C. R. Acad. Sci. Paris, Ser. I 356 (2018) 44–51

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Group theory/Logic

Varieties generated by unstable involution semigroups with continuum many subvarieties

Variétés engendrées par des demi-groupes involutifs instables, ayant un continuum de sous-variétés

Edmond W.H. Lee

Department of Mathematics, Nova Southeastern University, 3301 College Avenue, Fort Lauderdale, FL 33314, USA

ARTICLE INFO

Article history:
Received 10 May 2017
Accepted after revision 1 December 2017
Available online 13 December 2017

Presented by the Editorial Board

ABSTRACT

Over the years, several finite semigroups have been found to generate varieties with continuum many subvarieties. However, finite involution semigroups that generate varieties with continuum many subvarieties seem much rarer; in fact, only one example—an inverse semigroup of order 165—has so far been published. Nevertheless, it is shown in the present article that there are many smaller examples among involution semigroups that are *unstable* in the sense that the varieties they generate contain some involution semilattice with nontrivial unary operation. The most prominent examples are the unstable finite involution semigroups that are inherently non-finitely based, the smallest ones of which are of order six. It follows that the join of two finitely generated varieties of involution semigroups with finitely many subvarieties can contain continuum many subvarieties.

RÉSUMÉ

Avec le temps, on a découvert plusieurs demi-groupes finis engendrant des variétés contenant un ensemble continu de sous-variétés. Toutefois, les demi-groupes involutifs finis qui engendrent des variétés contenant autant de sous-variétés semblent beaucoup plus rares ; en fait, un seul exemple – un demi-groupe inversif d'ordre 165 – a été publié à ce jour. Nous montrons dans le présent article qu'il y a néanmoins beaucoup d'exemples plus petits parmi les demi-groupes involutifs qui sont *instables*, dans le sens que les variétés qu'ils engendrent contiennent un demi-réseau involutif avec une opération unaire non triviale. Les exemples les plus frappants sont les demi-groupes involutifs finis qui n'ont pas de base finie par essence, le plus petit étant d'ordre 6. Il s'ensuit que le joint de deux variétés engendrées par des demi-groupes involutifs finis et n'ayant qu'un nombre fini de sous-variétés peut contenir un ensemble continu de sous-variétés.

E-mail address: edmond.lee@nova.edu.

https://doi.org/10.1016/j.crma.2017.12.001
1631-073X/

1. Introduction

By the celebrated theorem of Oates and Powell [23], the variety $\mathsf{Var}\, G$ generated by any finite group G contains finitely many subvarieties. In contrast, the variety $\mathsf{Var}\, S$ generated by a finite semigroup S can contain continuum many subvarieties. A prominent source of such semigroups, due to Jackson [9], is the class of inherently non-finitely based finite semigroups. The multiplicative matrix semigroup

$$\mathcal{B}_2^1 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\},$$

commonly called the *Brandt monoid*, is the most well-known inherently non-finitely based semigroup [24]. Other finite semigroups [11,27], some of which are finitely based [7,9,19,20,29], have also been found to generate varieties with continuum many subvarieties.

A counterintuitive situation emerges for semigroups S that are reducts of involution semigroups $(S, {}^*)$. Recall that an *involution semigroup* or $*$-*semigroup* is a unary semigroup $(S, {}^*)$ that satisfies the equations

$$(x^*)^* \approx x \quad \text{and} \quad (xy)^* \approx y^* x^*, \tag{1}$$

and an *inverse semigroup* is a $*$-semigroup that satisfies the additional equations $xx^*x \approx x$ and $xx^*yy^* \approx yy^*xx^*$. Examples of inverse semigroups include groups $(G, {}^{-1})$ with inversion ${}^{-1}$, while the multiplicative $n \times n$ matrix semigroup $(\mathcal{M}_n, {}^\mathsf{T})$ over any field with usual matrix transposition ${}^\mathsf{T}$ is a $*$-semigroup that is not an inverse semigroup. Now when the Brandt monoid $\mathcal{B}_2^1$ is endowed with matrix transposition ${}^\mathsf{T}$, the resulting unary semigroup $(\mathcal{B}_2^1, {}^\mathsf{T})$ is an inverse subsemigroup of $(\mathcal{M}_2, {}^\mathsf{T})$ that generates a variety with only four subvarieties [14]; these subvarieties constitute the chain

$$\mathsf{Var}(\mathcal{E}, {}^\mathsf{T}) \subset \mathsf{Var}(\mathcal{S}\ell_2, {}^\mathsf{T}) \subset \mathsf{Var}(\mathcal{B}_2, {}^\mathsf{T}) \subset \mathsf{Var}(\mathcal{B}_2^1, {}^\mathsf{T}), \tag{2}$$

where $\mathcal{E} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$, $\mathcal{S}\ell_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$, and $\mathcal{B}_2 = \mathcal{B}_2^1 \backslash \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ are subsemigroups of $\mathcal{B}_2^1$. The contrast between the varieties $\mathsf{Var}\, \mathcal{B}_2^1$ and $\mathsf{Var}(\mathcal{B}_2^1, {}^\mathsf{T})$ naturally prompts the following question.

Question 1. For a $*$-semigroup $(S, {}^*)$, under what conditions will the variety $\mathsf{Var}(S, {}^*)$ contain continuum many subvarieties, given that the reduct variety $\mathsf{Var}\, S$ contains continuum many subvarieties?

Another difference between the Brandt monoid $\mathcal{B}_2^1$ and its inverse counterpart $(\mathcal{B}_2^1, {}^\mathsf{T})$ is that unlike the former, the latter is not inherently non-finitely based [25]. In view of this difference, the decrease from continuum many subvarieties in $\mathsf{Var}\, \mathcal{B}_2^1$ to only four subvarieties in $\mathsf{Var}(\mathcal{B}_2^1, {}^\mathsf{T})$ does not seem too surprising after all. However, it is instinctive to question if the inherent non-finite basis property is among the possibly many conditions that provide an answer to Question 1.

Question 2. Does every inherently non-finitely based finite $*$-semigroup generate a variety with continuum many subvarieties?

Several examples of inherently non-finitely based finite $*$-semigroups have so far been found, for example, the Brandt monoid $\mathcal{B}_2^1$ endowed with the *skewed transposition* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\mathsf{S} = \begin{bmatrix} d & b \\ c & a \end{bmatrix}$ and the multiplicative matrix semigroup

$$\mathcal{A}_2^1 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

endowed with the operation * that interchanges the second and fifth elements while fixing all other elements [1,3]. A positive answer to Question 2 would lead to many examples of finite $*$-semigroups that generate varieties with continuum many subvarieties. At the moment, the only explicit example is a certain non-finitely based inverse semigroup of order 165, due to Kad'ourek [13].

2. Main results

2.1. Unstable $*$-semigroups

Recall that a *semilattice* is a semigroup that is commutative and idempotent. Up to isomorphism, the smallest $*$-semilattice with nontrivial unary operation is the multiplicative matrix semigroup

$$\mathcal{S}\ell_3 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

with skewed transposition ${}^\mathsf{S}$. A variety of $*$-semigroups that contains the $*$-semilattice $(\mathcal{S}\ell_3, {}^\mathsf{S})$ is said to be *unstable*, and a $*$-semigroup is *unstable* if it generates an unstable variety. For instance, the $*$-semigroup $(\mathcal{B}_2^1, {}^\mathsf{S})$ is unstable because $(\mathcal{S}\ell_3, {}^\mathsf{S})$ is a $*$-subsemigroup, while the $*$-semigroup $(\mathcal{A}_2^1, {}^*)$ is also unstable because its $*$-subsemigroup generated by $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ modulo the ideal $\left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}$ is isomorphic to $(\mathcal{S}\ell_3, {}^{\mathsf{S}})$. In general, a finite $*$-semigroup of order n is unstable if and only if it violates the equation $\left(x^{n!}(x^*)^{n!}x^{n!}\right)^{n!} \approx x^{n!}$ [1]. On the other hand, since the regularity axiom $xx^*x \approx x$ of inverse semigroups is violated by $(\mathcal{S}\ell_3, {}^{\mathsf{S}})$, all finite inverse semigroups—in particular, the inverse Brandt monoid $(\mathcal{B}_2^1, {}^{\mathsf{T}})$—are not unstable.

The importance of unstable $*$-semigroups is evident from the pioneering work of Auinger et al. [1] on inherently non-finitely based finite $*$-semigroups.

Theorem 3 *(Auinger et al. [1]). Each unstable finite $*$-semigroup and its semigroup reduct are simultaneously inherently non-finitely based. More specifically, for any finite $*$-semigroup $(S, {}^*)$,*

(i) *if $(S, {}^*)$ is inherently non-finitely based, then S is also inherently non-finitely based;*
(ii) *if S is inherently non-finitely based and $(S, {}^*)$ is unstable, then $(S, {}^*)$ is also inherently non-finitely based.*

Since the inverse Brandt monoid $(\mathcal{B}_2^1, {}^{\mathsf{T}})$ is not inherently non-finitely based [25], it serves as a counterexample to the converse of Theorem 3(i). Recently, Theorem 3(ii) was deduced from a more general result: if the reduct S of an unstable $*$-semigroup $(S, {}^*)$ is non-finitely based, then the $*$-semigroup $(S, {}^*)$ is also non-finitely based [18].

The investigation of unstable $*$-semigroups is continued in the present article, with main focus on Questions 1 and 2. For the statement of the main result, it is convenient to call a variety *symmetric* if it is closed under anti-isomorphism. A semigroup is *symmetric* if it generates a symmetric variety.

Main Theorem. *Let $(S, {}^*)$ be any unstable $*$-semigroup. Suppose that the reduct variety* $\mathsf{VAR}\,S$ *contains continuum many symmetric subvarieties. Then the variety* $\mathsf{VAR}\,(S, {}^*)$ *contains continuum many subvarieties.*

Now since the variety $\mathsf{VAR}\,(\mathcal{S}\ell_3, {}^{\mathsf{S}})$ is an atom in the lattice of varieties of $*$-semigroups [8], the unstableness assumption in the main theorem is a weakest assumption—the only way to weaken it is to omit it altogether. On the other hand, unstableness is essential to the theorem due to the counterexample $(\mathcal{B}_2^1, {}^{\mathsf{T}})$.

The proof of the main theorem is established in Section 4, after some background material is first given in Section 3. The main argument of the proof is to show that the equational bases of the continuum many subvarieties of the reduct variety $\mathsf{VAR}\,S$ define distinct subvarieties of $\mathsf{VAR}\,(S, {}^*)$. These continuum many subvarieties of $\mathsf{VAR}\,S$ need to be symmetric simply because every variety of $*$-semigroups is symmetric; in fact, due to the axioms (1) of $*$-semigroups, every $*$-semigroup is anti-isomorphic to itself. In general, equational bases of nonsymmetric subvarieties of the reduct variety $\mathsf{VAR}\,S$ need not define unique subvarieties of $\mathsf{VAR}\,(S, {}^*)$, as shown in the following example.

Example 4. The rectangular band $\mathcal{RB}_4$ of order four, given as the direct product of the multiplicative matrix semigroups

$$\mathcal{LZ}_2 = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right\} \quad \text{and} \quad \mathcal{RZ}_2 = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right\},$$

is a $*$-semigroup under the operation $(x, y)^* = (y^{\mathsf{T}}, x^{\mathsf{T}})$. The symmetric variety $\mathsf{VAR}\,\mathcal{RB}_4$ contains two nonsymmetric subvarieties: the variety $\mathsf{VAR}\,\mathcal{LZ}_2$ of left zero semigroups and the variety $\mathsf{VAR}\,\mathcal{RZ}_2$ of right zero semigroups, with equational bases $\{xy \approx x\}$ and $\{xy \approx y\}$, respectively. These equational bases do not define distinct subvarieties of $\mathsf{VAR}\,(\mathcal{RB}_4, {}^*)$; in fact, each of them defines the trivial variety.

In Section 5, a result analogous to the main theorem is established for unstable $*$-monoids.

2.2. Inherently non-finitely based $*$-semigroups

Recall that a finite algebra is *inherently non-finitely based* if every locally finite variety containing it is non-finitely based. As observed in Section 1, for any inherently non-finitely based finite semigroup S, the variety $\mathsf{VAR}\,S$ contains continuum many subvarieties; the proof of this result in fact exhibits continuum many subvarieties of $\mathsf{VAR}\,S$ that are symmetric [9, proof of Theorem 3.2]. Consequently, a partial answer to Question 2 follows from Theorem 3 and the main theorem.

Theorem 5. *The variety generated by any unstable inherently non-finitely based finite $*$-semigroup contains continuum many subvarieties. Consequently, each of the varieties* $\mathsf{VAR}\,(\mathcal{A}_2^1, {}^*)$, $\mathsf{VAR}\,(\mathcal{B}_2^1, {}^{\mathsf{S}})$, *and* $\mathsf{VAR}\{(\mathcal{B}_2^1, {}^{\mathsf{T}}) \times (\mathcal{S}\ell_3, {}^{\mathsf{S}})\}$ *contains continuum many subvarieties.*

Let $(S, {}^*)$ be any inherently non-finitely based $*$-semigroup of order at most six. Then the reduct S is inherently non-finitely based by Theorem 3(i). By the solution to the finite basis problem for semigroups of order up to six [19,21,22], the semigroup S is isomorphic to either $\mathcal{A}_2^1$ or $\mathcal{B}_2^1$. It follows that $(S, {}^*)$ is isomorphic to either $(\mathcal{A}_2^1, {}^*)$ or $(\mathcal{B}_2^1, {}^{\mathsf{S}})$. The minimality and uniqueness of $(\mathcal{A}_2^1, {}^*)$ and $(\mathcal{B}_2^1, {}^{\mathsf{S}})$ are thus established.

Example 6. Up to isomorphism, $(\mathcal{A}_2^1, {}^*)$ and $(\mathcal{B}_2^1, {}^{\mathsf{S}})$ are the only smallest inherently non-finitely based $*$-semigroups.

Many other unstable inherently non-finitely based finite $*$-semigroups are also available [2,3]. It is unknown, however, if there exists an inherently non-finitely based finite $*$-semigroup that is not unstable.

2.3. Finitely based $*$-semigroups

Up to this point, all known examples of finite $*$-semigroups that generate varieties with continuum many subvarieties are non-finitely based, hence it is of interest to locate finitely based examples. For this task, it seems sufficient to locate a finitely based $*$-semigroup $(S, {}^*)$, where the reduct variety $\mathsf{Var}\,S$ contains continuum many symmetric subvarieties, since by the main theorem, the unstable variety $\mathsf{Var}\{(S, {}^*) \times (\mathcal{S}\ell_3, {}^{\mathsf{S}})\}$ contains continuum many subvarieties. The problem with this argument is that the direct product of a finitely based $*$-semigroup with $(\mathcal{S}\ell_3, {}^{\mathsf{S}})$ need not be finitely based in general.

Example 7 *(Lee [17,18]).* The multiplicative matrix semigroup

$$\mathcal{L} = \left\{ \begin{bmatrix} 0&0&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0\\0&1&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&0&1\\0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&0\\0&0&1\\0&0&1 \end{bmatrix} \right\}$$

is a $*$-semigroup under the operation * that interchanges the second and fifth elements while fixing all other elements. The $*$-semigroup $(\mathcal{L}, {}^*)$ is finitely based but the direct product $(\mathcal{L}, {}^*) \times (\mathcal{S}\ell_3, {}^{\mathsf{S}})$ is non-finitely based.

It turns out that a result of Crvenković et al. [5] is useful in constructing the required examples. Let S be any semigroup that excludes the symbol 0 and let $S^{\triangleleft} = \{a^{\triangleleft} \mid a \in S\}$ be the semigroup anti-isomorphic to S, that is, $x^{\triangleleft} y^{\triangleleft} = (yx)^{\triangleleft}$ for all $x, y \in S$. Then S and $S^{\triangleleft}$ can be amalgamated into the semigroup

$$\widehat{S} = S \cup S^{\triangleleft} \cup \{0\},$$

where $xy = yx = 0$ for all $x \in S \cup \{0\}$ and $y \in S^{\triangleleft} \cup \{0\}$. By defining $0^{\triangleleft} = 0$ and $(x^{\triangleleft})^{\triangleleft} = x$ for all $x \in S$, the unary semigroup $(\widehat{S}, {}^{\triangleleft})$ becomes a $*$-semigroup.

Lemma 8 *(Crvenković et al. [5, Theorem 7]). Let S be any symmetric semigroup. Suppose that S is finitely based. Then the $*$-semigroup $(\widehat{S}, {}^{\triangleleft})$ is also finitely based.*

Let S be any symmetric finitely based finite semigroup. Then the $*$-semigroup $(\widehat{S}, {}^{\triangleleft})$ is also finitely based by Lemma 8. The semigroup S, being finite, contains some idempotent e. Then the $*$-subsemigroup $(\{e, e^{\triangleleft}, 0\}, {}^{\triangleleft})$ of $(\widehat{S}, {}^{\triangleleft})$ is isomorphic to the $*$-semilattice $(\mathcal{S}\ell_3, {}^{\mathsf{S}})$, so that $(\widehat{S}, {}^{\triangleleft})$ is unstable. Now if the variety $\mathsf{Var}\,S$ contains continuum many symmetric subvarieties, then since $S \subset \widehat{S}$, it follows from the main theorem that the variety $\mathsf{Var}(\widehat{S}, {}^{\triangleleft})$ contains continuum many subvarieties. The following result is thus established.

Theorem 9. *Let S be any symmetric finitely based semigroup. Suppose that the variety $\mathsf{Var}\,S$ contains continuum many symmetric subvarieties. Then $(\widehat{S}, {}^{\triangleleft})$ is a finitely based $*$-semigroup that generates a variety with continuum many subvarieties.*

A prime example of a symmetric finitely based semigroup that generates a variety with continuum many symmetric subvarieties is the multiplicative matrix semigroup

$$\mathcal{J}_7 = \left\{ \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&1\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix} \right\}$$

from Jackson [9, Subsection 3.1]. By Theorem 9, the finitely based $*$-semigroup $(\widehat{\mathcal{J}_7}, {}^{\triangleleft})$ of order 15 generates a variety with continuum many subvarieties. It follows that for any symmetric finitely based finite semigroup S such that $\mathcal{J}_7 \in \mathsf{Var}\,S$, the finitely based finite $*$-semigroup $(\widehat{S}, {}^{\triangleleft})$ also generates a variety with continuum many subvarieties. Infinite classes of such semigroups S can be found in Lee [16, Proposition 4.1] and Sapir [26, Example 7.4], and a method from Jackson and Sapir [12, Corollary 3.1] can in fact be used to locate as many of them as desired.

2.4. Joins of varieties with finitely many subvarieties

In his study of varieties of semigroups with continuum many subvarieties, Jackson [9] exhibited varieties $\mathbb{V}_1$ and $\mathbb{V}_2$ with countably many subvarieties such that the join $\mathbb{V}_1 \vee \mathbb{V}_2$ contains continuum many subvarieties. In fact, one of the varieties $\mathbb{V}_1$ and $\mathbb{V}_2$ can be chosen to contain only finitely many subvarieties. This led Jackson to raise the following question for varieties of semigroups.

Question 10 *(Jackson [9, Question 3.15]).* Are there finitely generated varieties $\mathbb{V}_1$ and $\mathbb{V}_2$ with finitely many subvarieties such that the join $\mathbb{V}_1 \vee \mathbb{V}_2$ contains continuum many subvarieties?

It turns out that this question has a positive answer for varieties of $*$-semigroups: the varieties $\mathbb{V}_1 = \mathsf{VAR}\,(\mathcal{B}_2^1, {}^{\mathsf{T}})$ and $\mathbb{V}_2 = \mathsf{VAR}\,(\mathcal{S}\ell_3, {}^{\mathsf{S}})$ contain finitely many subvarieties [8,14], but by Theorem 5, the join $\mathbb{V}_1 \vee \mathbb{V}_2$ contains continuum many subvarieties.

Recently, Question 10 was also affirmatively answered for varieties of monoids [10]. But the question remains open for varieties of semigroups.

3. Preliminaries

Acquaintance with rudiments of universal algebra is assumed of the reader. Refer to the monograph of Burris and Sankappanavar [4] for more information.

3.1. Words, terms, and equations

Let $\mathscr{A}$ be a countably infinite alphabet and let $\mathscr{A}^* = \{x^* \mid x \in \mathscr{A}\}$ be a disjoint copy of $\mathscr{A}$. Elements of $\mathscr{A} \cup \mathscr{A}^*$ are called *variables*, elements of the free semigroup $(\mathscr{A} \cup \mathscr{A}^*)^+$ are called *words*, and words in $\mathscr{A}^+$ are called *plain words*. The set $\mathcal{T}(\mathscr{A})$ of *terms* over $\mathscr{A}$ is the smallest set containing $\mathscr{A}$ that is closed under concatenation and *. The proper inclusion $(\mathscr{A} \cup \mathscr{A}^*)^+ \subset \mathcal{T}(\mathscr{A})$ holds and the equations (1) can be used to convert each term into some unique word.

An *equation* is an expression $\mathbf{u} \approx \mathbf{v}$ formed by terms $\mathbf{u}, \mathbf{v} \in \mathcal{T}(\mathscr{A})$, a *word equation* is an equation $\mathbf{u} \approx \mathbf{v}$ formed by words $\mathbf{u}, \mathbf{v} \in (\mathscr{A} \cup \mathscr{A}^*)^+$, and a *plain equation* is an equation $\mathbf{u} \approx \mathbf{v}$ formed by plain words $\mathbf{u}, \mathbf{v} \in \mathscr{A}^+$. For any $*$-semigroup $(S, {}^*)$, let $\mathsf{Eq}\,(S, {}^*)$ denote the set of equations satisfied by $(S, {}^*)$, commonly called the *equational theory* of $(S, {}^*)$. Let $\mathsf{Eq}_\mathsf{W}\,(S, {}^*)$ and $\mathsf{Eq}_\mathsf{P}\,(S, {}^*)$ denote the subsets of $\mathsf{Eq}\,(S, {}^*)$ consisting of word equations and plain equations, respectively. It is clear that the inclusions $\mathsf{Eq}\,S = \mathsf{Eq}_\mathsf{P}\,(S, {}^*) \subset \mathsf{Eq}_\mathsf{W}\,(S, {}^*) \subset \mathsf{Eq}\,(S, {}^*)$ hold, where $\mathsf{Eq}\,S$ is the equational theory of the semigroup S.

3.2. Equational bases and varieties

For any class $\mathfrak{K}$ of algebras of a fixed type, let $\mathsf{Eq}\,\mathfrak{K}$ denote the set of equations satisfied by all algebras in $\mathfrak{K}$. A subset Σ of $\mathsf{Eq}\,\mathfrak{K}$ is an *equational basis* for $\mathfrak{K}$ if every equation in $\mathsf{Eq}\,\mathfrak{K}$ is deducible from Σ. A class $\mathfrak{K}$ of algebras is *finitely based* if it has some finite equational basis.

When referring to an equational basis Σ for a class of semigroups, it is unambiguous to take the associativity axiom

$$(xy)z \approx x(yz) \tag{3}$$

for granted without explicitly stating it and assume that all equations in Σ are plain equations. Similarly, in an equational basis Σ for a class of $*$-semigroups, the equations $\{(1), (3)\}$ need not be stated and equations in Σ can be chosen to be word equations.

A class $\mathfrak{K}$ of algebras of a fixed type is a *variety* if it is closed under the formation of homomorphic images, subalgebras, and arbitrary direct products. The variety *generated* by $\mathfrak{K}$, denoted by $\mathsf{VAR}\,\mathfrak{K}$, is the smallest variety containing $\mathfrak{K}$. A class $\mathfrak{K}$ of algebras and the variety $\mathsf{VAR}\,\mathfrak{K}$ it generates satisfy the same equations and so share the same equational bases.

Recall that a variety is *symmetric* if it is closed under anti-isomorphism. A symmetric variety that satisfies an equation $\mathbf{u} \approx \mathbf{v}$ also satisfies the *reverse equation* $\overleftarrow{\mathbf{u}} \approx \overleftarrow{\mathbf{v}}$ obtained from writing $\mathbf{u} \approx \mathbf{v}$ in reverse. Any equational basis Σ for a symmetric variety is hence *symmetric* in the sense that for all $\mathbf{u} \approx \mathbf{v} \in \Sigma$, the reverse equation $\overleftarrow{\mathbf{u}} \approx \overleftarrow{\mathbf{v}}$ is deducible from Σ.

3.3. Organized equational bases

The *content* of a word $\mathbf{u}$, denoted by $\mathsf{con}(\mathbf{u})$, is the set of variables occurring in $\mathbf{u}$. A word equation $\mathbf{u} \approx \mathbf{v}$ is *homogeneous* if $\mathsf{con}(\mathbf{u}) = \mathsf{con}(\mathbf{v})$. A word $\mathbf{u}$ is *mixed* if there exists some $x \in \mathscr{A}$ such that $x, x^* \in \mathsf{con}(\mathbf{u})$. A word equation $\mathbf{u} \approx \mathbf{v}$ is *mixed* if both $\mathbf{u}$ and $\mathbf{v}$ are mixed. For any $*$-semigroup $(S, {}^*)$, let $\mathsf{Eq}_\mathsf{mix}\,(S, {}^*)$ denote the set of mixed equations satisfied by $(S, {}^*)$.

A symmetric equational basis Σ for a $*$-semigroup is *organized* if each equation in Σ is either mixed or plain. In other words, an organized equational basis for a $*$-semigroup $(S, {}^*)$ is a symmetric equational basis of the form

$$\Sigma = \Sigma_\mathsf{mix} \cup \Sigma_\mathsf{P},$$

where $\Sigma_\mathsf{mix} \subseteq \mathsf{Eq}_\mathsf{mix}\,(S, {}^*)$ and $\Sigma_\mathsf{P} \subseteq \mathsf{Eq}_\mathsf{P}\,(S, {}^*)$.

Lemma 11 *(Lee [18, Lemma 12]). Let $(S, {}^*)$ be any unstable $*$-semigroup.*

(i) *The set $\mathsf{Eq}_\mathsf{mix}\,(S, {}^*) \cup \mathsf{Eq}_\mathsf{P}\,(S, {}^*)$ is an organized equational basis for $(S, {}^*)$. More generally, if $\Sigma_\mathsf{P} \subseteq \mathsf{Eq}\,S = \mathsf{Eq}_\mathsf{P}\,(S, {}^*)$ is any symmetric equational basis for S, then $\mathsf{Eq}_\mathsf{mix}\,(S, {}^*) \cup \Sigma_\mathsf{P}$ is an organized equational basis for $(S, {}^*)$.*

(ii) *If $\Sigma_\mathsf{mix} \cup \Sigma_\mathsf{P}$ is any organized equational basis for $(S, {}^*)$, then Σ_P is an equational basis for S.*

4. Proof of the main theorem

For any variety $\mathbb{V}$ and any set Ψ of equations, let $\mathbb{V}\Psi$ denote the subvariety of $\mathbb{V}$ defined by Ψ.

Lemma 12. *Let $(S, {}^*)$ be any unstable $*$-semigroup and let $\mathbb{S}^* = \mathrm{V}\textsc{ar}\,(S, {}^*)$ and $\mathbf{S} = \mathrm{V}\textsc{ar}\,S$. Suppose that Ψ is any symmetric set of homogeneous plain equations. Then the subvariety $\mathbb{S}^*\Psi$ of $\mathbb{S}^*$ is generated by some unstable $*$-semigroup. Further, if $\mathbb{S}^*\Psi = \mathrm{V}\textsc{ar}\,(U, {}^*)$ for some unstable $*$-semigroup $(U, {}^*)$, then $\mathbf{S}\Psi = \mathrm{V}\textsc{ar}\,U$.*

Proof. By assumption, $(\mathcal{S}\ell_3, {}^{\mathsf{S}}) \in \mathbb{S}^*$. Further, since the equations in Ψ are plain and homogeneous, they are satisfied by $(\mathcal{S}\ell_3, {}^{\mathsf{S}})$. Hence $(\mathcal{S}\ell_3, {}^{\mathsf{S}}) \in \mathbb{S}^*\Psi$ and the subvariety $\mathbb{S}^*\Psi$ of $\mathbb{S}^*$ is unstable. Now let $(U, {}^*)$ be any unstable $*$-semigroup such that $\mathbb{S}^*\Psi = \mathrm{V}\textsc{ar}\,(U, {}^*)$. By Lemma 11(i), the set $\mathsf{Eq}_{\mathsf{mix}}\,(S, {}^*) \cup \mathsf{Eq}_{\mathsf{P}}\,(S, {}^*)$ is an organized equational basis for $\mathbb{S}^*$. Therefore, $\mathsf{Eq}_{\mathsf{mix}}\,(S, {}^*) \cup \mathsf{Eq}_{\mathsf{P}}\,(S, {}^*) \cup \Psi$ is a symmetric equational basis for $(U, {}^*)$; this equational basis is in fact organized since $\mathsf{Eq}_{\mathsf{mix}}\,(S, {}^*) \subseteq \mathsf{Eq}_{\mathsf{mix}}\,(U, {}^*)$ and $\mathsf{Eq}_{\mathsf{P}}\,(S, {}^*) \cup \Psi \subseteq \mathsf{Eq}_{\mathsf{P}}\,(U, {}^*)$. Hence by Lemma 11(ii), the set $\mathsf{Eq}_{\mathsf{P}}\,(S, {}^*) \cup \Psi$ is an equational basis for the semigroup U. It is clear that $\mathsf{Eq}_{\mathsf{P}}\,(S, {}^*) \cup \Psi$ defines the variety $\mathbf{S}\Psi$, so that $\mathbf{S}\Psi = \mathrm{V}\textsc{ar}\,U$. □

The following examples show that Lemma 12 does not hold if either $(S, {}^*)$ is not unstable or Ψ is not symmetric.

Example 13. Let $\mathbb{S}^* = \mathrm{V}\textsc{ar}\,(\mathcal{B}_2, {}^{\mathsf{T}})$ and $\mathbf{S} = \mathrm{V}\textsc{ar}\,\mathcal{B}_2$, where the inverse Brandt semigroup $(\mathcal{B}_2, {}^{\mathsf{T}})$ is not unstable. Then $\Psi = \{xy \approx yx\}$ is a symmetric set such that $\mathbb{S}^*\Psi = \mathrm{V}\textsc{ar}\,(\mathcal{S}\ell_2, {}^{\mathsf{T}})$ and $\mathbf{S}\Psi \neq \mathrm{V}\textsc{ar}\,\mathcal{S}\ell_2$.

Proof. This is easily verified by referring to the lattice of subvarieties of $\mathbb{S}^*$ in (2). The inverse semigroup $(\mathcal{B}_2, {}^{\mathsf{T}})$ is not unstable because $(\mathcal{S}\ell_3, {}^{\mathsf{S}}) \notin \mathrm{V}\textsc{ar}\,(\mathcal{B}_2, {}^{\mathsf{T}})$. Since the noncommutative variety $\mathbb{S}^*$ covers the commutative variety $\mathrm{V}\textsc{ar}\,(\mathcal{S}\ell_2, {}^{\mathsf{T}})$, the equality $\mathbb{S}^*\Psi = \mathrm{V}\textsc{ar}\,(\mathcal{S}\ell_2, {}^{\mathsf{T}})$ holds. But $\mathbf{S}\Psi \neq \mathrm{V}\textsc{ar}\,\mathcal{S}\ell_2$ because the subvariety $\mathbf{S}\Psi$ of $\mathbf{S}$ is not even finitely generated [15, Corollary 6.8]. □

Example 14. Let $\mathbb{S}^* = \mathrm{V}\textsc{ar}(\mathcal{RB}_4 \times \mathcal{S}\ell_3, {}^*)$ and $\mathbf{S} = \mathrm{V}\textsc{ar}\{\mathcal{RB}_4 \times \mathcal{S}\ell_3\}$, where the operation * on $\mathcal{RB}_4 \times \mathcal{S}\ell_3$ is given by $(x, y)^* = (x^*, y^{\mathsf{S}})$. Then $\Psi = \{xyx \approx xy\}$ is a nonsymmetric set such that $\mathbb{S}^*\Psi = \mathrm{V}\textsc{ar}\,(\mathcal{S}\ell_3, {}^{\mathsf{S}})$ and $\mathbf{S}\Psi \neq \mathrm{V}\textsc{ar}\,\mathcal{S}\ell_3$.

Proof. As shown in Dolinka [6], the variety $\mathrm{V}\textsc{ar}\,(\mathcal{S}\ell_3, {}^{\mathsf{S}})$ is defined by the equations

$$x^2 \approx x, \quad xy \approx yx, \quad xx^*y \approx xx^*, \tag{4}$$

while the variety $\mathbb{S}^*$ is defined by the equations

$$x^2 \approx x, \quad axyb \approx ayxb, \quad axx^*yb \approx azz^*tb. \tag{5}$$

Since the variety $\mathbb{S}^*\Psi$ satisfies the equation $xyx \approx xy$ and every variety of $*$-semigroups is symmetric, $\mathbb{S}^*\Psi$ also satisfies the reverse equation $xyx \approx yx$. Hence $\mathbb{S}^*\Psi$ is commutative. Further, since

$$xx^*y \overset{(5)}{\approx} xxx^*y \overset{\Psi}{\approx} xxx^*yx^* \overset{(5)}{\approx} xxx^*x^*x^* \overset{(5)}{\approx} xx^*,$$

the variety $\mathbb{S}^*\Psi$ also satisfies the equation $xx^*y \approx xx^*$. Therefore, the variety $\mathbb{S}^*\Psi$ satisfies the equations (4), so that the inclusion $\mathbb{S}^*\Psi \subseteq \mathrm{V}\textsc{ar}\,(\mathcal{S}\ell_3, {}^{\mathsf{S}})$ holds. It is easily shown that $(\mathcal{S}\ell_3, {}^{\mathsf{S}})$ satisfies the equations (5) and Ψ, so that the inclusion $\mathrm{V}\textsc{ar}\,(\mathcal{S}\ell_3, {}^{\mathsf{S}}) \subseteq \mathbb{S}^*\Psi$ holds. Therefore, $\mathbb{S}^*\Psi = \mathrm{V}\textsc{ar}\,(\mathcal{S}\ell_3, {}^{\mathsf{S}})$. As for the variety $\mathbf{S}\Psi$, since it coincides with the variety of left normal bands, it cannot be generated by the semilattice $\mathcal{S}\ell_3$ alone. □

Lemma 15. *Let $(S, {}^*)$ be any unstable $*$-semigroup and let $\mathbb{S}^* = \mathrm{V}\textsc{ar}\,(S, {}^*)$ and $\mathbf{S} = \mathrm{V}\textsc{ar}\,S$. Suppose that Ψ_1 and Ψ_2 are symmetric sets of homogeneous plain equations. Then $\mathbf{S}\Psi_1 = \mathbf{S}\Psi_2$ if and only if $\mathbb{S}^*\Psi_1 = \mathbb{S}^*\Psi_2$.*

Proof. It is obvious that $\mathbf{S}\Psi_1 = \mathbf{S}\Psi_2$ implies that $\mathbb{S}^*\Psi_1 = \mathbb{S}^*\Psi_2$. Conversely, suppose that $\mathbb{S}^*\Psi_1 = \mathbb{S}^*\Psi_2$. Then by Lemma 12, there exist unstable $*$-semigroups $(U_1, {}^*)$ and $(U_2, {}^*)$ such that $\mathbb{S}^*\Psi_i = \mathrm{V}\textsc{ar}(U_i, {}^*)$ and $\mathbf{S}\Psi_i = \mathrm{V}\textsc{ar}\,U_i$. Therefore $\mathrm{V}\textsc{ar}(U_1, {}^*) = \mathrm{V}\textsc{ar}(U_2, {}^*)$, whence $\mathrm{V}\textsc{ar}\,U_1 = \mathrm{V}\textsc{ar}\,U_2$. Thus $\mathbf{S}\Psi_1 = \mathbf{S}\Psi_2$. □

It is easily verified that the only plain equations satisfied by a nontrivial semilattice are homogeneous ones. Therefore, it is not unreasonable to call a variety of semigroups *homogeneous* if it contains some nontrivial semilattice. Clearly, if $\mathbf{V}$ is a variety of semigroups that is not homogeneous, then the join $\mathbf{V} \vee \mathrm{V}\textsc{ar}\,\mathcal{S}\ell_2$ is homogeneous.

Lemma 16. *Let $\mathbf{V}_1$ and $\mathbf{V}_2$ be any nonhomogeneous varieties of semigroups. Then $\mathbf{V}_1 \vee \mathrm{V}\textsc{ar}\,\mathcal{S}\ell_2 = \mathbf{V}_2 \vee \mathrm{V}\textsc{ar}\,\mathcal{S}\ell_2$ if and only if $\mathbf{V}_1 = \mathbf{V}_2$.*

Proof. This result is well known. See, for example, Vernikov [28, Lemma 1.3]. □

Lemma 17. *Suppose that* **S** *is any homogeneous variety of semigroups that contains continuum many symmetric subvarieties. Then* **S** *contains continuum many symmetric homogeneous subvarieties.*

Proof. Suppose that **S** contains continuum many symmetric nonhomogeneous subvarieties, say $\{\mathbf{V}_i \mid i \in \Lambda\}$ is the set of these subvarieties. Then, since $\mathcal{S}\ell_2 \in \mathbf{S}$ by assumption and the variety $\mathsf{VAR}\,\mathcal{S}\ell_2$ is symmetric, each join $\mathbf{V}_i \vee \mathsf{VAR}\,\mathcal{S}\ell_2$ is a symmetric homogeneous subvariety of **S**. By Lemma 16, the varieties in the set $\{\mathbf{V}_i \vee \mathsf{VAR}\,\mathcal{S}\ell_2 \mid i \in \Lambda\}$ are distinct. □

The following is a restatement of the main theorem with finer details.

Theorem 18. *Let* $(S, {}^*)$ *be any unstable* $*$*-semigroup and let* $\mathbb{S}^* = \mathsf{VAR}\,(S, {}^*)$ *and* $\mathbf{S} = \mathsf{VAR}\,S$*. Suppose that the variety* **S** *contains continuum many symmetric subvarieties. Then the variety* $\mathbb{S}^*$ *contains continuum many subvarieties.*

Proof. Since the assumption $(\mathcal{S}\ell_3, {}^{\mathsf{S}}) \in \mathbb{S}^*$ implies that $\mathcal{S}\ell_3 \in \mathbf{S}$, the variety **S** is homogeneous. Let $\{\mathbf{V}_i \mid i \in \Lambda\}$ be any set of continuum many symmetric subvarieties of **S**. By Lemma 17, each variety $\mathbf{V}_i$ can be assumed homogeneous. Therefore, for each $i \in \Lambda$, there exists some symmetric set Ψ_i of homogeneous plain equations such that $\mathbf{V}_i = \mathbf{S}\Psi_i$. By Lemma 15, the varieties in $\{\mathbb{S}^*\Psi_i \mid i \in \Lambda\}$ are distinct subvarieties of $\mathbb{S}^*$. □

5. Varieties of $*$-monoids

The property of unstableness can be extended from varieties of $*$-semigroups to varieties of $*$-monoids in the obvious manner: a variety of $*$-monoids is *unstable* if it contains the $*$-monoid $(\mathcal{S}\ell_3^1, {}^{\mathsf{S}})$, and a $*$-monoid is *unstable* if it generates an unstable variety of $*$-monoids.

Theorem 19. *Let* $(M, {}^*)$ *be any unstable* $*$*-monoid. Suppose that the variety of monoids generated by* M *contains continuum many symmetric subvarieties. Then the variety of* $*$*-monoids generated by* $(M, {}^*)$ *contains continuum many subvarieties.*

Proof. It suffices to show that the variety $\mathbb{M}^* = \mathsf{VAR}\,(M, {}^*)$ of $*$-semigroups contains continuum many subvarieties that are generated by $*$-monoids. By assumption, the variety $\mathbf{M} = \mathsf{VAR}\,M$ contains continuum many symmetric subvarieties generated by monoids. The arguments in the proof of Lemma 17 can easily be repeated to show that **M** contains continuum many symmetric homogeneous subvarieties generated by monoids, say $\{\mathbf{V}_i \mid i \in \Lambda\}$ is the set of these subvarieties. Therefore for each $i \in \Lambda$, there exists some symmetric set Ψ_i of homogeneous plain equations such that $\mathbf{V}_i = \mathbf{M}\Psi_i$. Since $\mathbf{V}_i$ is generated by monoids, the set Ψ_i can be chosen to be *closed under deletion* in the sense that for any equation $\mathbf{u} \approx \mathbf{v}$ in Ψ_i and any substitution φ_x that maps the variable x to 1, the equation $\mathbf{u}\varphi_x \approx \mathbf{v}\varphi_x$ also belongs to Ψ_i. It thus follows from Lemma 15 that the varieties in $\{\mathbb{M}^*\Psi_i \mid i \in \Lambda\}$ are distinct subvarieties of $\mathbb{M}^*$ generated by $*$-monoids. □

Corollary 20. *Each of the* $*$*-monoids* $(\mathcal{A}_2^1, {}^*)$, $(\mathcal{B}_2^1, {}^{\mathsf{S}})$*, and* $(\mathcal{B}_2^1, {}^{\mathsf{T}}) \times (\mathcal{S}\ell_3, {}^{\mathsf{S}})$ *generates a variety of* $*$*-monoid with continuum many subvarieties.*

Proof. Each of the monoids $\mathcal{A}_2^1$ and $\mathcal{B}_2^1$ generates a variety of monoids with continuum many symmetric subvarieties [10]. Since the $*$-monoids $(\mathcal{A}_2^1, {}^*)$, $(\mathcal{B}_2^1, {}^{\mathsf{S}})$, and $(\mathcal{B}_2^1, {}^{\mathsf{T}}) \times (\mathcal{S}\ell_3, {}^{\mathsf{S}})$ are all unstable, the result follows from Theorem 19. □

Acknowledgement

The author is indebted to the reviewer for a number of suggestions.

References

[1] K. Auinger, I. Dolinka, T.V. Pervukhina, M.V. Volkov, Unary enhancements of inherently non-finitely based semigroups, Semigroup Forum 89 (2014) 41–51.
[2] K. Auinger, I. Dolinka, M.V. Volkov, Equational theories of semigroups with involution, J. Algebra 369 (2012) 203–225.
[3] K. Auinger, I. Dolinka, M.V. Volkov, Matrix identities involving multiplication and transposition, J. Eur. Math. Soc. 14 (2012) 937–969.
[4] S. Burris, H.P. Sankappanavar, A Course in Universal Algebra, Springer-Verlag, New York, 1981.
[5] S. Crvenković, I. Dolinka, M. Vinčić, Equational bases for some 0-direct unions of semigroups, Studia Sci. Math. Hung. 36 (2000) 423–431.
[6] I. Dolinka, On the lattice of varieties of involution semigroups, Semigroup Forum 62 (2001) 438–459.
[7] C.C. Edmunds, E.W.H. Lee, K.W.K. Lee, Small semigroups generating varieties with continuum many subvarieties, Order 27 (2010) 83–100.
[8] S. Fajtlowicz, Equationally complete semigroups with involution, Algebra Univers. 1 (1971) 355–358.
[9] M. Jackson, Finite semigroups whose varieties have uncountably many subvarieties, J. Algebra 228 (2000) 512–535.
[10] M. Jackson, E.W.H. Lee, Monoid varieties with extreme properties, Trans. Amer. Math. Soc., https://doi.org/10.1090/tran/7091, in press.
[11] M. Jackson, R. McKenzie, Interpreting graph colorability in finite semigroups, Int. J. Algebra Comput. 16 (2006) 119–140.
[12] M. Jackson, O. Sapir, Finitely based, finite sets of words, Int. J. Algebra Comput. 10 (2000) 683–708.
[13] J. Kad'ourek, A finitely generated variety of combinatorial inverse semigroups having uncountably many subvarieties, Proc. R. Soc. Edinb., Sect. A, Math. 132 (2002) 1373–1394.

[14] E.I. Kleĭman, On basis of identities of Brandt semigroups, Semigroup Forum 13 (1977) 209–218.
[15] E.W.H. Lee, Subvarieties of the variety generated by the five-element Brandt semigroup, Int. J. Algebra Comput. 16 (2006) 417–441.
[16] E.W.H. Lee, Inherently non-finitely generated varieties of aperiodic monoids with central idempotents, J. Math. Sci. (N.Y.) 209 (2015) 588–599; reprint of Zap. Nauč. Semin. POMI 423 (2014) 166–182.
[17] E.W.H. Lee, Finitely based finite involution semigroups with non-finitely based reducts, Quaest. Math. 39 (2016) 217–243.
[18] E.W.H. Lee, Equational theories of unstable involution semigroups, Electron. Res. Announc. Math. Sci. 24 (2017) 10–20.
[19] E.W.H. Lee, J.R. Li, Minimal non-finitely based monoids, Diss. Math. (Rozprawy Mat.) 475 (2011), 65 pp.
[20] E.W.H. Lee, J.R. Li, The variety generated by all monoids of order four is finitely based, Glas. Mat. Ser. III 50 (2015) 373–396.
[21] E.W.H. Lee, J.R. Li, W.T. Zhang, Minimal non-finitely based semigroups, Semigroup Forum 85 (2012) 577–580.
[22] E.W.H. Lee, W.T. Zhang, Finite basis problem for semigroups of order six, LMS J. Comput. Math. 18 (2015) 1–129.
[23] S. Oates, M.B. Powell, Identical relations in finite groups, J. Algebra 1 (1964) 11–39.
[24] M.V. Sapir, Problems of Burnside type and the finite basis property in varieties of semigroups, Math. USSR, Izv. 30 (1988) 295–314; translation of Izv. Akad. Nauk SSSR, Ser. Mat. 51 (1987) 319–340.
[25] M.V. Sapir, Identities of finite inverse semigroups, Int. J. Algebra Comput. 3 (1993) 115–124.
[26] O. Sapir, Non-finitely based monoids, Semigroup Forum 90 (2015) 557–586.
[27] A.N. Trahtman, A six-element semigroup that generates a variety with a continuum of subvarieties, Ural. Gos. Univ. Mat. Zap. 14 (3) (1988) 138–143 (in Russian).
[28] B.M. Vernikov, On modular elements of the lattice of semigroup varieties, Comment. Math. Univ. Carol. 48 (2007) 595–606.
[29] W.T. Zhang, Y.F. Luo, The variety generated by a certain transformation monoid, Int. J. Algebra Comput. 18 (2008) 1193–1201.