_0) - \chi(\tau;0,\chi_0)\| \\ &< \frac{1}{2}\eta_{22} + \|g_2(\widetilde{\tau}_2,\chi(\tau_2^-)) - g_2(\tau,\chi(\tau_2^-))\| \\ &\le \frac{1}{2}\eta_{22} + L_{g_2}|\widetilde{\tau}_2 - \tau_2| \\ &< \frac{1}{2}\eta_{22} + L_{g_2}\delta_{\tau_2} < \eta_{22} < \epsilon, \end{aligned}$$

i.e. $H(\widetilde{\chi}(\widetilde{\tau}_2+0;0,\widetilde{\chi}_0), r(\tau_2,\widetilde{\tau}_2]) < \epsilon$.

Note that $|\widetilde{\varsigma}_2 - \varsigma_2| \le |\widetilde{\tau}_2 - \tau_2| + |\widetilde{d}_{\varsigma_2} - d| < 2\delta_{\tau_2}$. For $\varsigma_2 < \tau \le \widetilde{\varsigma}_2$,

$$\begin{aligned}\|\chi(\varsigma_2;0,\chi_0) - \widetilde{\chi}(\tau;0,\widetilde{\chi}_0)\| &\le \|\chi(\varsigma_2;0,\chi_0) - \widetilde{\chi}(\varsigma_2;0,\widetilde{\chi}_0)\| + \|\widetilde{\chi}(\tau;0,\widetilde{\chi}_0) - \widetilde{\chi}(\varsigma_2;0,\widetilde{\chi}_0)\| \\ &< \frac{1}{2}\eta_{22} + \|g_2(\tau,\widetilde{\chi}(\widetilde{\tau}_2^-)) - g_2(\varsigma_2,\widetilde{\chi}(\widetilde{\tau}_2^-))\| \\ &\le \frac{1}{2}\eta_{22} + L_{g_2}|\widetilde{\varsigma}_2 - \varsigma_2| \\ &< \frac{1}{2}\eta_{22} + 2L_{g_2}\delta_{\tau_2} < \eta_{22} < \epsilon, \end{aligned}$$

i.e. $H(\chi(\varsigma_2;0,\chi_0), \widetilde{r}(\varsigma_2,\widetilde{\varsigma}_2]) < \epsilon$.

Let δ_{ς_2} be an arbitrary positive constant, put $\eta_{22} < 2L_{g_2}\delta_{\varsigma_2}$, and then $|\widetilde{\varsigma}_2 - \varsigma_2| < \delta_{\varsigma_2}$.
Hence $\forall\ \delta_{\varsigma_2} > 0,\ \exists\ \delta_{\tau_2} > 0,\ \forall\ \widetilde{d}_{\varsigma_2} > 0,\ |\widetilde{d}_{\varsigma_2} - d| < \delta_{\tau_2}$, then $H(\widetilde{r}(\widetilde{\tau}_2,\widetilde{\varsigma}_2], r(\tau_2,\varsigma_2]) < \epsilon$, $|\widetilde{\varsigma}_2 - \varsigma_2| < \delta_{\varsigma_2}$.

The Hausdorff distance about the trajectories $\widetilde{r}(\widetilde{\varsigma}_2,\widetilde{\tau}_3]$ and $r(\varsigma_2,\tau_3]$ is

$$\begin{aligned} &H\Big(\widetilde{r}(\widetilde{\varsigma}_2,\widetilde{\tau}_3], r(\varsigma_2,\tau_3]\Big) \\ &\le \max\Big\{R\Big(\widetilde{r}(\varsigma_2^{\max},\tau_3^{\min}], r(\varsigma_2^{\max},\tau_3^{\min}]\Big), H\Big(\chi(\varsigma_2+0;0,\chi_0), \widetilde{r}(\widetilde{\varsigma}_2,\varsigma_2]\Big), \\ &\quad H\Big(\widetilde{\chi}(\widetilde{\varsigma}_2+0;0,\widetilde{\chi}_0), r(\varsigma_2,\widetilde{\varsigma}_2]\Big), H\Big(\chi(\tau_3;0,\chi_0), \widetilde{r}(\tau_3,\widetilde{\tau}_3]\Big), H\Big(\widetilde{\chi}(\widetilde{\tau}_3;0,\widetilde{\chi}_0), r(\widetilde{\tau}_3,\tau_3]\Big)\Big\}. \end{aligned} \tag{15}$$

Since $(\widetilde{\varsigma}_2,\varsigma_2] = \emptyset$, $(\widetilde{\tau}_3,\tau_3] = \emptyset$, then $H(\chi(\varsigma_2+0;0,\chi_0), \widetilde{r}(\widetilde{\varsigma}_2,\varsigma_2]) = 0$, $H(\widetilde{\chi}(\widetilde{\tau}_3;0,\widetilde{\chi}_0), r(\widetilde{\tau}_3,\tau_3]) = 0$.

Let $0 < \eta_{23} < \epsilon$, and we have that
$\exists\ \delta_{\varsigma_2} > 0$, $\delta_{\varsigma_2} < \frac{\eta_{23}}{4M}$, $\forall\ \widetilde{\varsigma}_2 \in R^+$, $|\widetilde{\varsigma}_2 - \varsigma_2| < \delta_{\varsigma_2}$, $\forall\ \widetilde{d}_{\tau_3} > 0$, $|\widetilde{d}_{\tau_3} - d| < \delta_{\varsigma_2}$, then $\|\widetilde{\chi}(\tau; 0, \widetilde{\chi}_0) - \chi(\tau; 0, \chi_0)\| < \frac{1}{2}\eta_{23}$, $\varsigma_2^{\max} < \tau \le \tau_3^{\min}$, that is $R(\widetilde{r}(\varsigma_2^{\max}, \tau_3^{\min}], r(\varsigma_2^{\max}, \tau_3^{\min}]) < \frac{1}{2}\eta_{23} < \epsilon$.

For $\varsigma_2 < \tau \le \widetilde{\varsigma}_2$,

$$\begin{aligned}\|\widetilde{\chi}(\widetilde{\varsigma}_2 + 0; 0, \widetilde{\chi}_0) - \chi(\tau; 0, \chi_0)\| &\le \|\widetilde{\chi}(\widetilde{\varsigma}_2; 0, \widetilde{\chi}_0) - \chi(\widetilde{\varsigma}_2; 0, \chi_0)\| + \|\chi(\widetilde{\varsigma}_2; 0, \chi_0) - \chi(\tau; 0, \chi_0)\| \\ &< \frac{1}{2}\eta_{23} + \|(\mathbf{I}^1_{\tau, \widetilde{\varsigma}_2} f)(\widetilde{\varsigma}_2, \chi)\| \\ &\le \frac{1}{2}\eta_{23} + M|\widetilde{\varsigma}_2 - \varsigma_2| \\ &< \frac{1}{2}\eta_{23} + M\delta_{\varsigma_2} < \eta_{23} < \epsilon,\end{aligned}$$

i.e. $H(\widetilde{\chi}(\widetilde{\varsigma}_2 + 0; 0, \widetilde{\chi}_0), r(\varsigma_2, \widetilde{\varsigma}_2]) < \epsilon$.

Consider $|\widetilde{\tau}_3 - \tau_3| \le |\widetilde{\varsigma}_2 - \varsigma_2| + |\widetilde{d}_{\tau_3} - d| < 2\delta_{\varsigma_2}$. For $\tau_3 < \tau \le \widetilde{\tau}_3$,

$$\begin{aligned}\|\chi(\tau_3; 0, \chi_0) - \widetilde{\chi}(\tau; 0, \widetilde{\chi}_0)\| &\le \|\chi(\tau_3; 0, \chi_0) - \widetilde{\chi}(\tau_3; 0, \widetilde{\chi}_0)\| + \|\widetilde{\chi}(\tau; 0, \widetilde{\chi}_0) - \widetilde{\chi}(\tau_3; 0, \widetilde{\chi}_0)\| \\ &< \frac{1}{2}\eta_{23} + \|(\mathbf{I}^1_{\tau_3, \tau} f)(\tau, \widetilde{\chi})\| \\ &\le \frac{1}{2}\eta_{23} + M|\widetilde{\tau}_3 - \tau_3| \\ &< \frac{1}{2}\eta_{23} + 2M\delta_{\varsigma_2} < \eta_{23} < \epsilon,\end{aligned}$$

i.e. $H(\chi(\tau_3; 0, \chi_0), \widetilde{r}(\tau_3, \widetilde{\tau}_3]) < \epsilon$.

Set $\eta_{23} < 2M\delta_{\tau_3}$, and then $|\widetilde{\tau}_3 - \tau_3| < \delta_{\tau_3}$, where δ_{τ_3} denotes an arbitrary positive constant.
Therefore, $\forall\ \delta_{\tau_3} > 0$, $\exists\ \delta_{\varsigma_2} > 0$, $\forall\ \widetilde{\varsigma}_2 \in R^+$, $|\widetilde{\varsigma}_2 - \varsigma_2| < \delta_{\varsigma_2}$, $\forall\ \widetilde{d}_{\tau_3} > 0$, $|\widetilde{d}_{\tau_3} - d| < \delta_{\varsigma_2}$, then $H(\widetilde{r}(\widetilde{\varsigma}_2, \widetilde{\tau}_3], r(\varsigma_2, \tau_3]) < \epsilon$, $|\widetilde{\tau}_3 - \tau_3| < \delta_{\tau_3}$.

From the above procedure, we arrive at the conclusion:

$$\begin{aligned}&\forall\ \delta_{\varsigma_i} > 0,\ \exists\ \delta_{\tau_i} > 0,\ \forall\ \widetilde{d}_{\varsigma_i} > 0,\ |\widetilde{d}_{\varsigma_i} - d| < \delta_{\tau_i},\ \text{then} \\ &H(\widetilde{r}(\widetilde{\tau}_i, \widetilde{\varsigma}_i], r(\tau_i, \varsigma_i]) < \epsilon, \quad |\widetilde{\varsigma}_i - \varsigma_i| < \delta_{\varsigma_i}, \quad i = 1, 2, \cdots, k.\end{aligned} \tag{16}$$

$$\begin{aligned}&\forall\ \delta_{\tau_{i+1}} > 0,\ \exists\ \delta_{\varsigma_i} > 0,\ \forall\ \widetilde{\varsigma}_i \in R^+,\ |\widetilde{\varsigma}_i - \varsigma_i| < \delta_{\varsigma_i},\ \forall\ \widetilde{d}_{\tau_{i+1}} > 0,\ |\widetilde{d}_{\tau_{i+1}} - d| < \delta_{\varsigma_i}, \\ &\text{then} \quad H(\widetilde{r}(\widetilde{\varsigma}_i, \widetilde{\tau}_{i+1}], r(\varsigma_i, \tau_{i+1}]) < \epsilon, \quad |\widetilde{\tau}_{i+1} - \tau_{i+1}| < \delta_{\tau_{i+1}}, \quad i = 1, 2, \cdots, k-1.\end{aligned} \tag{17}$$

Finally, consider the trajectories $\widetilde{r}(\widetilde{\varsigma}_k, T]$ and $r(\varsigma_k, T]$,

$$\begin{aligned}H\Big(\widetilde{r}(\widetilde{\varsigma}_k, T], r(\varsigma_k, T]\Big) \le \max\Big\{&R\Big(\widetilde{r}(\varsigma_k^{\max}, T], r(\varsigma_k^{\max}, T]\Big), H\Big(\chi(\varsigma_k + 0; 0, \chi_0), \widetilde{r}(\widetilde{\varsigma}_k, \varsigma_k]\Big), \\ &H\Big(\widetilde{\chi}(\widetilde{\varsigma}_k + 0; 0, \widetilde{\chi}_0), r(\varsigma_k, \widetilde{\varsigma}_k]\Big)\Big\}.\end{aligned} \tag{18}$$

Since $(\widetilde{\varsigma}_k, \varsigma_k] = \emptyset$, $H(\chi(\varsigma_k + 0; 0, \chi_0), \widetilde{r}(\widetilde{\varsigma}_k, \varsigma_k]) = 0$.

One can deduce that $\forall\ \epsilon > 0$, $\exists\ \delta_{\varsigma_k}$, $0 < \delta_{\varsigma_k} < \frac{\epsilon}{2M}$, $\forall\ \widetilde{\varsigma}_k \in R^+$, $|\widetilde{\varsigma}_k - \varsigma_k| < \delta_{\varsigma_k}$, $\forall\ \widetilde{d}_{\tau_{k+1}} > 0$, $|\widetilde{d}_{\tau_{k+1}} - d| < \delta_{\varsigma_k}$, then $\|\widetilde{\chi}(\tau; 0, \widetilde{\chi}_0) - \chi(\tau; 0, \chi_0)\| < \frac{\epsilon}{2}$, $\varsigma_k^{\max} < \tau \le T$, that is $R(\widetilde{r}(\varsigma_k^{\max}, T], r(\varsigma_k^{\max}, T]) < \epsilon$.

For $\varsigma_k < \tau \le \widetilde{\varsigma}_k$,

$$\begin{aligned}\|\widetilde{\chi}(\widetilde{\varsigma}_k + 0; 0, \widetilde{\chi}_0) - \chi(\tau; 0, \chi_0)\| &\le \|\widetilde{\chi}(\widetilde{\varsigma}_k; 0, \widetilde{\chi}_0) - \chi(\widetilde{\varsigma}_k; 0, \chi_0)\| + \|\chi(\widetilde{\varsigma}_k; 0, \chi_0) - \chi(\tau; 0, \chi_0)\| \\ &< \frac{\epsilon}{2} + \|(\mathbf{I}^1_{\tau, \widetilde{\varsigma}_k} f)(\widetilde{\varsigma}_k, \chi)\| \\ &\le \frac{\epsilon}{2} + M|\widetilde{\varsigma}_k - \varsigma_k| \\ &< \frac{\epsilon}{2} + M\delta_{\varsigma_k} < \epsilon,\end{aligned}$$

i.e. $H(\widetilde{\chi}(\widetilde{\varsigma}_k + 0; 0, \widetilde{\chi}_0), r(\varsigma_k, \widetilde{\varsigma}_k]) < \epsilon$.

Therefore, $\forall\ \epsilon > 0$, $\exists\ \delta_{\varsigma_k} > 0$, $\forall\ \widetilde{\varsigma}_k \in R^+$, $|\widetilde{\varsigma}_k - \varsigma_k| < \delta_{\varsigma_k}$, $\forall\ \widetilde{d}_{\tau_k+1} > 0$, $|\widetilde{d}_{\tau_k+1} - d| < \delta_{\varsigma_k}$, then $H(\widetilde{r}(\widetilde{\varsigma}_k, T], r(\varsigma_k, T]) < \epsilon$.

Now $\delta_{\varsigma_k} = \delta_{\varsigma_k}(\epsilon)$, $\delta_{\tau_k} = \delta_{\tau_k}(\delta_{\varsigma_k}, \epsilon)$, $\delta_{\varsigma_{k-1}} = \delta_{\varsigma_{k-1}}(\delta_{\tau_k}, \epsilon)$, $\cdots$, $\delta_{\tau_1} = \delta_{\tau_1}(\delta_{\varsigma_1}, \epsilon)$ and $\delta_0 = \delta_0(\delta_T, \delta_{\tau_1}, \epsilon)$.

Consequently, one has the conclusion:

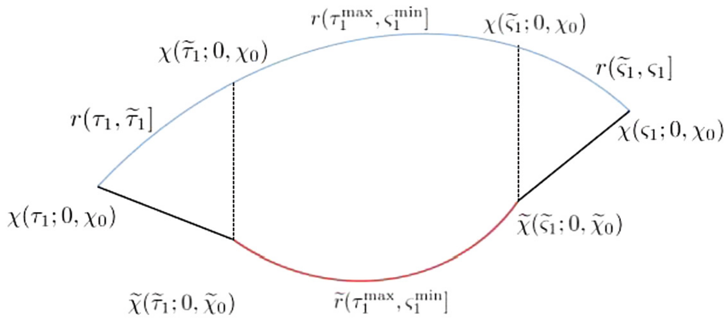

Fig. 5. The blue line denotes the orbital of the solution to (1) in the interval $(\tau_1, \varsigma_1]$, and the red line denotes the orbital of the solution to (3) in the interval $(\widetilde{\tau}_1, \widetilde{\varsigma}_1]$.

$\forall\ \epsilon > 0,\ \exists\ \delta_0 > 0,\ \forall\ \widetilde{\chi}_0 \in D,\ \|\widetilde{\chi}_0 - \chi_0\| < \delta_0,\ \forall\ \widetilde{d}_{\tau_i} > 0,\ \forall\ \widetilde{d}_{\varsigma_i} > 0,\ |\widetilde{d}_{\tau_i} - d| < \delta_0,\ |\widetilde{d}_{\varsigma_i} - d| < \delta_0,\ i = 1, 2, \cdots, k,$ then $H(\widetilde{r}[0, \widetilde{\tau}_1], r[0, \tau_1]) < \epsilon,\ H(\widetilde{r}(\widetilde{\tau}_i, \widetilde{\varsigma}_i], r(\tau_i, \varsigma_i]) < \epsilon,\ i = 1, 2, \cdots, k,\ H(\widetilde{r}(\widetilde{\varsigma}_i, \widetilde{\tau}_{i+1}], r(\varsigma_i, \tau_{i+1}]) < \epsilon,\ i = 1, 2, \cdots, k-1,$ $H(\widetilde{r}(\widetilde{\varsigma}_k, T], r(\varsigma_k, T]) < \epsilon$.

Note that

$$r[0, T] = r[0, \tau_1] \bigcup \Big(\bigcup_{i=1,2,\cdots,k} r(\tau_i, \varsigma_i] \Big) \bigcup \Big(\bigcup_{i=1,2,\cdots,k-1} r(\varsigma_i, \tau_{i+1}] \Big) \bigcup r(\varsigma_k, T],$$

$$\widetilde{r}[0, T] = \widetilde{r}[0, \widetilde{\tau}_1] \bigcup \Big(\bigcup_{i=1,2,\cdots,k} \widetilde{r}(\widetilde{\tau}_i, \widetilde{\varsigma}_i] \Big) \bigcup \Big(\bigcup_{i=1,2,\cdots,k-1} \widetilde{r}(\widetilde{\varsigma}_i, \widetilde{\tau}_{i+1}] \Big) \bigcup \widetilde{r}(\widetilde{\varsigma}_k, T].$$

Now apply Theorem 2.1, and then

$$\begin{aligned}
&H(\widetilde{r}[0, T], r[0, T]) \\
&= H\Big(\widetilde{r}[0, \widetilde{\tau}_1] \bigcup \Big(\bigcup_{i=1,2,\cdots,k} \widetilde{r}(\widetilde{\tau}_i, \widetilde{\varsigma}_i] \Big) \bigcup \Big(\bigcup_{i=1,2,\cdots,k-1} \widetilde{r}(\widetilde{\varsigma}_i, \widetilde{\tau}_{i+1}] \Big) \bigcup \widetilde{r}(\widetilde{\varsigma}_k, T], \\
&\qquad r[0, \tau_1] \bigcup \Big(\bigcup_{i=1,2,\cdots,k} r(\tau_i, \varsigma_i] \Big) \bigcup \Big(\bigcup_{i=1,2,\cdots,k-1} r(\varsigma_i, \tau_{i+1}] \Big) \bigcup r(\varsigma_k, T] \Big) \\
&\le \max \Big\{ H(\widetilde{r}[0, \widetilde{\tau}_1], r[0, \tau_1]),\ \ H(\widetilde{r}(\widetilde{\tau}_i, \widetilde{\varsigma}_i], r(\tau_i, \varsigma_i]),\ i = 1, 2, \cdots, k, \\
&\qquad H(\widetilde{r}(\widetilde{\varsigma}_i, \widetilde{\tau}_{i+1}], r(\varsigma_i, \tau_{i+1}]),\ i = 1, 2, \cdots, k-1,\ \ H(\widetilde{r}(\widetilde{\varsigma}_k, T], r(\varsigma_k, T]) \Big\} < \epsilon.
\end{aligned} \tag{19}$$

Case 2. Let $\tau_i^{\min} = \tau_i,\ \tau_i^{\max} = \widetilde{\tau}_i,\ \varsigma_i^{\min} = \widetilde{\varsigma}_i,\ \varsigma_i^{\max} = \varsigma_i,\ i = 1, 2, \cdots$; (the case $\tau_i^{\min} = \widetilde{\tau}_i,\ \tau_i^{\max} = \tau_i,\ \varsigma_i^{\min} = \varsigma_i,\ \varsigma_i^{\max} = \widetilde{\varsigma}_i,$ $i = 1, 2, \cdots$ can be considered analogously).

For the point $(0, \chi_0) \in [0, \infty) \times R^n$, let ϵ and T be positive constants. Since $\tau_i \to \infty$ $(i \to \infty)$, then $\exists\ k \in \Lambda \backslash \{0\}$ such that $2kd = \varsigma_k < T < \tau_{k+1} = (2k+1)d$.

In this case, we still have the formula (9) (see Fig. 2), the Hausdorff distance between the trajectories $\widetilde{r}[0, \widetilde{\tau}_1]$ and $r[0, \tau_1]$, and we have the same conclusion that $\forall\ \delta_{\tau_1} > 0,\ \exists\ \delta_0 > 0,\ \forall\ \widetilde{\chi}_0 \in D,\ \|\widetilde{\chi}_0 - \chi_0\| < \delta_0,\ \forall\ \widetilde{d}_{\tau_1} > 0,\ |\widetilde{d}_{\tau_1} - d| < \delta_0,$ then $H(\widetilde{r}[0, \widetilde{\tau}_1], r[0, \tau_1]) < \epsilon,\ |\widetilde{\tau}_1 - \tau_1| < \delta_{\tau_1}$.

For the trajectories $\widetilde{r}(\widetilde{\tau}_1, \widetilde{\varsigma}_1]$ and $r(\tau_1, \varsigma_1]$, we have the inequality (10). Since $(\widetilde{\tau}_1, \tau_1] = \emptyset$ and $(\varsigma_1, \widetilde{\varsigma}_1] = \emptyset$, then $H(\chi(\tau_1 + 0; 0, \chi_0), \widetilde{r}(\widetilde{\tau}_1, \tau_1]) = 0,\ H(\chi(\varsigma_1; 0, \chi_0), \widetilde{r}(\varsigma_1, \widetilde{\varsigma}_1]) = 0$ (see Fig. 5).

Let $0 < \eta_{11} < \epsilon$, and we have that
$\exists\ \delta_{\tau_1} > 0,\ \delta_{\tau_1} < \frac{\eta_{11}}{4L_{g_1}},\ \forall\ \widetilde{d}_{\varsigma_1} > 0,\ |d - \widetilde{d}_{\varsigma_1}| < \delta_{\tau_1},$ then $\|\widetilde{\chi}(\tau; 0, \widetilde{\chi}_0) - \chi(\tau; 0, \chi_0)\| < \frac{1}{2}\eta_{11},\ \tau_1^{\max} < \tau \le \varsigma_1^{\min}$, that is $R(\widetilde{r}(\tau_1^{\max}, \varsigma_1^{\min}], r(\tau_1^{\max}, \varsigma_1^{\min}]) < \frac{1}{2}\eta_{11} < \epsilon$.

For $\tau_1 < \tau \le \widetilde{\tau}_1$, similar to (11), we obtain $H(\widetilde{\chi}(\widetilde{\tau}_1 + 0; 0, \widetilde{\chi}_0), r(\tau_1, \widetilde{\tau}_1]) < \epsilon$.

Note that $|\varsigma_1 - \widetilde{\varsigma}_1| \le |\tau_1 - \widetilde{\tau}_1| + |d - \widetilde{d}_{\varsigma_1}| < 2\delta_{\tau_1}$. For $\widetilde{\varsigma}_1 < \tau \le \varsigma_1$,

$$\begin{aligned}
\|\widetilde{\chi}(\widetilde{\varsigma}_1; 0, \widetilde{\chi}_0) - \chi(\tau; 0, \chi_0)\| &\le \|\widetilde{\chi}(\widetilde{\varsigma}_1; 0, \widetilde{\chi}_0) - \chi(\widetilde{\varsigma}_1; 0, \chi_0)\| + \|\chi(\tau; 0, \chi_0) - \chi(\widetilde{\varsigma}_1; 0, \chi_0)\| \\
&< \frac{1}{2}\eta_{11} + \|g_1(\tau, \chi(\tau_1^-)) - g_1(\widetilde{\varsigma}_1, \chi(\tau_1^-))\| \\
&\le \frac{1}{2}\eta_{11} + L_{g_1}|\varsigma_1 - \widetilde{\varsigma}_1| \\
&< \frac{1}{2}\eta_{11} + 2L_{g_1}\delta_{\tau_1} < \eta_{11} < \epsilon,
\end{aligned} \tag{20}$$

i.e. $H(\widetilde{\chi}(\widetilde{\varsigma}_1; 0, \widetilde{\chi}_0), r(\widetilde{\varsigma}_1, \varsigma_1]) < \epsilon$.

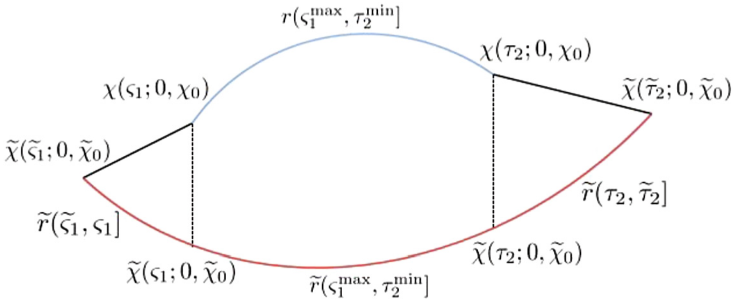

Fig. 6. The blue line denotes the orbital of the solution to (1) in the interval $(\varsigma_1, \tau_2]$, and the red line denotes the orbital of the solution to (3) in the interval $(\widetilde{\varsigma}_1, \widetilde{\tau}_2]$.

Let δ_{ς_1} denote an arbitrary positive constant, and assume that $\eta_{11} < 2L_{g_1}\delta_{\varsigma_1}$, and then $|\varsigma_1 - \widetilde{\varsigma}_1| < \delta_{\varsigma_1}$. Hence, $\forall\ \delta_{\varsigma_1} > 0,\ \exists\ \delta_{\tau_1} > 0,\ \forall\ \widetilde{d}_{\varsigma_1} > 0,\ |d - \widetilde{d}_{\varsigma_1}| < \delta_{\tau_1}$, then $H(\widetilde{r}(\widetilde{\tau}_1, \widetilde{\varsigma}_1], r(\tau_1, \varsigma_1]) < \epsilon,\ |\varsigma_1 - \widetilde{\varsigma}_1| < \delta_{\varsigma_1}$.

For the Hausdorff distance about the trajectories $\widetilde{r}(\widetilde{\varsigma}_1, \widetilde{\tau}_2]$ and $r(\varsigma_1, \tau_2]$ (see Fig. 6), we have the inequality (12). Since $(\varsigma_1, \widetilde{\varsigma}_1] = \emptyset$ and $(\widetilde{\tau}_2, \tau_2] = \emptyset$, then $H(\widetilde{\chi}(\widetilde{\varsigma}_1 + 0; 0, \widetilde{\chi}_0), r(\varsigma_1, \widetilde{\varsigma}_1]) = 0$. $H(\widetilde{\chi}(\widetilde{\tau}_2; 0, \widetilde{\chi}_0), r(\widetilde{\tau}_2, \tau_2]) = 0$.

Set $0 < \eta_{12} < \epsilon$; we have that $\exists\ \delta_{\varsigma_1} > 0,\ \delta_{\varsigma_1} < \frac{\eta_{12}}{4M},\ \forall\ \widetilde{\varsigma}_1 \in R^+,\ |\varsigma_1 - \widetilde{\varsigma}_1| < \delta_{\varsigma_1},\ \forall\ \widetilde{d}_{\tau_2} > 0,\ |d - \widetilde{d}_{\tau_2}| < \delta_{\varsigma_1}$, then $\|\widetilde{\chi}(\tau; 0, \widetilde{\chi}_0) - \chi(\tau; 0, \chi_0)\| < \frac{1}{2}\eta_{12},\ \varsigma_1^{\max} < \tau \leq \tau_2^{\min}$, that is $R(\widetilde{r}(\varsigma_1^{\max}, \tau_2^{\min}], r(\varsigma_1^{\max}, \tau_2^{\min}]) < \frac{1}{2}\eta_{12} < \epsilon$.

For $\widetilde{\varsigma}_1 < \tau \leq \varsigma_1$,

$$\begin{aligned}
\|\chi(\varsigma_1 + 0; 0, \chi_0) - \widetilde{\chi}(\tau; 0, \widetilde{\chi}_0)\| &\leq \|\chi(\varsigma_1; 0, \chi_0) - \widetilde{\chi}(\varsigma_1; 0, \widetilde{\chi}_0)\| + \|\widetilde{\chi}(\varsigma_1; 0, \widetilde{\chi}_0) - \widetilde{\chi}(\tau; 0, \widetilde{\chi}_0)\| \\
&< \frac{1}{2}\eta_{12} + \|(\mathbf{I}^1_{\tau,\varsigma_1} f)(\varsigma_1, \widetilde{\chi})\| \\
&\leq \frac{1}{2}\eta_{12} + M|\varsigma_1 - \widetilde{\varsigma}_1| \\
&< \frac{1}{2}\eta_{12} + M\delta_{\varsigma_1} < \eta_{12} < \epsilon,
\end{aligned}$$

i.e. $H(\chi(\varsigma_1 + 0; 0, \chi_0), \widetilde{r}(\widetilde{\varsigma}_1, \varsigma_1]) < \epsilon$.

For $\tau_2 < \tau \leq \widetilde{\tau}_2$, similar to (13), we get $H(\chi(\tau_2; 0, \chi_0), \widetilde{r}(\tau_2, \widetilde{\tau}_2]) < \epsilon$.

Set $\eta_{12} < 2M\delta_{\tau_2}$, and then $|\widetilde{\tau}_2 - \tau_2| < \delta_{\tau_2}$, where δ_{τ_2} is an arbitrary positive constant. Therefore, $\forall\ \delta_{\tau_2} > 0,\ \exists\ \delta_{\varsigma_1} > 0,\ \forall\ \widetilde{\varsigma}_1 \in R^+,\ |\varsigma_1 - \widetilde{\varsigma}_1| < \delta_{\varsigma_1},\ \forall\ \widetilde{d}_{\tau_2} > 0,\ |d - \widetilde{d}_{\tau_2}| < \delta_{\varsigma_1}$, then $H(\widetilde{r}(\widetilde{\varsigma}_1, \widetilde{\tau}_2], r(\varsigma_1, \tau_2]) < \epsilon$, $|\widetilde{\tau}_2 - \tau_2| < \delta_{\tau_2}$.

Repeating the procedure, we obtain the same conclusion as in (16) and (17).

Considering the trajectories $\widetilde{r}(\widetilde{\varsigma}_k, T]$ and $r(\varsigma_k, T]$, we still have the expression (18).

Since $(\varsigma_k, \widetilde{\varsigma}_k] = \emptyset$, $H(\widetilde{\chi}(\widetilde{\varsigma}_k + 0; 0, \widetilde{\chi}_0), r(\varsigma_k, \widetilde{\varsigma}_k]) = 0$.

We have that $\forall\ \epsilon > 0,\ \exists\ \delta_{\varsigma_k},\ 0 < \delta_{\varsigma_k} < \frac{\epsilon}{2M},\ \forall\ \widetilde{\varsigma}_k \in R^+,\ |\varsigma_k - \widetilde{\varsigma}_k| < \delta_{\varsigma_k},\ \forall\ \widetilde{d}_{\tau_{k+1}} > 0,\ |d - \widetilde{d}_{\tau_{k+1}}| < \delta_{\varsigma_k}$, then $\|\widetilde{\chi}(\tau; 0, \widetilde{\chi}_0) - \chi(\tau; 0, \chi_0)\| < \frac{\epsilon}{2},\ \varsigma_k^{\max} < \tau \leq T$, that is $R(\widetilde{r}(\varsigma_k^{\max}, T], r(\varsigma_k^{\max}, T]) < \epsilon$.

For $\widetilde{\varsigma}_k < \tau \leq \varsigma_k$,

$$\begin{aligned}
\|\chi(\varsigma_k + 0; 0, \chi_0) - \widetilde{\chi}(\tau; 0, \widetilde{\chi}_0)\| &\leq \|\chi(\varsigma_k; 0, \chi_0) - \widetilde{\chi}(\varsigma_k; 0, \widetilde{\chi}_0)\| + \|\widetilde{\chi}(\varsigma_k; 0, \widetilde{\chi}_0) - \widetilde{\chi}(\tau; 0, \widetilde{\chi}_0)\| \\
&< \frac{\epsilon}{2} + \|(\mathbf{I}^1_{\tau,\varsigma_k} f)(\varsigma_k, \widetilde{\chi})\| \\
&\leq \frac{\epsilon}{2} + M|\varsigma_k - \widetilde{\varsigma}_k| \\
&< \frac{\epsilon}{2} + M\delta_{\varsigma_k} < \epsilon,
\end{aligned}$$

i.e. $H(\chi(\varsigma_k + 0; 0, \chi_0), \widetilde{r}(\widetilde{\varsigma}_k, \varsigma_k]) < \epsilon$.

Therefore, $\forall\ \epsilon > 0,\ \exists\ \delta_{\varsigma_k} > 0,\ \forall\ \widetilde{\varsigma}_k \in R^+,\ |\varsigma_k - \widetilde{\varsigma}_k| < \delta_{\varsigma_k},\ \forall\ \widetilde{d}_{\tau_k+1} > 0,\ |d - \widetilde{d}_{\tau_k+1}| < \delta_{\varsigma_k}$, then $H(\widetilde{r}(\widetilde{\varsigma}_k, T], r(\varsigma_k, T]) < \epsilon$.

Now $\delta_{\varsigma_k} = \delta_{\varsigma_k}(\epsilon)$, $\delta_{\tau_k} = \delta_{\tau_k}(\delta_{\varsigma_k}, \epsilon)$, $\delta_{\varsigma_{k-1}} = \delta_{\varsigma_{k-1}}(\delta_{\tau_k}, \epsilon)$, $\cdots$, $\delta_{\tau_1} = \delta_{\tau_1}(\delta_{\varsigma_1}, \epsilon)$ and $\delta_0 = \delta_0(\delta_{\tau_1}, \epsilon)$.

Thus we infer that $\forall\ \epsilon > 0,\ \exists\ \delta_0 > 0,\ \forall\ \widetilde{\chi}_0 \in D,\ \|\widetilde{\chi}_0 - \chi_0\| < \delta_0,\ \forall\ \widetilde{d}_{\tau_i} > 0,\ \forall\ \widetilde{d}_{\varsigma_i} > 0,\ |\widetilde{d}_{\tau_i} - d| < \delta_0,\ |\widetilde{d}_{\varsigma_i} - d| < \delta_0,\ i = 1, 2, \cdots, k$, then $H(\widetilde{r}[0, \widetilde{\tau}_1], r[0, \tau_1]) < \epsilon,\ H(\widetilde{r}(\widetilde{\tau}_i, \widetilde{\varsigma}_i], r(\tau_i, \varsigma_i]) < \epsilon,\ i = 1, 2, \cdots, k,\ H(\widetilde{r}(\widetilde{\varsigma}_i, \widetilde{\tau}_{i+1}], r(\varsigma_i, \tau_{i+1}]) < \epsilon,\ i = 1, 2, \cdots, k-1$, $H(\widetilde{r}(\widetilde{\varsigma}_k, T], r(\varsigma_k, T]) < \epsilon$.

Note that

$$r[0, T] = r[0, \tau_1] \bigcup \left(\bigcup_{i=1,2,\cdots,k} r(\tau_i, \varsigma_i] \right) \bigcup \left(\bigcup_{i=1,2,\cdots,k-1} r(\varsigma_i, \tau_{i+1}] \right) \bigcup r(\varsigma_k, T],$$

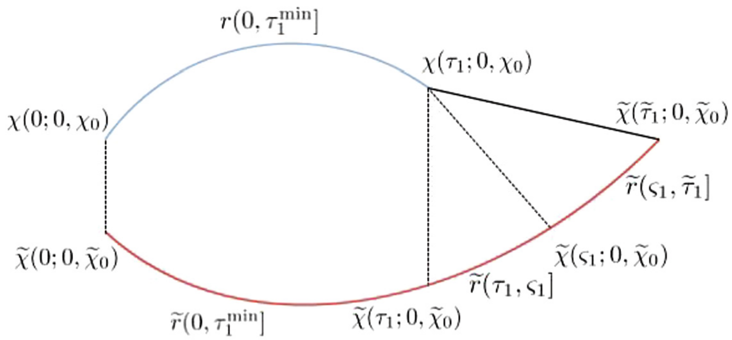

Fig. 7. The blue line denotes the orbital of the solution to (1) in the interval $(0,\tau_1]$, and the red line denotes the orbital of the solution to (3) in the interval $(0,\widetilde{\tau}_1]$.

$$\widetilde{r}[0,T]=\widetilde{r}[0,\widetilde{\tau}_1]\bigcup\left(\bigcup_{i=1,2,\cdots,k}\widetilde{r}(\widetilde{\tau}_i,\widetilde{\varsigma}_i]\right)\bigcup\left(\bigcup_{i=1,2,\cdots,k-1}\widetilde{r}(\widetilde{\varsigma}_i,\widetilde{\tau}_{i+1}]\right)\bigcup\widetilde{r}(\widetilde{\varsigma}_k,T].$$

Apply Theorem 2.1 and we have the result (19), that is $H(\widetilde{r}[0,T],r[0,T])<\epsilon$.

Case 3. Let $\tau_i^{\min}=\tau_i$, $\varsigma_i^{\min}=\varsigma_i$, $\tau_i^{\max}=\widetilde{\tau}_i$, $\varsigma_i^{\max}=\widetilde{\varsigma}_i$, $i=1,2,\cdots$; (the case $\tau_i^{\min}=\widetilde{\tau}_i$, $\varsigma_i^{\min}=\widetilde{\varsigma}_i$, $\tau_i^{\max}=\tau_i$, $\varsigma_i^{\max}=\varsigma_i$, $i=1,2,\cdots$ can be considered similarly).

For the point $(0,\chi_0)\in[0,\infty)\times R^n$, let ϵ and T be positive constants. Since $\tau_i\to\infty$ $(i\to\infty)$, then $\exists\ k\in\Lambda\backslash\{0\}$ such that $2kd=\varsigma_k<T<\tau_{k+1}=(2k+1)d$. Therefore, we can select a constant $\delta_T=\delta_T(d,T)>0$, which is sufficiently small, and then $\forall\ \widetilde{d}_{\tau_i},\ \widetilde{d}_{\varsigma_i}>0$, $|\widetilde{d}_{\tau_i}-d|<\delta_T$, $|\widetilde{d}_{\varsigma_i}-d|<\delta_T$ and $T<\varsigma_{k+1}^{\min}$, $\tau_i^{\min}<\varsigma_i^{\min}<\tau_i^{\max}<\varsigma_i^{\max}<\tau_{i+1}^{\min}$, $i=1,2,\cdots,k+1$.

Furthermore,

$\widetilde{\varsigma}_1<\tau_2\Leftrightarrow\widetilde{d}_{\tau_1}+\widetilde{d}_{\varsigma_1}<3d\Rightarrow\delta_T<\frac{d}{2}$;

$\widetilde{\varsigma}_2<\tau_3\Leftrightarrow\widetilde{d}_{\tau_1}+\widetilde{d}_{\varsigma_1}+\widetilde{d}_{\tau_2}+\widetilde{d}_{\varsigma_2}<5d\Rightarrow\delta_T<\frac{d}{4}$;

$\widetilde{\varsigma}_3<\tau_4\Leftrightarrow\widetilde{d}_{\tau_1}+\widetilde{d}_{\varsigma_1}+\cdots+\widetilde{d}_{\tau_3}+\widetilde{d}_{\varsigma_3}<7d\Rightarrow\delta_T<\frac{d}{6}$;

$\vdots$

$\widetilde{\varsigma}_k<T\Leftrightarrow\widetilde{d}_{\tau_1}+\widetilde{d}_{\varsigma_1}+\cdots+\widetilde{d}_{\tau_k}+\widetilde{d}_{\varsigma_k}<T\Rightarrow\delta_T<\frac{T-\varsigma_k}{2k}$.

From the inequalities, $0<\delta_T<\min\{\frac{d}{2k},\frac{T-\varsigma_k}{2k}\}$. With the property $H(\overline{X},\overline{Y})=H(X,Y)$ and Lemma 2.4, we consider the Hausdorff distance between the trajectories on the corresponding subintervals.

For the trajectories $\widetilde{r}[0,\widetilde{\tau}_1]$ and $r[0,\tau_1]$ (see Fig. 7),

$$\begin{aligned}H\Big(\widetilde{r}[0,\widetilde{\tau}_1],r[0,\tau_1]\Big)&=H\Big(\widetilde{r}(0,\widetilde{\tau}_1],r(0,\tau_1]\Big)\\&\le\max\Big\{R\Big(\widetilde{r}(0,\tau_1^{\min}],r(0,\tau_1^{\min}]\Big),H\Big(\chi(\tau_1;0,\chi_0),\widetilde{r}(\tau_1,\varsigma_1]\Big),\\&\qquad H\Big(\widetilde{r}(\varsigma_1,\widetilde{\tau}_1],r(\widetilde{\tau}_1,\varsigma_1]\Big),H\Big(\widetilde{\chi}(\widetilde{\tau}_1;0,\widetilde{\chi}_0),r(\varsigma_1,\tau_1]\Big)\Big\}.\end{aligned}\tag{21}$$

Since $(\widetilde{\tau}_1,\varsigma_1]=\emptyset$ and $(\varsigma_1,\tau_1]=\emptyset$, $H(\widetilde{r}(\varsigma_1,\widetilde{\tau}_1],r(\widetilde{\tau}_1,\varsigma_1])=0$, $H(\widetilde{\chi}(\widetilde{\tau}_1;0,\widetilde{\chi}_0),r(\varsigma_1,\tau_1])=0$.

Next, we will estimate the other two terms.

Let $0<\eta_{01}<\epsilon$, and we have that

$\exists\ \delta_0>0$, $\delta_0<\min\{\delta_T,\frac{\eta_{01}}{2M}\}$, $\forall\ \widetilde{\chi}_0\in D$, $\|\widetilde{\chi}_0-\chi_0\|<\delta_0$, $\forall\ \widetilde{d}_{\tau_1}>0$, $|\widetilde{d}_{\tau_1}-d|<\delta_0$, then $\|\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)-\chi(\tau;0,\chi_0)\|<\frac{1}{2}\eta_{01}$, $0<\tau\le\tau_1^{\min}$, that is $R(\widetilde{r}(0,\tau_1^{\min}],r(0,\tau_1^{\min}])<\frac{1}{2}\eta_{01}<\epsilon$.

Note $|\widetilde{\tau}_1-\tau_1|=|\widetilde{d}_{\tau_1}-d|<\delta_0<\frac{\eta_{01}}{2M}$. For $\tau_1<\tau\le\varsigma_1$,

$$\begin{aligned}\|\chi(\tau_1;0,\chi_0)-\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)\|&\le\|\chi(\tau_1;0,\chi_0)-\widetilde{\chi}(\tau_1;0,\widetilde{\chi}_0)\|+\|\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)-\widetilde{\chi}(\tau_1;0,\widetilde{\chi}_0)\|\\&<\frac{1}{2}\eta_{01}+\|(\mathbf{I}^1_{\tau_1,\tau}f)(\tau,\widetilde{\chi})\|\\&\le\frac{1}{2}\eta_{01}+M|\varsigma_1-\tau_1|<\frac{1}{2}\eta_{01}+M|\widetilde{\tau}_1-\tau_1|\\&<\frac{1}{2}\eta_{01}+M\delta_0<\eta_{01}<\epsilon,\end{aligned}$$

i.e. $H(\chi(\tau_1;0,\chi_0),\widetilde{r}(\tau_1,\varsigma_1])<\epsilon$.

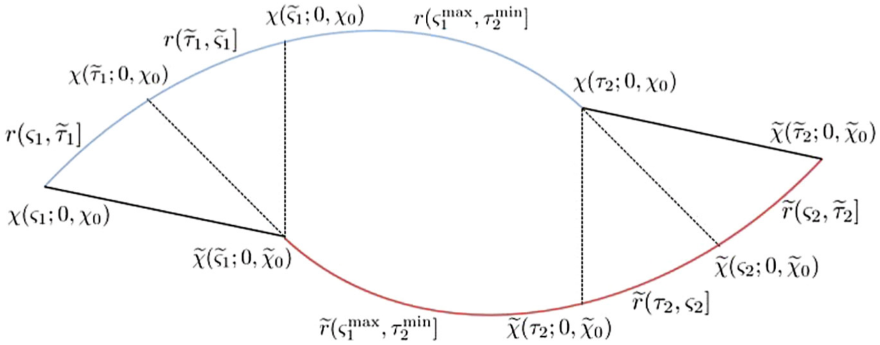

Fig. 8. The blue line denotes the orbital of the solution to (1) in the interval $(\varsigma_1,\tau_2]$, and the red line denotes the orbital of the solution to (3) in the interval $(\widetilde{\varsigma}_1,\widetilde{\tau}_2]$.

We assume that $\eta_{01} < 2M\delta_{\tau_1}$, and then $|\widetilde{\tau}_1-\tau_1| < \delta_{\tau_1}$, where δ_{τ_1} is an arbitrary positive constant.
Therefore, $\forall\ \delta_{\tau_1}>0,\ \exists\ \delta_0>0,\ \forall\ \widetilde{\chi}_0\in D,\ \|\widetilde{\chi}_0-\chi_0\|<\delta_0,\ \forall\ \widetilde{d}_{\tau_1}>0,\ |\widetilde{d}_{\tau_1}-d|<\delta_0$, then $H(\widetilde{r}[0,\widetilde{\tau}_1],r[0,\tau_1])<\epsilon,\ |\widetilde{\tau}_1-\tau_1|<\delta_{\tau_1}$.

Consider the trajectories $\widetilde{r}(\widetilde{\varsigma}_1,\widetilde{\tau}_2]$ and $r(\varsigma_1,\tau_2]$ (see Fig. 8), and the Hausdorff distance

$$\begin{aligned}
&H\Big(\widetilde{r}(\widetilde{\varsigma}_1,\widetilde{\tau}_2],r(\varsigma_1,\tau_2]\Big)\\
&\le \max\Big\{R\Big(\widetilde{r}(\varsigma_1^{\max},\tau_2^{\min}],r(\varsigma_1^{\max},\tau_2^{\min}]\Big),H\Big(\chi(\varsigma_1+0;0,\chi_0),\widetilde{r}(\widetilde{\varsigma}_1,\widetilde{\tau}_1]\Big),\\
&\qquad H\Big(r(\varsigma_1,\widetilde{\tau}_1],\widetilde{r}(\widetilde{\tau}_1,\varsigma_1]\Big),H\Big(\widetilde{\chi}(\widetilde{\varsigma}_1+0;0,\widetilde{\chi}_0),r(\widetilde{\tau}_1,\widetilde{\varsigma}_1]\Big),H\Big(\chi(\tau_2;0,\chi_0),\widetilde{r}(\tau_2,\varsigma_2]\Big),\\
&\qquad H\Big(\widetilde{r}(\varsigma_2,\widetilde{\tau}_2],r(\widetilde{\tau}_2,\varsigma_2]\Big),H\Big(\widetilde{\chi}(\widetilde{\tau}_2;0,\widetilde{\chi}_0),r(\varsigma_2,\tau_2]\Big)\Big\}.
\end{aligned}\tag{22}$$

Since $(\widetilde{\varsigma}_1,\widetilde{\tau}_1]=\emptyset$, $(\widetilde{\tau}_1,\varsigma_1]=\emptyset$, $(\widetilde{\tau}_2,\varsigma_2]=\emptyset$ and $(\varsigma_2,\tau_2]=\emptyset$, $H(\chi(\varsigma_1+0;0,\chi_0),\widetilde{r}(\widetilde{\varsigma}_1,\widetilde{\tau}_1])=0$, $H(r(\varsigma_1,\widetilde{\tau}_1],\widetilde{r}(\widetilde{\tau}_1,\varsigma_1])=0$, $H(\widetilde{r}(\varsigma_2,\widetilde{\tau}_2],r(\widetilde{\tau}_2,\varsigma_2])=0$ and $H(\widetilde{\chi}(\widetilde{\tau}_2;0,\widetilde{\chi}_0),r(\varsigma_2,\tau_2])=0$.

We need to estimate the other three parts in inequality (22).

Set $|\widetilde{d}_{\varsigma_1}-d|<\delta_{\tau_1}$, and then $|\widetilde{\varsigma}_1-\varsigma_1|\le|\widetilde{\tau}_1-\tau_1|+|\widetilde{d}_{\varsigma_1}-d|<2\delta_{\tau_1}$, so if δ_{τ_1} is sufficiently small, then $|\widetilde{\varsigma}_1-\varsigma_1|<\delta_{\varsigma_1}$, where δ_{ς_1} denotes an arbitrary positive number.

Let $0<\eta_{12}<\epsilon$, and we have that
$\exists\ \delta_{\varsigma_1}>0,\ \delta_{\varsigma_1}<\frac{\eta_{12}}{4M},\ \forall\ \widetilde{\varsigma}_1\in R^+,\ |\widetilde{\varsigma}_1-\varsigma_1|<\delta_{\varsigma_1},\ \forall\ \widetilde{d}_{\tau_2}>0,\ |\widetilde{d}_{\tau_2}-d|<\delta_{\varsigma_1}$, then $\|\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)-\chi(\tau;0,\chi_0)\|<\frac{1}{2}\eta_{12},\ \varsigma_1^{\max}<\tau\le\tau_2^{\min}$, that is $R(\widetilde{r}(\varsigma_1^{\max},\tau_2^{\min}],r(\varsigma_1^{\max},\tau_2^{\min}])<\frac{1}{2}\eta_{12}<\epsilon$.

For $\widetilde{\tau}_1<\tau\le\widetilde{\varsigma}_1$,

$$\begin{aligned}
\|\widetilde{\chi}(\widetilde{\varsigma}_1+0;0,\widetilde{\chi}_0)-\chi(\tau;0,\chi_0)\| &\le \|\widetilde{\chi}(\widetilde{\varsigma}_1;0,\widetilde{\chi}_0)-\chi(\widetilde{\varsigma}_1;0,\chi_0)\|+\|\chi(\widetilde{\varsigma}_1;0,\chi_0)-\chi(\tau;0,\chi_0)\|\\
&< \frac{1}{2}\eta_{12}+\|(\mathbf{I}^1_{\tau,\widetilde{\varsigma}_1}f)(\widetilde{\varsigma}_1,\chi)\|\\
&\le \frac{1}{2}\eta_{12}+M|\widetilde{\varsigma}_1-\widetilde{\tau}_1|<\frac{1}{2}\eta_{12}+M|\widetilde{\varsigma}_1-\varsigma_1|\\
&< \frac{1}{2}\eta_{12}+M\delta_{\varsigma_1}<\eta_{12}<\epsilon,
\end{aligned}$$

i.e. $H(\widetilde{\chi}(\widetilde{\varsigma}_1+0;0,\widetilde{\chi}_0),r(\widetilde{\tau}_1,\widetilde{\varsigma}_1])<\epsilon$.

Now $|\widetilde{\tau}_2-\tau_2|\le|\widetilde{\varsigma}_1-\varsigma_1|+|\widetilde{d}_{\tau_2}-d|<2\delta_{\varsigma_1}$. For $\tau_2<\tau\le\varsigma_2$,

$$\begin{aligned}
\|\chi(\tau_2;0,\chi_0)-\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)\| &\le \|\chi(\tau_2;0,\chi_0)-\widetilde{\chi}(\tau_2;0,\widetilde{\chi}_0)\|+\|\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)-\widetilde{\chi}(\tau_2;0,\widetilde{\chi}_0)\|\\
&< \frac{1}{2}\eta_{12}+\|(\mathbf{I}^1_{\tau_2,\tau}f)(\tau,\widetilde{\chi})\|\\
&\le \frac{1}{2}\eta_{12}+M|\varsigma_2-\tau_2|<\frac{1}{2}\eta_{12}+M|\widetilde{\tau}_2-\tau_2|\\
&< \frac{1}{2}\eta_{12}+2M\delta_{\varsigma_1}<\eta_{12}<\epsilon,
\end{aligned}$$

i.e. $H(\chi(\tau_2;0,\chi_0),\widetilde{r}(\tau_2,\varsigma_2])<\epsilon$.

Let $\delta_{\tau_2} > 0$ be an arbitrary number, set $\eta_{12} < 2M\delta_{\tau_2}$, and then $|\widetilde{\tau}_2 - \tau_2| < \delta_{\tau_2}$.
Therefore, $\forall\ \delta_{\tau_2} > 0,\ \exists\ \delta_{\varsigma_1} > 0,\ \forall\ \widetilde{\varsigma}_1 \in R^+,\ |\widetilde{\varsigma}_1 - \varsigma_1| < \delta_{\varsigma_1},\ \forall\ \widetilde{d}_{\tau_2} > 0,\ |\widetilde{d}_{\tau_2} - d| < \delta_{\varsigma_1}$, then $H(\widetilde{r}(\widetilde{\varsigma}_1, \widetilde{\tau}_2], r(\varsigma_1, \tau_2]) < \epsilon$, $|\widetilde{\tau}_2 - \tau_2| < \delta_{\tau_2}$.

From the process above, we have the conclusion:

$$\begin{aligned}&\forall\,\delta_{\tau_{i+1}} > 0,\ \exists\,\delta_{\varsigma_i} > 0,\ \forall\,\widetilde{\varsigma}_i \in R^+,\ |\widetilde{\varsigma}_i - \varsigma_i| < \delta_{\varsigma_i},\ \forall\,\widetilde{d}_{\tau_{i+1}} > 0,\ |\widetilde{d}_{\tau_{i+1}} - d| < \delta_{\varsigma_i},\\ &\text{then}\quad H(\widetilde{r}(\widetilde{\varsigma}_i, \widetilde{\tau}_{i+1}], r(\varsigma_i, \tau_{i+1}]) < \epsilon, \quad |\widetilde{\tau}_{i+1} - \tau_{i+1}| < \delta_{\tau_{i+1}}, \quad i = 1, 2, \cdots, k-1.\end{aligned} \tag{23}$$

For the trajectories $\widetilde{r}(\widetilde{\varsigma}_k, T]$ and $r(\varsigma_k, T]$,

$$\begin{aligned}&H\Big(\widetilde{r}(\widetilde{\varsigma}_k, T], r(\varsigma_k, T]\Big)\\ &\le \max\Big\{R\Big(\widetilde{r}(\varsigma_k^{\max}, T], r(\varsigma_k^{\max}, T]\Big), H\Big(\chi(\varsigma_k + 0; 0, \chi_0), \widetilde{r}(\widetilde{\varsigma}_k, \widetilde{\tau}_k]\Big),\\ &\qquad H\Big(r(\varsigma_k, \widetilde{\tau}_k], \widetilde{r}(\widetilde{\tau}_k, \varsigma_k]\Big), H\Big(\widetilde{\chi}(\widetilde{\varsigma}_k + 0; 0, \widetilde{\chi}_0), r(\widetilde{\tau}_k, \widetilde{\varsigma}_k]\Big)\Big\}.\end{aligned} \tag{24}$$

Since $(\widetilde{\varsigma}_k, \widetilde{\tau}_k] = \emptyset$, $(\widetilde{\tau}_k, \varsigma_k] = \emptyset$, then $H(\chi(\varsigma_k + 0; 0, \chi_0), \widetilde{r}(\widetilde{\varsigma}_k, \widetilde{\tau}_k] = 0$, $H(r(\varsigma_k, \widetilde{\tau}_k], \widetilde{r}(\widetilde{\tau}_k, \varsigma_k] = 0$.

Set $|\widetilde{d}_{\varsigma_k} - d| < \delta_{\tau_k}$, $|\widetilde{\varsigma}_k - \varsigma_k| \le |\widetilde{\tau}_k - \tau_k| + |\widetilde{d}_{\varsigma_k} - d| < 2\delta_{\tau_k}$, let $\delta_{\varsigma_k} > 0$ denote an arbitrary number, so if δ_{τ_k} is sufficiently small, then $|\widetilde{\varsigma}_k - \varsigma_k| < \delta_{\varsigma_k}$.

We have that $\forall\ \epsilon > 0$, $\exists\ \delta_{\varsigma_k}$, $0 < \delta_{\varsigma_k} < \frac{\epsilon}{2M}$, $\forall\ \widetilde{\varsigma}_k \in R^+$, $|\widetilde{\varsigma}_k - \varsigma_k| < \delta_{\varsigma_k}$, $\forall\ \widetilde{d}_{\tau_{k+1}} > 0$, $|\widetilde{d}_{\tau_{k+1}} - d| < \delta_{\varsigma_k}$, then $\|\widetilde{\chi}(\tau; 0, \widetilde{\chi}_0) - \chi(\tau; 0, \chi_0)\| < \frac{\epsilon}{2}$, $\varsigma_k^{\max} < \tau \le T$, that is $R(\widetilde{r}(\varsigma_k^{\max}, T], r(\varsigma_k^{\max}, T]) < \epsilon$.

For $\widetilde{\tau}_k < \tau \le \widetilde{\varsigma}_k$,

$$\begin{aligned}\|\widetilde{\chi}(\widetilde{\varsigma}_k + 0; 0, \widetilde{\chi}_0) - \chi(\tau; 0, \chi_0)\| &\le \|\widetilde{\chi}(\widetilde{\varsigma}_k; 0, \widetilde{\chi}_0) - \chi(\widetilde{\varsigma}_k; 0, \chi_0)\| + \|\chi(\widetilde{\varsigma}_k; 0, \chi_0) - \chi(\tau; 0, \chi_0)\|\\ &< \frac{\epsilon}{2} + \|(\mathbf{I}^1_{\tau, \widetilde{\varsigma}_k} f)(\widetilde{\varsigma}_k, \chi)\|\\ &< \frac{\epsilon}{2} + M|\widetilde{\varsigma}_k - \varsigma_k|\\ &< \frac{\epsilon}{2} + M\delta_{\varsigma_k} < \epsilon,\end{aligned}$$

i.e. $H(\widetilde{\chi}(\widetilde{\varsigma}_k + 0; 0, \widetilde{\chi}_0), r(\widetilde{\tau}_k, \widetilde{\varsigma}_k]) < \epsilon$.

Therefore, $\forall\ \epsilon > 0,\ \exists\ \delta_{\varsigma_k} > 0,\ \forall\ \widetilde{\varsigma}_k \in R^+,\ |\widetilde{\varsigma}_k - \varsigma_k| < \delta_{\varsigma_k},\ \forall\ \widetilde{d}_{\tau_k+1} > 0,\ |\widetilde{d}_{\tau_k+1} - d| < \delta_{\varsigma_k}$, then $H(\widetilde{r}(\widetilde{\varsigma}_k, T], r(\varsigma_k, T]) < \epsilon$.

Now $\delta_{\varsigma_k} = \delta_{\varsigma_k}(\epsilon)$, $\delta_{\tau_k} = \delta_{\tau_k}(\delta_{\varsigma_k}, \epsilon)$, $\delta_{\varsigma_{k-1}} = \delta_{\varsigma_{k-1}}(\delta_{\tau_k}, \epsilon)$, $\cdots$, $\delta_{\tau_1} = \delta_{\tau_1}(\delta_{\varsigma_1}, \epsilon)$ and $\delta_0 = \delta_0(\delta_T, \delta_{\tau_1}, \epsilon)$.

Consequently, we have the conclusion:

$\forall\ \epsilon > 0,\ \exists\ \delta_0 > 0,\ \forall\ \widetilde{\chi}_0 \in D,\ \|\widetilde{\chi}_0 - \chi_0\| < \delta_0,\ \forall\ \widetilde{d}_{\tau_i} > 0,\ \forall\ \widetilde{d}_{\varsigma_i} > 0,\ |\widetilde{d}_{\tau_i} - d| < \delta_0,\ |\widetilde{d}_{\varsigma_i} - d| < \delta_0,\ i = 1, 2, \cdots, k$, then $H(\widetilde{r}[0, \widetilde{\tau}_1], r[0, \tau_1]) < \epsilon$, $H(\widetilde{r}(\widetilde{\varsigma}_i, \widetilde{\tau}_{i+1}], r(\varsigma_i, \tau_{i+1}]) < \epsilon$, $i = 1, 2, \cdots, k-1$, $H(\widetilde{r}(\widetilde{\varsigma}_k, T], r(\varsigma_k, T]) < \epsilon$.

Consider that

$$\begin{aligned}r[0, T] &= r[0, \tau_1] \bigcup \Big(\bigcup_{i=1,2,\cdots,k} r(\tau_i, \varsigma_i]\Big) \bigcup \Big(\bigcup_{i=1,2,\cdots,k-1} r(\varsigma_i, \tau_{i+1}]\Big) \bigcup r(\varsigma_k, T],\\ \widetilde{r}[0, T] &= \widetilde{r}[0, \widetilde{\tau}_1] \bigcup \Big(\bigcup_{i=1,2,\cdots,k} \widetilde{r}(\widetilde{\tau}_i, \widetilde{\varsigma}_i]\Big) \bigcup \Big(\bigcup_{i=1,2,\cdots,k-1} \widetilde{r}(\widetilde{\varsigma}_i, \widetilde{\tau}_{i+1}]\Big) \bigcup \widetilde{r}(\widetilde{\varsigma}_k, T].\end{aligned}$$

Now apply Theorem 2.1 and [13, Theorem 1.3], and then

$$\begin{aligned}&H(\widetilde{r}[0, T], r[0, T])\\ &= H\Big(\widetilde{r}[0, \widetilde{\tau}_1] \bigcup \Big(\bigcup_{i=1,2,\cdots,k} \widetilde{r}(\widetilde{\tau}_i, \widetilde{\varsigma}_i]\Big) \bigcup \Big(\bigcup_{i=1,2,\cdots,k-1} \widetilde{r}(\widetilde{\varsigma}_i, \widetilde{\tau}_{i+1}]\Big) \bigcup \widetilde{r}(\widetilde{\varsigma}_k, T],\\ &\qquad r[0, \tau_1] \bigcup \Big(\bigcup_{i=1,2,\cdots,k} r(\tau_i, \varsigma_i]\Big) \bigcup \Big(\bigcup_{i=1,2,\cdots,k-1} r(\varsigma_i, \tau_{i+1}]\Big) \bigcup r(\varsigma_k, T]\Big)\\ &\le H\Big(\widetilde{r}[0, \widetilde{\tau}_1] \bigcup \Big(\bigcup_{i=1,2,\cdots,k-1} \widetilde{r}(\widetilde{\varsigma}_i, \widetilde{\tau}_{i+1}]\Big) \bigcup \widetilde{r}(\widetilde{\varsigma}_k, T],\\ &\qquad r[0, \tau_1] \bigcup \Big(\bigcup_{i=1,2,\cdots,k-1} r(\varsigma_i, \tau_{i+1}]\Big) \bigcup r(\varsigma_k, T]\Big)\end{aligned}$$

$$\le \max\Big\{H(\tilde r[0,\tilde\tau_1], r[0,\tau_1]),\ H(\tilde r(\tilde\varsigma_i,\tilde\tau_{i+1}], r(\varsigma_i,\tau_{i+1}]),\ i=1,2,\cdots,k-1,$$
$$H(\tilde r(\tilde\varsigma_k,T], r(\varsigma_k,T])\Big\} < \epsilon.$$

The proof is complete. □

Next, we present the orbital Hausdorff dependence on the initial condition and the difference between the impulsive points $\tau_i, i=1,2\cdots$ and the junction points $\varsigma_i, i=1,2\cdots$ of solutions to the fractional order impulsive differential equation (2).

Theorem 3.3. *Assume that conditions $[H_1]-[H_4]$ are fulfilled. Then, the solution to problem (2) is orbital Hausdorff dependent on the initial condition and the difference between the impulsive points $\tau_i, i=1,2\cdots$ and the junction points $\varsigma_i, i=1,2\cdots$.*

Proof. Consider the possible location of the distribution of the impulsive points τ_i, $\tilde\tau_i$ and the junction points ς_i, $\tilde\varsigma_i$, and we divide our proofs into several cases.

Case 1. Let $\tau_i^{\min}=\tau_i$, $\tau_i^{\max}=\tilde\tau_i$, $\varsigma_i^{\min}=\varsigma_i$, $\varsigma_i^{\max}=\tilde\varsigma_i$, $i=1,2,\cdots$; (the case $\tau_i^{\min}=\tilde\tau_i$, $\tau_i^{\max}=\tau_i$, $\varsigma_i^{\min}=\tilde\varsigma_i$, $\varsigma_i^{\max}=\varsigma_i$, $i=1,2,\cdots$ can be considered similarly). In this case, we have the same conclusion with δ_T in case 1 of Theorem 3.2.

Next, we consider the Hausdorff distance between the trajectories on the corresponding subintervals.

For $\tilde r[0,\tilde\tau_1]$ and $r[0,\tau_1]$, the inequality (9) is satisfied and further $H(\tilde\chi(\tilde\tau_1;0,\tilde\chi_0), r(\tilde\tau_1,\tau_1])=0$. We now estimate the other two terms.

Set $0<\eta_{01}<\epsilon$, and we have that

$\exists\ \delta_0>0,\ \delta_0<\min\{\delta_T, (\frac{\Gamma(\alpha+1)\eta_{01}}{6M})^{\frac{1}{\alpha}}\}$, $\forall\ \tilde\chi_0\in D$, $\|\tilde\chi_0-\chi_0\|<\delta_0$, $\forall\ \tilde d_{\tau_1}>0$, $|\tilde d_{\tau_1}-d|<\delta_0$, then $\|\tilde\chi(\tau;0,\tilde\chi_0)-\chi(\tau;0,\chi_0)\|<\frac12\eta_{01}$, $0<\tau\le\tau_1^{\min}$, that is $R(\tilde r(0,\tau_1^{\min}], r(0,\tau_1^{\min}])<\frac12\eta_{01}<\epsilon$.

Now $|\tilde\tau_1-\tau_1|=|\tilde d_{\tau_1}-d|<\delta_0<(\frac{\Gamma(\alpha+1)\eta_{01}}{6M})^{\frac1\alpha}$. For $\tau_1<\tau\le\tilde\tau_1$,

$$\begin{aligned}
&\|\chi(\tau_1;0,\chi_0)-\tilde\chi(\tau;0,\tilde\chi_0)\|\\
&\le\|\chi(\tau_1;0,\chi_0)-\tilde\chi(\tau_1;0,\tilde\chi_0)\|+\|\tilde\chi(\tau;0,\tilde\chi_0)-\tilde\chi(\tau_1;0,\tilde\chi_0)\|\\
&<\frac12\eta_{01}+\|(\mathbf{I}^\alpha_{0,\tau}f)(\tau,\tilde\chi)-(\mathbf{I}^\alpha_{0,\tau_1}f)(\tau_1,\tilde\chi)\|\\
&\le\frac12\eta_{01}+\frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}|(\tau-\varsigma)^{\alpha-1}-(\tau_1-\varsigma)^{\alpha-1}|\|f(\varsigma,\tilde\chi(\varsigma))\|\,\mathrm{d}s+\|(\mathbf{I}^\alpha_{\tau_1,\tau}f)(\tau,\tilde\chi)\|\\
&\le\frac12\eta_{01}+\frac{3M}{\Gamma(\alpha+1)}(\tilde\tau_1-\tau_1)^\alpha\\
&<\frac12\eta_{01}+\frac{3M}{\Gamma(\alpha+1)}\delta_0^\alpha<\eta_{01}<\epsilon,
\end{aligned}$$

i.e. $H(\chi(\tau_1;0,\chi_0),\tilde r(\tau_1,\tilde\tau_1])<\epsilon$.

Let $\delta_{\tau_1}>0$ be an arbitrary constant, assume that $\eta_{01}<\frac{6M}{\Gamma(\alpha+1)}\delta^\alpha_{\tau_1}$, and then $|\tilde\tau_1-\tau_1|<\delta_{\tau_1}$.

Therefore, $\forall\ \delta_{\tau_1}>0$, $\exists\ \delta_0>0$, $\forall\ \tilde\chi_0\in D$, $\|\tilde\chi_0-\chi_0\|<\delta_0$, $\forall\ \tilde d_{\tau_1}>0$, $|\tilde d_{\tau_1}-d|<\delta_0$, then $H(\tilde r[0,\tilde\tau_1], r[0,\tau_1])<\epsilon$, $|\tilde\tau_1-\tau_1|<\delta_{\tau_1}$.

For the trajectories $\tilde r(\tilde\tau_i,\tilde\varsigma_i]$ and $r(\tau_i,\varsigma_i]$, $i=1,2,\cdots,k$,

$$\begin{aligned}
&H\Big(\tilde r(\tilde\tau_i,\tilde\varsigma_i], r(\tau_i,\varsigma_i]\Big)\\
&\le\max\Big\{R\Big(\tilde r(\tau_i^{\max},\varsigma_i^{\min}], r(\tau_i^{\max},\varsigma_i^{\min}]\Big), H\Big(\chi(\tau_i+0;0,\chi_0),\tilde r(\tilde\tau_i,\tau_i]\Big),\\
&\quad H\Big(\tilde\chi(\tilde\tau_i+0;0,\tilde\chi_0), r(\tau_i,\tilde\tau_i]\Big), H\Big(\chi(\varsigma_i;0,\chi_0),\tilde r(\varsigma_i,\tilde\varsigma_i]\Big), H\Big(\tilde\chi(\tilde\varsigma_i;0,\tilde\chi_0), r(\tilde\varsigma_i,\varsigma_i]\Big)\Big\}.
\end{aligned}$$

Since $(\tilde\tau_i,\tau_i]=\emptyset$ and $(\tilde\varsigma_i,\varsigma_i]=\emptyset$, $H(\chi(\tau_i+0;0,\chi_0),\tilde r(\tilde\tau_i,\tau_i])=0$, $H(\tilde\chi(\tilde\varsigma_i;0,\tilde\chi_0), r(\tilde\varsigma_i,\varsigma_i])=0$.

Let $0<\eta_{ii}<\epsilon$, and we have that

$\exists\ \delta_{\tau_i}>0$, $\delta_{\tau_i}<\frac{\eta_{ii}}{4L_{g_i}}$, $\forall\ \tilde d_{\varsigma_i}>0$, $|\tilde d_{\varsigma_i}-d|<\delta_{\tau_i}$, then $\|\tilde\chi(\tau;0,\tilde\chi_0)-\chi(\tau;0,\chi_0)\|<\frac12\eta_{ii}$, $\tau_i^{\max}<\tau\le\varsigma_i^{\min}$, that is $R(\tilde r(\tau_i^{\max},\varsigma_i^{\min}], r(\tau_i^{\max},\varsigma_i^{\min}])<\frac12\eta_{ii}<\epsilon$.

For $\tau_i < \tau \le \widetilde{\tau}_i$,

$$\begin{aligned}\|\widetilde{\chi}(\widetilde{\tau}_i+0;0,\widetilde{\chi}_0)-\chi(\tau;0,\chi_0)\| &\le \|\widetilde{\chi}(\widetilde{\tau}_i;0,\widetilde{\chi}_0)-\chi(\widetilde{\tau}_i;0,\chi_0)\|+\|\chi(\widetilde{\tau}_i;0,\chi_0)-\chi(\tau;0,\chi_0)\|\\ &< \frac{1}{2}\eta_{ii}+\|g_i(\widetilde{\tau}_i,\chi(\tau_i^-))-g_i(\tau,\chi(\tau_i^-))\|\\ &\le \frac{1}{2}\eta_{ii}+L_{g_i}|\widetilde{\tau}_i-\tau_i|\\ &< \frac{1}{2}\eta_{ii}+L_{g_i}\delta_{\tau_i}<\eta_{ii}<\epsilon,\end{aligned}$$

i.e. $H(\widetilde{\chi}(\widetilde{\tau}_i+0;0,\widetilde{\chi}_0),r(\tau_i,\widetilde{\tau}_i])<\epsilon$.

Since $|\widetilde{\varsigma}_i-\varsigma_i|\le|\widetilde{\tau}_i-\tau_i|+|\widetilde{d}_{\varsigma_i}-d|<2\delta_{\tau_i}$, for $\varsigma_i<\tau\le\widetilde{\varsigma}_i$,

$$\begin{aligned}\|\chi(\varsigma_i;0,\chi_0)-\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)\| &\le \|\chi(\varsigma_i;0,\chi_0)-\widetilde{\chi}(\varsigma_i;0,\widetilde{\chi}_0)\|+\|\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)-\widetilde{\chi}(\varsigma_i;0,\widetilde{\chi}_0)\|\\ &< \frac{1}{2}\eta_{ii}+\|g_i(\tau,\widetilde{\chi}(\widetilde{\tau}_i^-))-g_i(\varsigma_i,\widetilde{\chi}(\widetilde{\tau}_i^-))\|\\ &\le \frac{1}{2}\eta_{ii}+L_{g_i}|\widetilde{\varsigma}_i-\varsigma_i|\\ &< \frac{1}{2}\eta_{ii}+2L_{g_i}\delta_{\tau_i}<\eta_{ii}<\epsilon,\end{aligned}$$

i.e. $H(\chi(\varsigma_i;0,\chi_0),\widetilde{r}(\varsigma_i,\widetilde{\varsigma}_i])<\epsilon$.

Set $\eta_{ii}<2L_{g_i}\delta_{\varsigma_i}$, and then $|\widetilde{\varsigma}_i-\varsigma_i|<\delta_{\varsigma_i}$, where δ_{ς_i} denotes arbitrary positive constants.

Therefore, $\forall\ \delta_{\varsigma_i}>0,\ \exists\ \delta_{\tau_i}>0,\ \forall\ \widetilde{d}_{\varsigma_i}>0,\ |\widetilde{d}_{\varsigma_i}-d|<\delta_{\tau_i}$, then $H(\widetilde{r}(\widetilde{\tau}_i,\widetilde{\varsigma}_i],r(\tau_i,\varsigma_i])<\epsilon,\ |\widetilde{\varsigma}_i-\varsigma_i|<\delta_{\varsigma_i}$.

For $\widetilde{r}(\widetilde{\varsigma}_1,\widetilde{\tau}_2]$ and $r(\varsigma_1,\tau_2]$, we have the inequality (12) and further $H(\chi(\varsigma_1+0;0,\chi_0),\widetilde{r}(\widetilde{\varsigma}_1,\varsigma_1])=0$, $H(\widetilde{\chi}(\widetilde{\tau}_2;0,\widetilde{\chi}_0),r(\widetilde{\tau}_2,\tau_2])=0$.

Let $0<\eta_{12}<\epsilon$, and we have that

$\exists\ \delta_{\varsigma_1},\ 0<\delta_{\varsigma_1}<(\frac{\Gamma(\alpha+1)\eta_{12}}{3M2^{\alpha+1}})^{\frac{1}{\alpha}},\ \forall\ \widetilde{\varsigma}_1\in R^+,\ |\widetilde{\varsigma}_1-\varsigma_1|<\delta_{\varsigma_1},\ \forall\ \widetilde{d}_{\tau_2}>0,\ |\widetilde{d}_{\tau_2}-d|<\delta_{\varsigma_1}$, then $\|\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)-\chi(\tau;0,\chi_0)\|<\frac{1}{2}\eta_{12}$, $\varsigma_1^{\max}<\tau\le\tau_2^{\min}$, that is $R(\widetilde{r}(\varsigma_1^{\max},\tau_2^{\min}],r(\varsigma_1^{\max},\tau_2^{\min}])<\frac{\eta_{12}}{2}<\epsilon$.

For $\varsigma_1<\tau\le\widetilde{\varsigma}_1$,

$$\begin{aligned}&\|\widetilde{\chi}(\widetilde{\varsigma}_1+0;0,\widetilde{\chi}_0)-\chi(\tau;0,\chi_0)\|\\ &\le \|\widetilde{\chi}(\widetilde{\varsigma}_1;0,\widetilde{\chi}_0)-\chi(\widetilde{\varsigma}_1;0,\chi_0)\|+\|\chi(\widetilde{\varsigma}_1;0,\chi_0)-\chi(\tau;0,\chi_0)\|\\ &< \frac{1}{2}\eta_{12}+\|(\mathbf{I}^{\alpha}_{\varsigma_1,\widetilde{\varsigma}_1}f)(\widetilde{\varsigma}_1,\chi)-(\mathbf{I}^{\alpha}_{\varsigma_1,\tau}f)(\tau,\chi)\|\\ &\le \frac{1}{2}\eta_{12}+\frac{1}{\Gamma(\alpha)}\int_{\varsigma_1}^{\tau}|(\widetilde{\varsigma}_1-\varsigma)^{\alpha-1}-(\tau-\varsigma)^{\alpha-1}|\|f(\varsigma,\chi(\varsigma))\|\,\mathrm{d}s+\|(\mathbf{I}^{\alpha}_{\tau,\widetilde{\varsigma}_1}f)(\widetilde{\varsigma}_1,\chi)\|\\ &\le \frac{1}{2}\eta_{12}+\frac{3M}{\Gamma(\alpha+1)}(\widetilde{\varsigma}_1-\varsigma_1)^{\alpha}\\ &< \frac{1}{2}\eta_{12}+\frac{3M}{\Gamma(\alpha+1)}\delta_{\varsigma_1}^{\alpha}<\eta_{12}<\epsilon,\end{aligned}$$

i.e. $H(\widetilde{\chi}(\widetilde{\varsigma}_1+0;0,\widetilde{\chi}_0),r(\varsigma_1,\widetilde{\varsigma}_1])<\epsilon$.

Consider $|\widetilde{\tau}_2-\tau_2|\le|\widetilde{\varsigma}_1-\varsigma_1|+|\widetilde{d}_{\tau_2}-d|<2\delta_{\varsigma_1}$. For $\tau_2<\tau\le\widetilde{\tau}_2$,

$$\begin{aligned}\|\chi(\tau_2;0,\chi_0)-\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)\| &\le \|\chi(\tau_2;0,\chi_0)-\widetilde{\chi}(\tau_2;0,\widetilde{\chi}_0)\|+\|\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)-\widetilde{\chi}(\tau_2;0,\widetilde{\chi}_0)\|\\ &< \frac{1}{2}\eta_{12}+\|(\mathbf{I}^{\alpha}_{\widetilde{\varsigma}_1,\tau}f)(\tau,\widetilde{\chi})-(\mathbf{I}^{\alpha}_{\widetilde{\varsigma}_1,\tau_2}f)(\tau_2,\widetilde{\chi})\|\\ &\le \frac{1}{2}\eta_{12}+\frac{3M}{\Gamma(\alpha+1)}(\widetilde{\tau}_2-\tau_2)^{\alpha}\\ &< \frac{1}{2}\eta_{12}+\frac{3M}{\Gamma(\alpha+1)}2^{\alpha}\delta_{\varsigma_1}^{\alpha}<\eta_{12}<\epsilon,\end{aligned}$$

i.e. $H(\chi(\tau_2;0,\chi_0),\widetilde{r}(\tau_2,\widetilde{\tau}_2])<\epsilon$.

Let $\delta_{\tau_2}>0$ denote an arbitrary constant, set $\eta_{12}<\frac{6M}{\Gamma(\alpha+1)}\delta_{\tau_2}^{\alpha}$, and then $|\widetilde{\tau}_2-\tau_2|<\delta_{\tau_2}$.

Therefore, $\forall\ \delta_{\tau_2}>0,\ \exists\ \delta_{\varsigma_1}>0,\ \forall\ \widetilde{\varsigma}_1\in R^+,\ |\widetilde{\varsigma}_1-\varsigma_1|<\delta_{\varsigma_1},\ \forall\ \widetilde{d}_{\tau_2}>0,\ |\widetilde{d}_{\tau_2}-d|<\delta_{\varsigma_1}$, then $H(\widetilde{r}(\widetilde{\varsigma}_1,\widetilde{\tau}_2],r(\varsigma_1,\tau_2])<\epsilon$, $|\widetilde{\tau}_2-\tau_2|<\delta_{\tau_2}$.

For $\widetilde{r}(\widetilde{\varsigma}_2, \widetilde{\tau}_3]$ and $r(\varsigma_2, \tau_3]$, inequality (15) is satisfied and we have $H(\chi(\varsigma_2+0;0,\chi_0), \widetilde{r}(\widetilde{\varsigma}_2, \varsigma_2]) = 0$ and $H(\widetilde{\chi}(\widetilde{\tau}_3;0,\widetilde{\chi}_0), r(\widetilde{\tau}_3, \tau_3]) = 0$.

Let $0 < \eta_{23} < \epsilon$, and we have that

$\exists\ \delta_{\varsigma_2}$, $0 < \delta_{\varsigma_2} < (\frac{\Gamma(\alpha+1)\eta_{23}}{3M2^{\alpha+1}})^{\frac{1}{\alpha}}$, $\forall\ \widetilde{\varsigma}_2 \in R^+$, $|\widetilde{\varsigma}_2 - \varsigma_2| < \delta_{\varsigma_2}$, $\forall\ \widetilde{d}_{\tau_3} > 0$, $|\widetilde{d}_{\tau_3} - d| < \delta_{\varsigma_2}$, then $\|\widetilde{\chi}(\tau;0,\widetilde{\chi}_0) - \chi(\tau;0,\chi_0)\| < \frac{1}{2}\eta_{23}$, $\varsigma_2^{\max} < \tau \le \tau_3^{\min}$, that is $R(\widetilde{r}(\varsigma_2^{\max}, \tau_3^{\min}], r(\varsigma_2^{\max}, \tau_3^{\min}]) < \frac{\eta_{23}}{2} < \epsilon$.

For $\varsigma_2 < \tau \le \widetilde{\varsigma}_2$,

$$
\begin{aligned}
&\|\widetilde{\chi}(\widetilde{\varsigma}_2+0;0,\widetilde{\chi}_0) - \chi(\tau;0,\chi_0)\| \\
&\le \|\widetilde{\chi}(\widetilde{\varsigma}_2;0,\widetilde{\chi}_0) - \chi(\widetilde{\varsigma}_2;0,\chi_0)\| + \|\chi(\widetilde{\varsigma}_2;0,\chi_0) - \chi(\tau;0,\chi_0)\| \\
&< \frac{1}{2}\eta_{23} + \|(\mathbf{I}^{\alpha}_{\varsigma_2,\widetilde{\varsigma}_2} f)(\widetilde{\varsigma}_2,\chi) - (\mathbf{I}^{\alpha}_{\varsigma_1,\tau} f)(\tau,\chi)\| \\
&\le \frac{1}{2}\eta_{23} + \frac{1}{\Gamma(\alpha)}\int_{\varsigma_2}^{\tau} |(\widetilde{\varsigma}_2-\varsigma)^{\alpha-1} - (\tau-\varsigma)^{\alpha-1}|\|f(\varsigma,\chi(\varsigma))\|\,\mathrm{d}s + \|(\mathbf{I}^{\alpha}_{\tau,\widetilde{\varsigma}_2} f)(\widetilde{\varsigma}_2,\chi)\| \\
&\le \frac{1}{2}\eta_{23} + \frac{3M}{\Gamma(\alpha+1)}(\widetilde{\varsigma}_2 - \varsigma_2)^{\alpha} \\
&< \frac{1}{2}\eta_{23} + \frac{3M}{\Gamma(\alpha+1)}\delta^{\alpha}_{\varsigma_2} < \eta_{23} < \epsilon,
\end{aligned}
$$

i.e. $H(\widetilde{\chi}(\widetilde{\varsigma}_2+0;0,\widetilde{\chi}_0), r(\varsigma_2,\widetilde{\varsigma}_2]) < \epsilon$.

Consider $|\widetilde{\tau}_3 - \tau_3| \le |\widetilde{\varsigma}_2 - \varsigma_2| + |\widetilde{d}_{\tau_3} - d| < 2\delta_{\varsigma_2}$, and for $\tau_3 < \tau \le \widetilde{\tau}_3$,

$$
\begin{aligned}
\|\chi(\tau_3;0,\chi_0) - \widetilde{\chi}(\tau;0,\widetilde{\chi}_0)\| &\le \|\chi(\tau_3;0,\chi_0) - \widetilde{\chi}(\tau_3;0,\widetilde{\chi}_0)\| + \|\widetilde{\chi}(\tau;0,\widetilde{\chi}_0) - \widetilde{\chi}(\tau_3;0,\widetilde{\chi}_0)\| \\
&< \frac{1}{2}\eta_{23} + \|(\mathbf{I}^{\alpha}_{\widetilde{\varsigma}_2,\tau} f)(\tau,\widetilde{\chi}) - (\mathbf{I}^{\alpha}_{\widetilde{\varsigma}_2,\tau_3} f)(\tau_3,\widetilde{\chi})\| \\
&\le \frac{1}{2}\eta_{23} + \frac{3M}{\Gamma(\alpha+1)}(\widetilde{\tau}_3 - \tau_3)^{\alpha} \\
&< \frac{1}{2}\eta_{23} + \frac{3M}{\Gamma(\alpha+1)}2^{\alpha}\delta^{\alpha}_{\varsigma_2} < \eta_{23} < \epsilon,
\end{aligned}
$$

i.e. $H(\chi(\tau_3;0,\chi_0), \widetilde{r}(\tau_3,\widetilde{\tau}_3]) < \epsilon$.

Set $\eta_{23} < \frac{6M}{\Gamma(\alpha+1)}\delta^{\alpha}_{\tau_3}$, and then $|\widetilde{\tau}_3 - \tau_3| < \delta_{\tau_3}$; here δ_{τ_3} denotes an arbitrary positive constant.

Therefore, $\forall\ \delta_{\tau_3} > 0$, $\exists\ \delta_{\varsigma_2} > 0$, $\forall\ \widetilde{\varsigma}_2 \in R^+$, $|\widetilde{\varsigma}_2 - \varsigma_2| < \delta_{\varsigma_2}$, $\forall\ \widetilde{d}_{\tau_3} > 0$, $|\widetilde{d}_{\tau_3} - d| < \delta_{\varsigma_2}$, then $H(\widetilde{r}(\widetilde{\varsigma}_2,\widetilde{\tau}_3], r(\varsigma_2,\tau_3]) < \epsilon$, $|\widetilde{\tau}_3 - \tau_3| < \delta_{\tau_3}$.

Repeat the above procedure, and we obtain the same conclusion as in (16) and (17).

Consider the trajectories $\widetilde{r}(\widetilde{\varsigma}_k, T]$ and $r(\varsigma_k, T]$, and we have the inequality (18) and further $H(\chi(\varsigma_k+0;0,\chi_0), \widetilde{r}(\widetilde{\varsigma}_k, \varsigma_k]) = 0$.

One can deduce that $\forall\ \epsilon > 0$, $\exists\ \delta_{\varsigma_k}$, $0 < \delta_{\varsigma_k} < (\frac{\Gamma(\alpha+1)\epsilon}{6M})^{\frac{1}{\alpha}}$, $\forall\ \widetilde{\varsigma}_k \in R^+$, $|\widetilde{\varsigma}_k - \varsigma_k| < \delta_{\varsigma_k}$, $\forall\ \widetilde{d}_{\tau_{k+1}} > 0$, $|\widetilde{d}_{\tau_{k+1}} - d| < \delta_{\varsigma_k}$, then $\|\widetilde{\chi}(\tau;0,\widetilde{\chi}_0) - \chi(\tau;0,\chi_0)\| < \frac{\epsilon}{2}$, $\varsigma_k^{\max} < \tau \le T$, that is $R(\widetilde{r}(\varsigma_k^{\max}, T], r(\varsigma_k^{\max}, T]) < \epsilon$.

For $\varsigma_k < \tau \le \widetilde{\varsigma}_k$,

$$
\begin{aligned}
\|\widetilde{\chi}(\widetilde{\varsigma}_k+0;0,\widetilde{\chi}_0) - \chi(\tau;0,\chi_0)\| &\le \|\widetilde{\chi}(\widetilde{\varsigma}_k;0,\widetilde{\chi}_0) - \chi(\widetilde{\varsigma}_k;0,\chi_0)\| + \|\chi(\widetilde{\varsigma}_k;0,\chi_0) - \chi(\tau;0,\chi_0)\| \\
&< \frac{\epsilon}{2} + \|(\mathbf{I}^{\alpha}_{\varsigma_k,\widetilde{\varsigma}_k} f)(\widetilde{\varsigma}_k,\chi) - (\mathbf{I}^{\alpha}_{\varsigma_k,\tau} f)(\tau,\chi)\| \\
&\le \frac{\epsilon}{2} + \frac{3M}{\Gamma(\alpha+1)}(\widetilde{\varsigma}_k - \varsigma_k)^{\alpha} \\
&< \frac{\epsilon}{2} + \frac{3M}{\Gamma(\alpha+1)}\delta^{\alpha}_{\varsigma_k} < \epsilon,
\end{aligned}
$$

i.e. $H(\widetilde{\chi}(\widetilde{\varsigma}_k+0;0,\widetilde{\chi}_0), r(\varsigma_k,\widetilde{\varsigma}_k]) < \epsilon$.

Therefore, $\forall\ \epsilon > 0$, $\exists\ \delta_{\varsigma_k} > 0$, $\forall\ \widetilde{\varsigma}_k \in R^+$, $|\widetilde{\varsigma}_k - \varsigma_k| < \delta_{\varsigma_k}$, $\forall\ \widetilde{d}_{\tau_k+1} > 0$, $|\widetilde{d}_{\tau_k+1} - d| < \delta_{\varsigma_k}$, then $H(\widetilde{r}(\widetilde{\varsigma}_k, T], r(\varsigma_k, T]) < \epsilon$.

Similar to case 1 of Theorem 3.2, we have $\forall\ \epsilon > 0$, $\exists\ \delta_0 > 0$, $\forall\ \widetilde{\chi}_0 \in D$, $\|\widetilde{\chi}_0 - \chi_0\| < \delta_0$, $\forall\ \widetilde{d}_{\tau_i} > 0$, $\forall\ \widetilde{d}_{\varsigma_i} > 0$, $|\widetilde{d}_{\tau_i} - d| < \delta_0$, $|\widetilde{d}_{\varsigma_i} - d| < \delta_0$, $i = 1, 2, \cdots, k$, then $H(\widetilde{r}[0,\widetilde{\tau}_1], r[0,\tau_1]) < \epsilon$, $H(\widetilde{r}(\widetilde{\tau}_i,\widetilde{\varsigma}_i], r(\tau_i,\varsigma_i]) < \epsilon$, $i = 1, 2, \cdots, k$, $H(\widetilde{r}(\widetilde{\varsigma}_i,\widetilde{\tau}_{i+1}], r(\varsigma_i,\tau_{i+1}]) < \epsilon$, $i = 1, 2, \cdots, k-1$, $H(\widetilde{r}(\widetilde{\varsigma}_k, T], r(\varsigma_k, T]) < \epsilon$.

Furthermore,

$$\begin{aligned}&H(\widetilde{r}[0,T],r[0,T])\\&\le \max\Big\{H(\widetilde{r}[0,\widetilde{\tau}_1],r[0,\tau_1]),\ \ H(\widetilde{r}(\widetilde{\tau}_i,\widetilde{\varsigma}_i],r(\tau_i,\varsigma_i]),\ i=1,2,\cdots,k,\\&\qquad H(\widetilde{r}(\widetilde{\varsigma}_i,\widetilde{\tau}_{i+1}],r(\varsigma_i,\tau_{i+1}]),\ i=1,2,\cdots,k-1,\ \ H(\widetilde{r}(\widetilde{\varsigma}_k,T],r(\varsigma_k,T])\Big\}<\epsilon.\end{aligned}$$

Case 2. Let $\tau_i^{\min}=\tau_i$, $\tau_i^{\max}=\widetilde{\tau}_i$, $\varsigma_i^{\min}=\widetilde{\varsigma}_i$, $\varsigma_i^{\max}=\varsigma_i$, $i=1,2,\cdots$; (the case $\tau_i^{\min}=\widetilde{\tau}_i$, $\tau_i^{\max}=\tau_i$, $\varsigma_i^{\min}=\varsigma_i$, $\varsigma_i^{\max}=\widetilde{\varsigma}_i$, $i=1,2,\cdots$ can be considered analogously).

In this case, we still have formula (9), and we follow the proof in case 1 of Theorem 3.3 for the trajectories $\widetilde{r}[0,\widetilde{\tau}_1]$ and $r[0,\tau_1]$, and we obtain the same conclusion that $\forall\ \delta_{\tau_1}>0,\ \exists\ \delta_0>0,\ \forall\ \widetilde{\chi}_0\in D,\ \|\widetilde{\chi}_0-\chi_0\|<\delta_0,\ \forall\ \widetilde{d}_{\tau_1}>0,\ |\widetilde{d}_{\tau_1}-d|<\delta_0$, then $H(\widetilde{r}[0,\widetilde{\tau}_1],r[0,\tau_1])<\epsilon,\ |\widetilde{\tau}_1-\tau_1|<\delta_{\tau_1}$.

Consider the trajectories $\widetilde{r}(\widetilde{\tau}_1,\widetilde{\varsigma}_1]$ and $r(\tau_1,\varsigma_1]$, and we also have the inequality (10). Since $(\widetilde{\tau}_1,\tau_1]=\emptyset$ and $(\varsigma_1,\widetilde{\varsigma}_1]=\emptyset$, then $H(\chi(\tau_1+0;0,\chi_0),\widetilde{r}(\widetilde{\tau}_1,\tau_1])=0$, $H(\chi(\varsigma_1;0,\chi_0),\widetilde{r}(\varsigma_1,\widetilde{\varsigma}_1])=0$.

Let $0<\eta_{11}<\epsilon$, and we have that
$\exists\ \delta_{\tau_1}>0,\ \delta_{\tau_1}<\frac{\eta_{11}}{4L_{g_1}},\ \forall\ \widetilde{\chi}_0\in D,\ \|\widetilde{\chi}_0-\chi_0\|<\delta_0,\ \forall\ \widetilde{d}_{\varsigma_1}>0,\ |d-\widetilde{d}_{\varsigma_1}|<\delta_{\tau_1}$, then $\|\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)-\chi(\tau;0,\chi_0)\|<\frac{1}{2}\eta_{11},\ \tau_1^{\max}<\tau\le\varsigma_1^{\min}$, that is $R(\widetilde{r}(\tau_1^{\max},\varsigma_1^{\min}],r(\tau_1^{\max},\varsigma_1^{\min}])<\frac{1}{2}\eta_{11}<\epsilon$.

For $\tau_1<\tau\le\widetilde{\tau}_1$, similar to (11), we get $H(\widetilde{\chi}(\widetilde{\tau}_1+0;0,\widetilde{\chi}_0),r(\tau_1,\widetilde{\tau}_1])<\epsilon$.

Now $|\varsigma_1-\widetilde{\varsigma}_1|\le|\tau_1-\widetilde{\tau}_1|+|d-\widetilde{d}_{\varsigma_1}|<2\delta_{\tau_1}$. For $\widetilde{\varsigma}_1<\tau\le\varsigma_1$, similar to (20) we have that $H(\widetilde{\chi}(\widetilde{\varsigma}_1;0,\widetilde{\chi}_0),r(\widetilde{\varsigma}_1,\varsigma_1])<\epsilon$.

Therefore, $\forall\ \delta_{\varsigma_1}>0,\ \exists\ \delta_{\tau_1}>0,\ \forall\ \widetilde{\chi}_0\in D,\ \|\widetilde{\chi}_0-\chi_0\|<\delta_0,\ \forall\ \widetilde{d}_{\varsigma_1}>0,\ |d-\widetilde{d}_{\varsigma_1}|<\delta_{\tau_1}$, then $H(\widetilde{r}(\widetilde{\tau}_1,\widetilde{\varsigma}_1],r(\tau_1,\varsigma_1])<\epsilon$, $|\varsigma_1-\widetilde{\varsigma}_1|<\delta_{\varsigma_1}$, where δ_{ς_1} denotes an arbitrary positive constant.

For the Hausdorff distance about the trajectories $\widetilde{r}(\widetilde{\varsigma}_1,\widetilde{\tau}_2]$ and $r(\varsigma_1,\tau_2]$, the inequality (12) holds. Since $(\varsigma_1,\widetilde{\varsigma}_1]=\emptyset$ and $(\widetilde{\tau}_2,\tau_2]=\emptyset$, then $H(\widetilde{\chi}(\widetilde{\varsigma}_1+0;0,\widetilde{\chi}_0),r(\varsigma_1,\widetilde{\varsigma}_1])=0$, $H(\widetilde{\chi}(\widetilde{\tau}_2;0,\widetilde{\chi}_0),r(\widetilde{\tau}_2,\tau_2])=0$.

Let $0<\eta_{12}<\epsilon$, and we have that
$\exists\ \delta_{\varsigma_1}>0,\ \delta_{\varsigma_1}<(\frac{\Gamma(\alpha+1)\eta_{12}}{3M2^{\alpha+1}})^{\frac{1}{\alpha}},\ \forall\ \widetilde{\varsigma}_1\in R^+,\ |\varsigma_1-\widetilde{\varsigma}_1|<\delta_{\varsigma_1},\ \forall\ \widetilde{d}_{\tau_2}>0,\ |\widetilde{d}_{\tau_2}-d|<\delta_{\varsigma_1}$, then $\|\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)-\chi(\tau;0,\chi_0)\|<\frac{1}{2}\eta_{12}$, $\varsigma_1^{\max}<\tau\le\tau_2^{\min}$, that is $R(\widetilde{r}(\varsigma_1^{\max},\tau_2^{\min}],r(\varsigma_1^{\max},\tau_2^{\min}])<\frac{\eta_{12}}{2}<\epsilon$.

For $\widetilde{\varsigma}_1<\tau\le\varsigma_1$,

$$\begin{aligned}\|\chi(\varsigma_1+0;0,\chi_0)-\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)\|&\le\|\chi(\varsigma_1;0,\chi_0)-\widetilde{\chi}(\varsigma_1;0,\widetilde{\chi}_0)\|+\|\widetilde{\chi}(\varsigma_1;0,\widetilde{\chi}_0)-\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)\|\\&<\frac{1}{2}\eta_{12}+\|(\mathbf{I}^{\alpha}_{\widetilde{\varsigma}_1,\varsigma_1}f)(\varsigma_1,\widetilde{\chi})-(\mathbf{I}^{\alpha}_{\widetilde{\varsigma}_1,\tau}f)(\tau,\widetilde{\chi})\|\\&\le\frac{1}{2}\eta_{12}+\frac{3M}{\Gamma(\alpha+1)}(\varsigma_1-\widetilde{\varsigma}_1)^{\alpha}\\&<\frac{1}{2}\eta_{12}+\frac{3M}{\Gamma(\alpha+1)}\delta^{\alpha}_{\varsigma_1}<\eta_{12}<\epsilon,\end{aligned}$$

i.e. $H(\chi(\varsigma_1+0;0,\chi_0),\widetilde{r}(\widetilde{\varsigma}_1,\varsigma_1])<\epsilon$.

For $\tau_2<\tau\le\widetilde{\tau}_2$, similar to (13), we have that $H(\chi(\tau_2;0,\chi_0),\widetilde{r}(\tau_2,\widetilde{\tau}_2])<\epsilon$.

Let δ_{τ_2} be an arbitrary positive constant, assume that $\eta_{12}<\frac{6M}{\Gamma(\alpha+1)}\delta^{\alpha}_{\tau_2}$, and then $|\widetilde{\tau}_2-\tau_2|<\delta_{\tau_2}$.

Therefore, $\forall\ \delta_{\tau_2}>0,\ \exists\ \delta_{\varsigma_1}>0,\ \forall\ \widetilde{\varsigma}_1\in R^+,\ |\varsigma_1-\widetilde{\varsigma}_1|<\delta_{\varsigma_1},\ \forall\ \widetilde{d}_{\tau_2}>0,\ |\widetilde{d}_{\tau_2}-d|<\delta_{\varsigma_1}$, then $H(\widetilde{r}(\widetilde{\varsigma}_1,\widetilde{\tau}_2],r(\varsigma_1,\tau_2])<\epsilon$, $|\widetilde{\tau}_2-\tau_2|<\delta_{\tau_2}$.

Repeat the procedure, and we obtain the same conclusion as in (16) and (17).

Considering the trajectories $\widetilde{r}(\widetilde{\varsigma}_k,T]$ and $r(\varsigma_k,T]$, we still have the expression (18) and $H(\widetilde{\chi}(\widetilde{\varsigma}_k+0;0,\widetilde{\chi}_0),r(\varsigma_k,\widetilde{\varsigma}_k])=0$.

We have that $\forall\ \epsilon>0,\ \exists\ \delta_{\varsigma_k},\ 0<\delta_{\varsigma_k}<(\frac{\Gamma(\alpha+1)\epsilon}{6M})^{\frac{1}{\alpha}},\ \forall\ \widetilde{\varsigma}_k\in R^+,\ |\varsigma_k-\widetilde{\varsigma}_k|<\delta_{\varsigma_k},\ \forall\ \widetilde{d}_{\tau_{k+1}}>0,\ |\widetilde{d}_{\tau_{k+1}}-d|<\delta_{\varsigma_k}$, then $\|\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)-\chi(\tau;0,\chi_0)\|<\frac{\epsilon}{2},\ \varsigma_k^{\max}<\tau\le T$, that is $R(\widetilde{r}(\varsigma_k^{\max},T],r(\varsigma_k^{\max},T])<\epsilon$.

For $\widetilde{\varsigma}_k<\tau\le\varsigma_k$,

$$\begin{aligned}\|\chi(\varsigma_k+0;0,\chi_0)-\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)\|&\le\|\chi(\varsigma_k;0,\chi_0)-\widetilde{\chi}(\varsigma_k;0,\widetilde{\chi}_0)\|+\|\widetilde{\chi}(\varsigma_k;0,\widetilde{\chi}_0)-\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)\|\\&<\frac{\epsilon}{2}+\|(\mathbf{I}^{\alpha}_{\widetilde{\varsigma}_k,\varsigma_k}f)(\varsigma_k,\widetilde{\chi})-(\mathbf{I}^{\alpha}_{\widetilde{\varsigma}_k,\tau}f)(\tau,\widetilde{\chi})\|\\&\le\frac{\epsilon}{2}+\frac{3M}{\Gamma(\alpha+1)}(\varsigma_k-\widetilde{\varsigma}_k)^{\alpha}\\&<\frac{\epsilon}{2}+\frac{3M}{\Gamma(\alpha+1)}\delta^{\alpha}_{\varsigma_k}<\epsilon,\end{aligned}$$

i.e. $H(\chi(\varsigma_k+0;0,\chi_0),\widetilde{r}(\widetilde{\varsigma}_k,\varsigma_k])<\epsilon$.

Therefore, $\forall\ \epsilon>0,\ \exists\ \delta_{\varsigma_k}>0,\ \forall\ \widetilde{\varsigma}_k\in R^+,\ |\varsigma_k-\widetilde{\varsigma}_k|<\delta_{\varsigma_k},\ \forall\ \widetilde{d}_{\tau_k+1}>0,\ |\widetilde{d}_{\tau_k+1}-d|<\delta_{\varsigma_k}$, then $H(\widetilde{r}(\widetilde{\varsigma}_k,T],r(\varsigma_k,T])<\epsilon$.

Now $\delta_{\varsigma_k}=\delta_{\varsigma_k}(\epsilon)$, $\delta_{\tau_k}=\delta_{\tau_k}(\delta_{\varsigma_k},\epsilon)$, $\delta_{\varsigma_{k-1}}=\delta_{\varsigma_{k-1}}(\delta_{\tau_k},\epsilon)$, $\cdots$, $\delta_{\tau_1}=\delta_{\tau_1}(\delta_{\varsigma_1},\epsilon)$ and $\delta_0=\delta_0(\delta_{\tau_1},\epsilon)$.

Consequently, one has the conclusion:

$\forall\ \epsilon>0,\ \exists\ \delta_0>0,\ \forall\ \widetilde{\chi}_0\in D,\ \|\widetilde{\chi}_0-\chi_0\|<\delta_0,\ \forall\ \widetilde{d}_{\tau_i}>0,\ \forall\ \widetilde{d}_{\varsigma_i}>0,\ |\widetilde{d}_{\tau_i}-d|<\delta_0,\ |d-\widetilde{d}_{\varsigma_i}|<\delta_0,\ i=1,2,\cdots,k$, then $H(\widetilde{r}[0,\widetilde{\tau}_1],r[0,\tau_1])<\epsilon$, $H(\widetilde{r}(\widetilde{\tau}_i,\widetilde{\varsigma}_i],r(\tau_i,\varsigma_i])<\epsilon$, $i=1,2,\cdots,k$, $H(\widetilde{r}(\widetilde{\varsigma}_i,\widetilde{\tau}_{i+1}],r(\varsigma_i,\tau_{i+1}])<\epsilon$, $i=1,2,\cdots,k-1$, $H(\widetilde{r}(\widetilde{\varsigma}_k,T],r(\varsigma_k,T])<\epsilon$.

Now apply Theorem 2.1 and [13, Theorem 1.3] , and we have the result (19), that is $H(\widetilde{r}[0,T],r[0,T])<\epsilon$.

Case 3. Let $\tau_i^{\min}=\tau_i$, $\varsigma_i^{\min}=\varsigma_i$, $\tau_i^{\max}=\widetilde{\tau}_i$, $\varsigma_i^{\max}=\widetilde{\varsigma}_i$, $i=1,2,\cdots$; (the case $\tau_i^{\min}=\widetilde{\tau}_i$, $\varsigma_i^{\min}=\widetilde{\varsigma}_i$, $\tau_i^{\max}=\tau_i$, $\varsigma_i^{\max}=\varsigma_i$, $i=1,2,\cdots$ can be considered similarly). In this case, we have the same conclusion, with δ_T in case 3 of Theorem 3.2.

For $\widetilde{r}[0,\widetilde{\tau}_1]$ and $r[0,\tau_1]$, we still have the inequality (21) and further $H(\widetilde{r}(\varsigma_1,\widetilde{\tau}_1],r(\widetilde{\tau}_1,\varsigma_1])=0$, $H(\widetilde{\chi}(\widetilde{\tau}_1;0,\widetilde{\chi}_0),r(\varsigma_1,\tau_1])=0$.

Set $0<\eta_{01}<\epsilon$, and we have that

$\exists\ \delta_0,\ 0<\delta_0<\min\{\delta_T,(\frac{\Gamma(\alpha+1)\eta_{01}}{6M})^{\frac{1}{\alpha}}\}$, $\forall\ \widetilde{\chi}_0\in D$, $\|\widetilde{\chi}_0-\chi_0\|<\delta_0$, $\forall\ \widetilde{d}_{\tau_1}>0$, $|\widetilde{d}_{\tau_1}-d|<\delta_0$, then $\|\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)-\chi(\tau;0,\chi_0)\|<\frac{1}{2}\eta_{01}$, $0<\tau\le\tau_1^{\min}$, that is $R(\widetilde{r}(0,\tau_1^{\min}],r(0,\tau_1^{\min}])<\frac{1}{2}\eta_{01}<\epsilon$.

Now $|\widetilde{\tau}_1-\tau_1|=|\widetilde{d}_{\tau_1}-d|<\delta_0<(\frac{\Gamma(\alpha+1)\eta_{01}}{6M})^{\frac{1}{\alpha}}$. For $\tau_1<\tau\le\varsigma_1$,

$$\begin{aligned}\|\chi(\tau_1;0,\chi_0)-\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)\|&\le\|\chi(\tau_1;0,\chi_0)-\widetilde{\chi}(\tau_1;0,\widetilde{\chi}_0)\|+\|\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)-\widetilde{\chi}(\tau_1;0,\widetilde{\chi}_0)\|\\&<\frac{1}{2}\eta_{01}+\|(\mathbf{I}^{\alpha}_{0,\tau}f)(\tau,\widetilde{\chi})-(\mathbf{I}^{\alpha}_{0,\tau_1}f)(\tau_1,\widetilde{\chi})\|\\&\le\frac{1}{2}\eta_{01}+\frac{3M}{\Gamma(\alpha+1)}(\varsigma_1-\tau_1)^{\alpha}<\frac{1}{2}\eta_{01}+\frac{3M}{\Gamma(\alpha+1)}(\widetilde{\tau}_1-\tau_1)^{\alpha}\\&<\frac{1}{2}\eta_{01}+\frac{3M}{\Gamma(\alpha+1)}\delta_0^{\alpha}<\eta_{01}<\epsilon,\end{aligned}$$

i.e. $H(\chi(\tau_1;0,\chi_0),\widetilde{r}(\tau_1,\varsigma_1])<\epsilon$.

Assume that $\eta_{01}<\frac{6M}{\Gamma(\alpha+1)}\delta_{\tau_1}^{\alpha}$, and then $|\widetilde{\tau}_1-\tau_1|<\delta_{\tau_1}$, where δ_{τ_1} is an arbitrary positive constant. Therefore, $\forall\ \delta_{\tau_1}>0$, $\exists\ \delta_0>0,\ \forall\ \widetilde{\chi}_0\in D,\ \|\widetilde{\chi}_0-\chi_0\|<\delta_0,\ \forall\ \widetilde{d}_{\tau_1}>0,\ |\widetilde{d}_{\tau_1}-d|<\delta_0$, then $H(\widetilde{r}[0,\widetilde{\tau}_1],r[0,\tau_1])<\epsilon$, $|\widetilde{\tau}_1-\tau_1|<\delta_{\tau_1}$.

For $\widetilde{r}(\widetilde{\varsigma}_1,\widetilde{\tau}_2]$ and $r(\varsigma_1,\tau_2]$, the inequality (22) is satisfied, and further $H(\chi(\varsigma_1+0;0,\chi_0),\widetilde{r}(\widetilde{\varsigma}_1,\widetilde{\tau}_1])=0$, $H(r(\varsigma_1,\widetilde{\tau}_1],\widetilde{r}(\widetilde{\tau}_1,\varsigma_1])=0$, $H(\widetilde{r}(\varsigma_2,\widetilde{\tau}_2],r(\widetilde{\tau}_2,\varsigma_2])=0$ and $H(\widetilde{\chi}(\widetilde{\tau}_2;0,\widetilde{\chi}_0),r(\varsigma_2,\tau_2])=0$.

Set $|d_{\varsigma_1}-d|<\delta_{\tau_1}$, and then $|\widetilde{\varsigma}_1-\varsigma_1|\le|\widetilde{\tau}_1-\tau_1|+|d_{\varsigma_1}-d|<2\delta_{\tau_1}$, so if δ_{τ_1} is sufficiently small, then $|\widetilde{\varsigma}_1-\varsigma_1|<\delta_{\varsigma_1}$, where δ_{ς_1} denotes an arbitrary positive number.

Let $0<\eta_{12}<\epsilon$, and we have that

$\exists\ \delta_{\varsigma_1}>0,\ \delta_{\varsigma_1}<(\frac{\Gamma(\alpha+1)\eta_{12}}{3M2^{\alpha+1}})^{\frac{1}{\alpha}}$, $\forall\ \widetilde{\varsigma}_1\in R^+$, $|\widetilde{\varsigma}_1-\varsigma_1|<\delta_{\varsigma_1}$, $\forall\ \widetilde{d}_{\tau_2}>0$, $|\widetilde{d}_{\tau_2}-d|<\delta_{\varsigma_1}$, then $\|\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)-\chi(\tau;0,\chi_0)\|<\frac{1}{2}\eta_{12}$, $\varsigma_1^{\max}<\tau\le\tau_2^{\min}$, that is $R(\widetilde{r}(\varsigma_1^{\max},\tau_2^{\min}],r(\varsigma_1^{\max},\tau_2^{\min}])<\frac{\eta_{12}}{2}<\epsilon$.

For $\widetilde{\tau}_1<\tau\le\widetilde{\varsigma}_1$,

$$\begin{aligned}\|\widetilde{\chi}(\widetilde{\varsigma}_1+0;0,\widetilde{\chi}_0)-\chi(\tau;0,\chi_0)\|&\le\|\widetilde{\chi}(\widetilde{\varsigma}_1;0,\widetilde{\chi}_0)-\chi(\widetilde{\varsigma}_1;0,\chi_0)\|+\|\chi(\widetilde{\varsigma}_1;0,\chi_0)-\chi(\tau;0,\chi_0)\|\\&<\frac{1}{2}\eta_{12}+\|(\mathbf{I}^{\alpha}_{\varsigma_1,\widetilde{\varsigma}_1}f)(\widetilde{\varsigma}_1,\chi)-(\mathbf{I}^{\alpha}_{\varsigma_1,\tau}f)(\tau,\chi)\|\\&\le\frac{1}{2}\eta_{12}+\frac{3M}{\Gamma(\alpha+1)}(\widetilde{\varsigma}_1-\widetilde{\tau}_1)^{\alpha}<\frac{1}{2}\eta_{12}+\frac{3M}{\Gamma(\alpha+1)}(\widetilde{\varsigma}_1-\varsigma_1)^{\alpha}\\&<\frac{1}{2}\eta_{12}+\frac{3M}{\Gamma(\alpha+1)}\delta_{\varsigma_1}^{\alpha}<\eta_{12}<\epsilon,\end{aligned}$$

i.e. $H(\widetilde{\chi}(\widetilde{\varsigma}_1+0;0,\widetilde{\chi}_0),r(\widetilde{\tau}_1,\widetilde{\varsigma}_1])<\epsilon$.

Now $|\widetilde{\tau}_2-\tau_2|\le|\widetilde{\varsigma}_1-\varsigma_1|+|\widetilde{d}_{\tau_2}-d|<2\delta_{\varsigma_1}$. For $\tau_2<\tau\le\varsigma_2$,

$$\begin{aligned}\|\chi(\tau_2;0,\chi_0)-\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)\|&\le\|\chi(\tau_2;0,\chi_0)-\widetilde{\chi}(\tau_2;0,\widetilde{\chi}_0)\|+\|\widetilde{\chi}(\tau;0,\widetilde{\chi}_0)-\widetilde{\chi}(\tau_2;0,\widetilde{\chi}_0)\|\\&<\frac{1}{2}\eta_{12}+\|(\mathbf{I}^{\alpha}_{\widetilde{\varsigma}_1,\tau}f)(\tau,\widetilde{\chi})-(\mathbf{I}^{\alpha}_{\widetilde{\varsigma}_1,\tau_2}f)(\tau_2,\widetilde{\chi})\|\\&\le\frac{1}{2}\eta_{12}+\frac{3M}{\Gamma(\alpha+1)}(\varsigma_2-\tau_2)^{\alpha}<\frac{1}{2}\eta_{12}+\frac{3M}{\Gamma(\alpha+1)}(\widetilde{\tau}_2-\tau_2)^{\alpha}\\&<\frac{1}{2}\eta_{12}+\frac{3M}{\Gamma(\alpha+1)}2^{\alpha}\delta_{\varsigma_1}^{\alpha}<\eta_{12}<\epsilon,\end{aligned}$$

i.e. $H(\chi(\tau_2;0,\chi_0),\widetilde{r}(\tau_2,\varsigma_2])<\epsilon$.

Let δ_{τ_2} be an arbitrary positive number, assume that $\eta_{12}<\frac{6M}{\Gamma(\alpha+1)}\delta_{\tau_2}^{\alpha}$, and then $|\widetilde{\tau}_2-\tau_2|<\delta_{\tau_2}$.

Therefore, $\forall\ \delta_{\tau_2}>0,\ \exists\ \delta_{\varsigma_1}>0,\ \forall\ \widetilde{\varsigma}_1\in R^+,\ |\widetilde{\varsigma}_1-\varsigma_1|<\delta_{\varsigma_1},\ \forall\ \widetilde{d}_{\tau_2}>0,\ |\widetilde{d}_{\tau_2}-d|<\delta_{\varsigma_1}$, then $H(\widetilde{r}(\widetilde{\varsigma}_1,\widetilde{\tau}_2],r(\varsigma_1,\tau_2])<\epsilon$, $|\widetilde{\tau}_2-\tau_2|<\delta_{\tau_2}$.

Consequently, the same conclusion in (23) follows.

For the trajectories $\widetilde{r}(\widetilde{\varsigma}_k, T]$ and $r(\varsigma_k, T]$, the inequality (24) holds, and furthermore we have $H(\chi(\varsigma_k+0;0,\chi_0), \widetilde{r}(\widetilde{\varsigma}_k, \widetilde{\tau}_k] = 0$ and $H(r(\varsigma_k, \widetilde{\tau}_k], \widetilde{r}(\widetilde{\tau}_k, \varsigma_k] = 0$.

Set $|\widetilde{d}_{\varsigma_k} - d| < \delta_{\tau_k}$, and then $|\widetilde{\varsigma}_k - \varsigma_k| \le |\widetilde{\tau}_k - \tau_k| + |\widetilde{d}_{\varsigma_k} - d| < 2\delta_{\tau_k}$, and let δ_{ς_k} denote an arbitrary positive number, so if δ_{τ_k} is sufficiently small, then $|\widetilde{\varsigma}_k - \varsigma_k| < \delta_{\varsigma_k}$.

We deduce that $\forall\ \epsilon > 0$, $\exists\ \delta_{\varsigma_k}$, $0 < \delta_{\varsigma_k} < (\frac{\Gamma(\alpha+1)\epsilon}{6M})^{\frac{1}{\alpha}}$, $\forall\ \widetilde{\varsigma}_k \in R^+$, $|\widetilde{\varsigma}_k - \varsigma_k| < \delta_{\varsigma_k}$, $\forall\ \widetilde{d}_{\tau_{k+1}} > 0$, $|\widetilde{d}_{\tau_{k+1}} - d| < \delta_{\varsigma_k}$, then $\|\widetilde{\chi}(\tau;0,\widetilde{\chi}_0) - \chi(\tau;0,\chi_0)\| < \frac{\epsilon}{2}$, $\varsigma_k^{\max} < \tau \le T$, that is $R(\widetilde{r}(\varsigma_k^{\max}, T], r(\varsigma_k^{\max}, T]) < \epsilon$.

For $\widetilde{\tau}_k < \tau \le \widetilde{\varsigma}_k$,

$$
\begin{aligned}
\|\widetilde{\chi}(\widetilde{\varsigma}_k+0;0,\widetilde{\chi}_0) - \chi(\tau;0,\chi_0)\| &\le \|\widetilde{\chi}(\widetilde{\varsigma}_k;0,\widetilde{\chi}_0) - \chi(\widetilde{\varsigma}_k;0,\chi_0)\| + \|\chi(\widetilde{\varsigma}_k;0,\chi_0) - \chi(\tau;0,\chi_0)\| \\
&< \frac{\epsilon}{2} + \|(\mathbf{I}^{\alpha}_{\varsigma_k,\widetilde{\varsigma}_k} f)(\widetilde{\varsigma}_k,\chi) - (\mathbf{I}^{\alpha}_{\varsigma_k,\tau} f)(\tau,\chi)\| \\
&< \frac{\epsilon}{2} + \frac{3M}{\Gamma(\alpha+1)}(\widetilde{\varsigma}_k - \varsigma_k)^{\alpha} \\
&< \frac{\epsilon}{2} + \frac{3M}{\Gamma(\alpha+1)}\delta^{\alpha}_{\varsigma_k} < \epsilon,
\end{aligned}
$$

i.e. $H(\widetilde{\chi}(\widetilde{\varsigma}_k+0;0,\widetilde{\chi}_0), r(\widetilde{\tau}_k, \widetilde{\varsigma}_k]) < \epsilon$.

Therefore, $\forall\ \epsilon > 0$, $\exists\ \delta_{\varsigma_k} > 0$, $\forall\ \widetilde{\varsigma}_k \in R^+$, $|\widetilde{\varsigma}_k - \varsigma_k| < \delta_{\varsigma_k}$, $\forall\ \widetilde{d}_{\tau_k+1} > 0$, $|\widetilde{d}_{\tau_k+1} - d| < \delta_{\varsigma_k}$, then $H(\widetilde{r}(\widetilde{\varsigma}_k, T], r(\varsigma_k, T]) < \epsilon$.

From an argument similar to that in case 3 of Theorem 3.2, we obtain $\forall\ \epsilon > 0$, $\exists\ \delta_0 > 0$, $\forall\ \widetilde{\chi}_0 \in D$, $\|\widetilde{\chi}_0 - \chi_0\| < \delta_0$, $\forall\ \widetilde{d}_{\tau_i} > 0$, $\forall\ \widetilde{d}_{\varsigma_i} > 0$, $|\widetilde{d}_{\tau_i} - d| < \delta_0$, $|\widetilde{d}_{\varsigma_i} - d| < \delta_0$, $i = 1, 2, \cdots, k$, then $H(\widetilde{r}[0, \widetilde{\tau}_1], r[0, \tau_1]) < \epsilon$, $H(\widetilde{r}(\widetilde{\varsigma}_i, \widetilde{\tau}_{i+1}], r(\varsigma_i, \tau_{i+1}]) < \epsilon$, $i = 1, 2, \cdots, k-1$, $H(\widetilde{r}(\widetilde{\varsigma}_k, T], r(\varsigma_k, T]) < \epsilon$.

Apply Theorem 2.1 and [13, Theorem 1.3], and then

$$
\begin{aligned}
& H(\widetilde{r}[0,T], r[0,T]) \\
&\le H\Bigg(\widetilde{r}[0,\widetilde{\tau}_1] \bigcup \Bigg(\bigcup_{i=1,2,\cdots,k-1} \widetilde{r}(\widetilde{\varsigma}_i, \widetilde{\tau}_{i+1}]\Bigg) \bigcup \widetilde{r}(\widetilde{\varsigma}_k, T], \\
&\qquad r[0,\tau_1] \bigcup \Bigg(\bigcup_{i=1,2,\cdots,k-1} r(\varsigma_i, \tau_{i+1}]\Bigg) \bigcup r(\varsigma_k, T]\Bigg) \\
&\le \max\Bigg\{H(\widetilde{r}[0,\widetilde{\tau}_1], r[0,\tau_1]), H(\widetilde{r}(\widetilde{\varsigma}_i, \widetilde{\tau}_{i+1}], r(\varsigma_i, \tau_{i+1}]),\ i = 1, 2, \cdots, k-1, \\
&\qquad H(\widetilde{r}(\widetilde{\varsigma}_k, T], r(\varsigma_k, T])\Bigg\} < \epsilon.
\end{aligned}
$$

The proof is complete. □

4. Examples

Set $\tau_0 = \varsigma_0 = 0$, $\tau_i = 2i - 1$ and $\varsigma_i = 2i$, $i \in \Lambda$. Clearly, $\varsigma_i < \tau_{i+1} \to \infty$ $(i \to \infty)$.

Example 4.1.

Consider the following NIDEs of integer order

$$
\begin{cases}
\chi'(\tau) = \arctan\chi(\tau),\ \tau \in (2i, 2i+1],\ i \in \Lambda, \\
\chi((2i-1)^+) = \frac{2i-1+|\chi((2i-1)^-)|}{2i+|\chi((2i-1)^-)|},\ i \in \Lambda \setminus \{0\}, \\
\chi(\tau) = \frac{\tau+|\chi((2i-1)^-)|}{1+\tau+|\chi((2i-1)^-)|},\ \tau \in (2i-1, 2i],\ i \in \Lambda \setminus \{0\}, \\
\chi(0) = \chi_0.
\end{cases}
\tag{25}
$$

One can derive the solution to (25), namely

$$\chi(\tau)=\begin{cases}\chi_0+\int_0^\tau \arctan\chi(s)\,ds, \text{ for } \tau\in(0,1],\\ \frac{\tau+|\chi(1^-)|}{1+\tau+|\chi(1^-)|}, \text{ for } \tau\in(1,2],\\ \frac{2+|\chi(1^-)|}{3+|\chi(1^-)|}+\int_2^\tau \arctan\chi(s)\,ds, \text{ for } \tau\in(2,3],\\ \vdots\\ \frac{\tau+|\chi((2\gamma-1)^-)|}{1+\tau+|\chi((2\gamma-1)^-)|}, \text{ for } \tau\in(2\gamma-1,2\gamma],\\ \frac{2\gamma+|\chi((2\gamma-1)^-)|}{1+2\gamma+|\chi((2\gamma-1)^-)|}+\int_{2\gamma}^\tau \arctan\chi(s)\,ds, \text{ for } \tau\in(2\gamma,2\gamma+1],\\ \vdots\end{cases} \tag{26}$$

Set $f(\tau,\chi)=\arctan\chi$, $g_i(\tau,\chi)=\frac{\tau+|\chi|}{1+\tau+|\chi|}$. Note $g_i\in C([2i-1,2i]\times D,R^n)$, $i=1,2,\cdots$.

Let $\tau\in(2i,2i+1]$. Clearly, $\|f(\tau,\chi)-f(\tau,\psi)\|\le\|\chi-\psi\|$ and $\|f(\tau,\chi)\|\le M:=\frac{\pi}{2}$, $\forall\ \chi\in R^n$. In addition, $\|g_i(\tau_1,\chi)-g_i(\tau_2,\psi)\|\le\|\tau_1-\tau_2\|+\|\chi-\psi\|, \forall\ \chi,\psi\in R^n$, so choose $L_{g_i}=1$. Thus, $[H_1]-[H_4]$ holds. Therefore Theorem 3.2 can be applied to (25).

Example 4.2.

Consider the following NIDEs of fractional order

$$\begin{cases}{}^c\mathbf{D}_{2i,\tau}^{\frac12}\chi(\tau)=\arctan\chi(\tau),\ \tau\in(2i,2i+1],\ i\in\Lambda,\ \alpha=\frac12\\ \chi((2i-1)^+)=\frac{2i-1+|\chi((2i-1)^-)|}{2i+|\chi((2i-1)^-)|},\ i\in\Lambda\setminus\{0\},\\ \chi(\tau)=\frac{\tau+|\chi((2i-1)^-)|}{1+\tau+|\chi((2i-1)^-)|},\ \tau\in(2i-1,2i],\ i\in\Lambda\setminus\{0\},\\ \chi(0)=\chi_0.\end{cases} \tag{27}$$

Clearly, $f(\tau,\chi)=\arctan\chi$, $g_i(\tau,\chi)=\frac{\tau+|\chi|}{1+\tau+|\chi|}$, which are the same as in Example 4.1. One can derive the solution to (27), namely

$$\chi(\tau)=\begin{cases}\chi_0+\frac{1}{\sqrt\pi}\int_0^\tau(\tau-s)^{-\frac12}\arctan\chi(s)\,ds,\quad \tau\in(0,1],\\ \frac{\tau+|\chi(1^-)|}{1+\tau+|\chi(1^-)|}, \text{ for } \tau\in(1,2],\\ \frac{2+|\chi(1^-)|}{3+|\chi(1^-)|}+\frac{1}{\sqrt\pi}\int_2^\tau(\tau-s)^{-\frac12}\arctan\chi(s)\,ds, \text{ for } \tau\in(2,3],\\ \vdots\\ \frac{\tau+|\chi((2\gamma-1)^-)|}{1+\tau+|\chi((2\gamma-1)^-)|}, \text{ for } \tau\in(2\gamma-1,2\gamma],\\ \frac{2\gamma+|\chi((2\gamma-1)^-)|}{1+2\gamma+|\chi((2\gamma-1)^-)|}+\frac{1}{\sqrt\pi}\int_{2\gamma}^\tau(\tau-s)^{-\frac12}\arctan\chi(s)\,ds, \text{ for } \tau\in(2\gamma,2\gamma+1],\\ \vdots\end{cases} \tag{28}$$

Note $[H_1]$–$[H_4]$ hold. Therefore Theorem 3.3 can be applied to (27).

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