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Number theory

Geometric sequences and zero-free region of the zeta function

Suites géométriques et région sans zéro de la fonction zêta

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ABSTRACT

Let $\mathcal{N}$ be the linear space of functions $\sum_{k=1}^{n} a_k \rho(\theta_k/x)$ with a condition $\sum_{k=1}^{n} a_k \theta_k = 0$ for $0 < \theta_k \leq 1$. Here $\rho(x)$ denotes the fractional part of x. Beurling pointed out that the problem of how well a constant function can be approximated by functions in $\mathcal{N}$ is closely related to the zero-free region of the Riemann zeta function. More precisely, Báez-Duarte gave a zero-free region related to a L^p-norm estimation of a constant function by using the Dirichlet series for the zeta function. In this paper, we consider the L^∞-norm estimation of a constant function and give a wider zero-free region than that of the Báez-Duarte result.

RÉSUMÉ

Soit $\mathcal{N}$ l'espace vectoriel de fonctions $\sum_{k=1}^{n} a_k \rho(\theta_k/x)$ satisfaisant la condition $\sum_{k=1}^{n} a_k \theta_k = 0$ pour $0 < \theta_k \leq 1$, où $\rho(x)$ désigne la partie fractionnaire de x. Beurling a indiqué que le problème d'approximation d'une fonction constante par fonctions dans $\mathcal{N}$ est étroitement lié à la région sans zéro de la fonction zêta de Riemann. Plus précisement, Báez-Duarte a donné une région sans zéro liée à une estimation de la norme L^p d'une fonction constante en utilisant les séries de Dirichlet pour la fonction zêta. Dans cet article, nous considerons une estimation de la norme L^∞ d'une fonction constante et donnons une région sans zéro plus large que celle du résultat de Báez-Duarte.

1. Introduction

Let $\rho(x)$ be the fractional part of x. The Nyman space $\mathcal{N}$ consists of all functions of the form

$$\sum_{k=1}^{n} a_k \rho\left(\frac{\theta_k}{x}\right)$$

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for any natural number n, which satisfies the condition $\sum_{k=1}^{n} a_k\theta_k = 0$ for $0 < \theta_k \le 1$. In many approaches to solve the Riemann hypothesis, Beurling [5] and Nyman [8] found a connection between the existence of the nontrivial zeros of the Riemann zeta function and a density of a function space $\mathcal{N}$ in $L^p(0,1)$. More precisely, the fact that $\mathcal{N}$ is dense in $L^p(0,1)$ is equivalent to that the zeta function is zero-free on the Re $s > 1/p$. In his paper [5], Beurling also pointed out that the problem of how well a function χ can be approached by functions in $\mathcal{N}$ is closely related to the distribution of the primes even in case ζ has zeros close to the line Re $s = 1$. Here χ denotes the characteristic function on $(0,1)$.

In [1], Báez-Duarte gave an explicit result about Beurling's remark.

Theorem 1.1. *If $f \in \mathcal{N}$, $1 < p \le 2$, and $\epsilon = \|\chi - f\|_p$, then ζ does not vanish in the closed triangle with vertices at the points $\{1/p, 1, 1 + (\mathrm{i}/2)\epsilon^{-1}\}$.*

Though the Theorem 1.1 gives each $f \in \mathcal{N}$ to a zero-free region for ζ, the region is angled towards the line Re $s = 1$. In this paper we give a different zero-free region using L^∞-norm.

We first introduce function spaces to work on. For $0 \le \delta < 1$, we define $\mathcal{X}_\delta$ by

$$\mathcal{X}_\delta := \{f \in \overline{\mathcal{N}} : f(x) = 1 \text{ for } \delta < x < 1\},$$

where $\overline{\mathcal{N}}$ is the closure of $\mathcal{N}$ in $L^2(0,1)$. Concrete functions in $\mathcal{X}_\delta$ are presented in Section 3.

The following is our main theorem.

Theorem 1.2. *For $0 < \delta < 0.043$, suppose that $f \in \mathcal{X}_\delta$ and $\epsilon = \|\chi - f\|_\infty$. Then $\zeta(\sigma + \mathrm{i}t)$ does not vanish in a region given by*

$$|t| < \frac{C}{\epsilon\delta^\sigma}$$

on the critical strip. Here $C = \pi/4\,\mathrm{e}^{2\pi}$.

As a consequence of Theorem 1.2, we see that the region

$$|t| < \frac{C}{\epsilon\sqrt{\delta}}$$

is free from zero, which is more regular than Báez-Duarte's result.

2. Proof of the theorem

For $f \in \mathcal{N}$ as

$$f(x) = \sum_{k=1}^{n} a_k \rho\left(\frac{\theta_k}{x}\right)$$

with a condition $\sum_{k=1}^{n} a_k\theta_k = 0$ for $0 < \theta_k \le 1$, we get

$$\mathrm{Re}\, f(x) = \sum_{k=1}^{n} \mathrm{Re}(a_k)\rho\left(\frac{\theta_k}{x}\right) \quad \text{and} \quad \mathrm{Im}\, f(x) = \sum_{k=1}^{n} \mathrm{Im}(a_k)\rho\left(\frac{\theta_k}{x}\right).$$

Since $\sum_{k=1}^{n} \mathrm{Re}(a_k)\theta_k = \sum_{k=1}^{n} \mathrm{Im}(a_k)\theta_k = 0$, both Re$f$ and Imf also belong to $\mathcal{N}$. So we may assume that f is a real-valued function without loss of generality. Moreover note that

$$f(x) = 0 \qquad \text{for} \qquad \max\theta_k \le x.$$

Thus, a contraction operator T_v defined by

$$T_v f(x) := \begin{cases} f(x/v), & 0 < x \le v \\ 0, & v < x < 1 \end{cases}$$

for $0 < v < 1$, is closed on $\mathcal{N}$. As a result, T_v is closed on $\overline{\mathcal{N}}$.

In [4], Bercovici and Foias obtained the following equivalent form for $\overline{\mathcal{N}}$ using the Mellin transform;

$$\overline{\mathcal{N}} = \left\{ f \in L^2(0,1) : \frac{Mf(s)}{\zeta(s)} \text{ is analytic on Re } s > \frac{1}{2} \right\}. \tag{2.1}$$

Here Mf is the Mellin transform defined by

$$Mf(s) := \frac{1}{\sqrt{2\pi}} \int_0^1 f(x) x^{s-1}\, \mathrm{d}x$$

for $f \in L^2(0,1)$. By considering orthogonals in (2.1), Balazard and Saias pointed out that the Bercovici–Foias theorem gives a complete characterization for the complement space of $\mathcal{N}$ in $L^2(0,1)$. More precisely, we have the following theorem.

Theorem 2.1. *Let $\mathcal{N}^\perp$ be the orthogonal complement of $\mathcal{N}$ in $L^2(0,1)$. Then we have*

$$\mathcal{N}^\perp = span_{L^2(0,1)} \left\{ x \to x^{s-1} \log^k x, \zeta(s) = 0 \text{ with } \mathrm{Re}\, s > 1/2 \right\}, \tag{2.2}$$

where $0 \le k \le$ multiplicity of s.

See [2,3,11,12] for more results of $\mathcal{N}$ and $\mathcal{N}^\perp$. In (2.2), we put

$$\varphi_s(x) := \mathrm{Im}(x^{s-1}).$$

Clearly we have

$$\varphi_s(x) = x^{\sigma-1} \sin(t \log x),$$

where $s = \sigma + \mathrm{i}t$. The graph of φ_s rapidly oscillate near the origin. So φ_s has infinitely many zeros near the origin. The zeros of φ_s on $(0,1]$, listed in decreasing order, are

$$r_n := r^n \quad \text{with} \quad r := \mathrm{e}^{-\pi/t} \tag{2.3}$$

for $n = 0, 1, \ldots$. For each natural number n, the area of φ_s on $[r_n, r_{n-1}]$ is given by

$$A_n := \int_{r_n}^{r_{n-1}} |\varphi_s(x)|\, \mathrm{d}x = \int_{r_n}^{r_{n-1}} |\mathrm{Im}(x^{s-1})|\, \mathrm{d}x$$
$$= \left| \mathrm{Im} \left(\int_{r_n}^{r_{n-1}} x^{s-1}\, \mathrm{d}x \right) \right| = \frac{t}{\sigma^2 + t^2} \frac{1 + r^\sigma}{r^\sigma} (r^\sigma)^n.$$

Consequently, we obtain two geometric sequences r_n and A_n for each φ_s, which are crucial in the proof of our main theorem.

The following lemma can be easily proved by elementary calculus.

Lemma 2.2. *For $0 < x \le 1$ we have*

$$\frac{\mathrm{e}^{2\pi x}}{\mathrm{e}^{\pi x} - 1} \le \frac{c}{x},$$

where $c = \mathrm{e}^{2\pi}/\pi$.

Now we prove our main result.

Proof. Let $f \in \mathcal{X}_\delta$ for a sufficiently small $\delta > 0$ and $\epsilon = \|\chi - f\|_\infty$. We borrow the well-known fact that $\zeta(s) \ne 0$ with $|t| < 1$ in the critical strip. Assume that there is a zero $s_0 = \sigma_0 + \mathrm{i}t_0$, with

$$1 < t_0 < \frac{C}{\epsilon \delta^{\sigma_0}} \quad \text{and} \quad \sigma_0 > 1/2,$$

where $C = \pi/4\, \mathrm{e}^{2\pi}$. We will complete the proof by deriving a contradiction.

Let r_n and A_n be the geometric sequences corresponding to φ_{s_0}. From $t_0 > 1$, we have

$$r = \mathrm{e}^{-\pi/t_0} > \mathrm{e}^{-\pi} \approx 0.043,$$

where r is defined in (2.3) for φ_{s_0}. So we can choose the positive integer N such that

$$r^N \le \delta < r^{N-1}. \tag{2.4}$$

Then we consider a function $f - T_r f$. From $f \in \mathcal{X}_\delta$, we get

$$(f - T_r f)(x) = \begin{cases} 1, & r < x < 1 \\ 0, & \delta < x \le r \\ \text{absolute value} \le 2\epsilon, & 0 < x \le \delta. \end{cases}$$

By Theorem 2.1, we have

$$0 = \int_0^1 (f - T_r f) \cdot \varphi_{s_0} = \int_0^\delta (f - T_r f) \cdot \varphi_{s_0} + \int_r^1 \varphi_{s_0}.$$

Thus we get

$$A_1 = \left| \int_0^\delta (f - T_r f) \cdot \varphi_{s_0} \right| \le 2\epsilon \cdot \int_0^{r^{N-1}} |\varphi_{s_0}| = 2\epsilon \cdot \sum_{n=N}^{\infty} A_n.$$

Moreover, we have

$$1 = \frac{2\epsilon \cdot \sum_{n=N}^{\infty} A_n}{A_1} = \frac{2\epsilon r^{\sigma_0 N}}{r^{\sigma_0}(1 - r^{\sigma_0})} \le \frac{2\epsilon \delta^{\sigma_0}}{r^{\sigma_0}(1 - r^{\sigma_0})};$$

the last inequality holds by (2.4). By Lemma 2.2,

$$\frac{1}{r^{\sigma_0}(1 - r^{\sigma_0})} = \frac{e^{2\pi\sigma_0/t_0}}{e^{\pi\sigma_0/t_0} - 1} \le \frac{e^{2\pi}}{\pi} \cdot \frac{t_0}{\sigma_0} \le \frac{2e^{2\pi}}{\pi} t_0$$

Consequently, we obtain that

$$1 \le \frac{2\,\epsilon\,\delta^{\sigma_0}}{r^{\sigma_0}(1 - r^{\sigma_0})} \le \frac{4\,e^{2\pi}}{\pi} \epsilon \delta^{\sigma_0} t_0 < 1,$$

which is impossible. Thus we finish the proof. □

3. Remark and question

For an example function in $\mathcal{X}_\delta$, we define the natural approximation f_n by

$$f_n(x) = n g(n) \rho\left(\frac{1}{nx}\right) - \sum_{k=1}^{n} \mu(k) \rho\left(\frac{1}{kx}\right), \quad \text{where} \quad g(n) := \sum_{k=1}^{n} \frac{\mu(k)}{k}.$$

Here μ denotes the Möbius function. Then, the fact that f_n belongs to $\mathcal{X}_{1/n}$ follows from the well-known one:

$$\sum_{k=1}^{\infty} \mu(k) \left[\frac{1}{kx}\right] = 1 \quad \text{for} \quad 0 < x \le 1.$$

See [1] for more results. As a summatory function of μ, let

$$M(n) := \sum_{k=1}^{n} \mu(k).$$

The properties of the functions μ and M are central in the theory of prime numbers. There is an exhaustive list of results of μ, M. We refer the reader to [6,7,9,10] for related work.

The oscillating property of M is known by Pintz, see [10]. More precisely, M changes signs infinitely many times. In case of g, it is known that

$$\lim_{n \to \infty} g(n) = 0.$$

However, the oscillating property of g is not known yet. If g also changes signs infinitely often, then we obtain

$$|g(n)| \le 1/n$$

for infinitely many n's. As a result, we have

$$\left\| ng(n)\rho\left(\frac{1}{nx}\right)\right\|_\infty \le 1, \quad \text{for infinitely many } n\text{'s.}$$

So we only need to consider the second term of f_n for $\|\chi - f_n\|_\infty$ on $(0, 1/n)$. Thus the following is an interesting question.

Question 3.1. *Does the sequence*

$$g(n) = \sum_{k=1}^{n} \frac{\mu(k)}{k}$$

has infinitely many sign-changing solutions?

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