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# Classification of $b^m$-Nambu structures of top degree ☆

## *Classification de structures $b^m$-Nambu de degré maximal*

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A B S T R A C T

In this paper, we classify $b^m$-Nambu structures via $b^m$-cohomology. The complex of $b^m$-forms is an extension of the De Rham complex, which allows us to consider *singular* forms. $b^m$-Cohomology is well understood thanks to Scott (2016) [12], and it can be expressed in terms of the De Rham cohomology of the manifold and of the critical hypersurface using a Mazzeo–Melrose-type formula. Each of the terms in $b^m$-Mazzeo–Melrose formula acquires a geometrical interpretation in this classification. We also give equivariant versions of this classification scheme.



R É S U M É

On classifie les structures $b^m$-Nambu de degré maximal en utilisant la $b^m$-cohomologie. Le complexe des $b^m$-formes est une extension du complexe de De Rham et permet considérer des formes *singulières*. La $b^m$-cohomologie est bien comprise grâce à Scott (2016) [12], et elle peut être exprimée en termes de la cohomologie de De Rham de la variété et de l'hypersurface critique en utilisant une formule de type Mazzeo–Melrose. Chacun des termes dans la formule de $b^m$-Mazzeo–Melrose acquiert une interpretation géométrique dans cette classification. On donne aussi des versions équivariantes des théorèmes de classification.



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## 1. Introduction

In this article, we focus our attention on $b^m$-Nambu structures. Nambu structures were introduced by Nambu [11] and Takhtajan [13] as a generalization of Poisson structures. Unlike the domain of Poisson geometry, Nambu geometry is not so well explored. In this short note, we give a classification theorem for a class of Nambu structures using a generalization of De Rham cohomology called $b^m$-cohomology. Our result generalizes a former classification theorem by Martínez-Torres for generic Nambu structures of top degree [8].

Recently, a class of Poisson structures called in the literature $b$-Poisson structures (see for instance, [3], [4], [6] and [2]) has been widely studied. A $b$-Poisson manifold is an even dimensional Poisson manifold $(M^{2n}, \Pi)$ where the Poisson structure $\Pi$ satisfies the following transversality condition: $\Pi^n$ cuts the zero section of the bundle $\Lambda^{2n}T(M^{2n})$ transversally. As a consequence the vanishing set of $\Pi^n$ is a smooth submanifold of codimension 1, which is called *critical hypersurface*.

The transversality condition can be relaxed in a way the critical hypersurface is still a smooth submanifold. This is the case of $b^m$-Poisson manifolds introduced by Scott [12] and later investigated by Guillemin, Miranda and Weitsman in [5]. In this paper, we generalize this setting to the Nambu world and classify these structures. This class of singular Nambu structures was already considered by Arnold in [1]. The classification theorem we prove here is an extension of Moser's classification theorem [10] for volume forms on a manifold. As an outcome of this classification scheme, a geometrical interpretation is given to the Mazzeo–Melrose decomposition theorem (see section 2.16 in [9] for $m = 1$ and [12] for general $m$), which expresses $b^m$-cohomology in terms of the classical De Rham cohomology groups of the manifold and of the critical hypersurface.

## 2. Constructions and classification of $b^m$-Nambu structures

Nambu structures of $b^m$-type can be described using forms that are singular along a smooth hypersurface. These forms, called $b^m$-forms, were studied by Scott [12] in his thesis. We start introducing the language of $b^m$-forms: we follow [12] for these definitions and main properties. The set-up in Scott [12] allows us to consider smooth hypersurfaces without a globally defining function. For the sake of simplicity in this paper, we will consider $Z$ a smooth hypersurfaces (not necessarily connected) and attach to it a defining function $f$.

Take a local set of coordinates $(x, \dots, x_{n-1})$ in a neighborhood of a point $p$ in the critical set, the $b^m$-tangent bundle can be defined as the bundle whose sections are locally generated by:

$$\{x^m \frac{\partial}{\partial x}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n-1}}\}, \tag{1}$$

with $x$ such that $|x| = \lambda$, and $\lambda$ is the distance function to $z$. For globally defining functions $f = x$.

As done in the case $m = 1$ in [3], we can define the dual bundle, the $b^m$-cotangent bundle ${}^{b^m}T^*(M)$. Sections of powers of these bundles are called $b^m$-forms.

A **Laurent series** of a closed $b^m$-form $\omega$ is a decomposition of $\omega$ in a tubular neighborhood $U$ of the critical set $Z$ of the form

$$\omega = \frac{\mathrm{d}x}{x^m} \wedge (\sum_{i=0}^{m-1} \pi^*(\alpha_i)x^i) + \beta \tag{2}$$

with $\pi : U \to Z$ the projection of the tubular neighborhood onto $Z$, $\alpha_i$ a closed smooth De Rham form on $Z$ and $\beta$ a De Rham form on $M$.

In [12], it is proved that in a neighborhood of $Z$, every closed $b^m$-form $\omega$ can be written in a Laurent form of type (2) once a defining function has been fixed.

The complex of $b^m$-forms endowed with a natural extension of De Rham differential defines $b^m$-cohomology. The following theorem tells us that $b^m$-cohomology can be read off from de Rham cohomology, thus generalizing the classical Mazzeo–Melrose decomposition theorem in Section 2.16 in [9].

**Theorem 2.1** *($b^m$-Mazzeo–Melrose, [12]). The $b^m$-cohomology groups can be determined from De Rham cohomology groups as follows:*

$$ {}^{b^m}H^p(M) \cong H^p(M) \oplus (H^{p-1}(Z))^m. \tag{3}$$

We now introduce $b^m$-Nambu structures of top degree,

**Definition 2.2.** A $b^m$-Nambu structure of top degree on a pair $(M^n, Z)$ with $Z$ a smooth hypersurface is given by a smooth $n$-multivector field $\Lambda$ such that there exists a local system of coordinates for which

$$\Lambda = x_1^m \frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_n} \tag{4}$$

and $Z$ is defined by $x_1 = 0$ in a neighborhood of $Z$.

Dualizing the local expression of the Nambu structure, we obtain the form

$$\Theta = \frac{1}{x_1^m} \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n \tag{5}$$

(which is not a smooth de Rham form), but it is a $b^m$-form of degree $n$ defined on a $b^m$-manifold. As it is done in [3], we can check that this dual form is non-degenerate. So we may define a $b^m$-Nambu form as follows.

Mimicking the same condition as for $b^m$-symplectic forms, we can talk about non-degenerate $b^m$-forms of top degree. This means that seen as a section of $\Lambda^n({}^bT^*M)$, the form does not vanish.

**Notation:** We will denote by $\Lambda$ the Nambu multivectorfield and by $\Theta$ its dual.

**Definition 2.3.** A $b^m$-Nambu form is a non-degenerate $b^m$-form of top degree.

We first include a collection of motivating examples, and then prove an equivariant classification theorem.

### 2.1. Examples

(i) **$b^m$-Symplectic surfaces:** any $b^m$-symplectic surface is a $b^m$-Nambu manifold with Nambu structure of top degree.

(ii) **$b^m$-Symplectic manifolds as $b^m$-Nambu manifolds:** let $(M^{2n}, \omega)$ be a $b^m$-symplectic manifold, then $(M^{2n}, \underbrace{\omega \wedge \ldots \wedge \omega}_{n})$ is automatically $b^m$-Nambu.

(iii) **Orientable manifolds:** let $(M^n, \Omega)$ be any orientable manifold (with $\Omega$ a volume form) and let $f$ be a defining function for $Z$, then $(1/f^m)\Omega$ defines a $b^m$-Nambu structure of top degree having $Z$ as critical set.
Any Nambu structure can be written in this way if the hypersurface can be globally described as the vanishing set of a smooth function.

(iv) **Spheres:** in [8], it was given special importance to the example $(S^n, \sqcup_i S_i^{(n-1)})$ because of the Schoenflies theorem,[1] which imposes the associated graph to be a tree. The nice feature of this example is that $O(n)$ acts on the $b^m$-manifold $(S^n, S^{(n-1)})$, and it makes sense to consider its classification under these symmetries. This also works for other homogeneous spaces of type $(G_1/G_2, G_2/G_3)$ with $G_2$ and $G_3$ with codimension 1 in $G_1$ and $G_2$, respectively.

### 2.2. $b^m$-Nambu structures of top degree and orientability

We start proving:

**Theorem 2.4.** *A compact $n$-dimensional manifold $M$ admitting a $b^{2k}$-Nambu structure is orientable.*

**Proof.** Consider a collar of charts for the $b^{2k}$-Nambu structure such that in local coordinates the Nambu structure can be written as $x_1^{2k} \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n}$ with compatible orientations in a neighborhood of each connected component of $Z$.

Consider a 2:1 orientable covering $(\tilde{M}, \tilde{Z})$ of the manifold, and denote by $\rho : \mathbb{Z}/2\mathbb{Z} \times \tilde{M} \to \tilde{M}$ the deck transformation. For each point $p \in \tilde{Z}$, take a neighborhood $U_p$ that does not contain other points identified by $\rho$ thus $U_p \cong \pi(U_p) =: V_p$, and $\Theta = \frac{1}{x^{2k}} \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$. This form defines an orientation on $V_p \setminus \pi(Z)$. Take a symmetric covering of such neighborhoods to define a collar of $Z$ with compatible orientations, and compatible with the covering. The compatible orientations and the symmetric coverings descend to $(M, Z)$, thus defining an orientation in $(M, Z)$. Thus, we have an orientation in $V \setminus Z$. By perturbing $\Theta$ in $V$, we obtain a volume form on $V$, $\tilde{\omega}$, and thus an orientation in $V$. These can be glued to define an orientation via the volume form $\tilde{\Theta}$ on the whole $M$, proving that $M$ is oriented. □

### 2.3. Classification of $b^m$-Nambu structures of top degree and $b^m$-cohomology

We present the definitions contained in [8] of modular period attached to the connected component of an orientable Nambu structure using the language of $b^m$-forms.

Let $\Theta$ be the dual to the multivectorfield $\Lambda$ defining a Nambu structure. From the general decomposition of $b^m$-forms as it was set in Equation (2), we may write:

$$\Theta = \Theta_0 \wedge \frac{\mathrm{d}f}{f^m}$$

with $\Theta_0 \in \Omega^{n-1}(M)$.

[1] The nature of this theorem is purely topological in dimension equal or greater than four, and so is its construction.

This decomposition is valid in a neighborhood of $Z$ whenever the defining function is well defined. For non-orientable manifolds, a similar decomposition can be proved by replacing the defining function $f$ by an adapted distance (see [7]).

With this language in mind, the **modular** $(n-1)$**-vector field** in [8] of $\Theta$ along $Z$ is the dual of the form $\Theta_0$ in the decomposition above which is indeed the **modular** $(n-1)$**-form** along $Z$ in [8].

Recall the following from [8] in our language.

**Definition 2.5.** The **modular period** $T_\Lambda^Z$ of the component $Z$ of the zero locus of $\Lambda$ is

$$T_\Lambda^Z := \int_Z \Theta_0 > 0.$$

In fact, this positive number determines the Nambu structure in a neighborhood of $Z$ up to isotopy, as it was proved in [8].

The following theorem gives a classification of $b^m$-Nambu structures.

**Theorem 2.6.** *Let $\Theta_0$ and $\Theta_1$ be two $b^m$-Nambu forms of degree $n$ on a compact orientable manifold $M^n$. If $[\Theta_0] = [\Theta_1]$ in $b^m$-cohomology, then there exists a diffeomorphism $\phi$ such that $\phi^*\Theta_1 = \Theta_0$.*

**Proof.** We will apply the techniques of [10] with the only difference that we work with $b^m$-volume forms instead of volume forms.

Since $\Theta_0$ and $\Theta_1$ are non-degenerate $b^m$-forms, both of them are a multiple of a volume form and thus the linear path $\Theta_t = (1-t)\Theta_0 + t\Theta_1$ is a path of non-degenerate $b^m$-forms.

Because $\Theta_0$ and $\Theta_1$ determine the same cohomology class:

$$\Theta_1 - \Theta_0 = \mathrm{d}\beta$$

with d the $b^m$-De Rham differential and $\beta$ a $b^m$-form of degree $n-1$.

Now consider the Moser equation:

$$\iota_{X_t}\Theta_t = -\beta. \tag{6}$$

Observe that since $\beta$ is a $b^m$-form and $\Theta_t$ is non-degenerate. The vector field $X_t$ is a $b^m$-vector field. Let $\phi_t$ be the $t$-dependent flow integrating $X_t$.

The $\phi_t$ gives the desired diffeomorphism $\phi_t : M \to M$, leaving $Z$ invariant (since $X_t$ is tangent to $Z$) and $\phi_t^*\Theta_t = \Theta_0$. $\square$

In particular, we recover the classification of $b$-Nambu structures of top degree in [8].

**Theorem 2.7** *(Classification of b-Nambu structures of top degree, [8]). A generic b-Nambu structure $\Theta$ is determined, up to orientation preserving diffeomorphism, by the following three invariants: the diffeomorphism type of the oriented pair $(M, Z)$, the modular periods and the regularized Liouville volume.*

By Theorem 2.1,

$${}^bH^n(M) \cong H^n(M) \oplus H^{n-1}(Z).$$

The first term on the right-hand side is the Liouville volume image by the De Rham theorem, as it was done in [4] for $b$-symplectic forms. The second term collects the periods of the modular vector field. So if the three invariants coincide, then they determine the same $b$-cohomology class.

In other words, the statement in [8] is equivalent to the following theorem in the language of $b$-cohomology.

**Theorem 2.8.** *Let $\Theta_1$ and $\Theta_2$ be two b-Nambu forms on an orientable manifold $M$. If $[\Theta_1] = [\Theta_2]$ in b-cohomology, then there exists a diffeomorphism $\phi$ such that $\phi^*\Theta_1 = \Theta_2$.*

This global Moser theorem for $b^m$-Nambu structures admits an equivariant version:

**Theorem 2.9.** *Let $\Theta_0$ and $\Theta_1$ be two $b^m$-Nambu forms of degree n on a compact orientable manifold $M^n$ and let $\rho : G \times M \longrightarrow M$ be a compact Lie group action preserving both $b^m$-forms. If $[\Theta_0] = [\Theta_1]$ in $b^m$-cohomology, then there exists an equivariant diffeomorphism $\phi$ such that $\phi^*\Theta_1 = \Theta_0$.*

**Proof.** As in the former proof, write

$$\Theta_1 - \Theta_0 = \mathrm{d}\beta$$

with d the $b^m$-De Rham differential and $\beta$ a $b^m$-form of degree $n-1$. Observe that the path $\Theta_t = (1-t)\Theta_0 + t\Theta_1$ is a path of invariant $b^m$-forms.

Now consider Moser's equation:

$$\iota_{X_t}\Theta_t = -\beta. \tag{7}$$

Since $\Theta_t$ is invariant, we can find an invariant $\tilde{\beta}$. For instance, take $\tilde{\beta} = \int_G \rho_g^*(\beta)\,\mathrm{d}\mu$ with $\mu$ a de Haar measure on $G$ and $\rho_g$ the induced diffeomorphism $\rho_g(x) := \rho(g, x)$.

Now replace $\beta$ by $\tilde{\beta}$ to obtain,

$$\iota_{X_t^G}\Theta_t = -\tilde{\beta} \tag{8}$$

with $X_t^G = \int_G \rho_{g_*} X_t\,\mathrm{d}\mu$. The vector field $X_t^G$ is an invariant $b$-vector field. Its flow $\phi_t^G$ preserves the action and $\phi_t^{G*}\Theta_t = \Theta_0$. □

Playing the equivariant $b^m$-Moser trick using the 2:1 cover of a non-orientable manifold and taking as $G$ the group of deck transformations, we obtain the following corollary.

**Corollary 2.10.** *Let $\Theta_0$ and $\Theta_1$ be two $b^m$-Nambu forms of degree n on a manifold $M^n$ (not necessarily oriented). If $[\Theta_0] = [\Theta_1]$ in $b^m$-cohomology, then there exists a diffeomorphism $\phi$ such that $\phi^*\Theta_1 = \Theta_0$.*

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