



Ordinary differential equations

Almost automorphic solutions to logistic equations with discrete and continuous delay



Solutions presque automorphes des équations logistiques avec retard discret et continu

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ABSTRACT

We obtain sufficient conditions for the existence and uniqueness of a positive compact almost automorphic solution to a logistic equation with discrete and continuous delay. Moreover, we provide a counterexample to some results in literature which deal with the uniqueness of almost periodic solutions to logistic type equations.

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R É S U M É

Nous obtenons des conditions suffisantes pour l'existence et l'unicité d'une solution positive et compacte presque automorphe, d'une équation logistique avec retard discret et continu. De plus, nous donnons un contre-exemple à des résultats publiés, qui traitent l'unicité des solutions presque périodiques des équations de type logistique.

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1. Introduction

We consider the following equation:

$$N'(t) = N(t) \left(a(t) - b(t)N(t - \tau_1) - c(t) \int_0^{\tau_2} k(s)N(t - s) ds \right) \quad \text{for } t \in \mathbb{R}, \quad (1.1)$$

where $k : [0, \infty) \rightarrow [0, \infty)$ is piecewise continuous, and the functions $a(\cdot)$, $b(\cdot)$ and $c(\cdot)$ are almost automorphic and satisfy the following conditions

$$0 < a_0 \leq a(t) \leq a_1, \quad 0 < b_0 \leq b(t) \leq b_1 \quad \text{and} \quad 0 < c_0 \leq c(t) \leq c_1 \quad \text{for } t \in \mathbb{R}, \quad (1.2)$$

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where $a_0, a_1, b_0, b_1, c_0, c_1$ are positive constants. This equation can model the dynamics of population of a species in a time-fluctuating environment. Even if the birth rate $a(t)$ and death rates $b(t), c(t)$ are periodic with respect to time t , the overall time dependence may not be periodic; i.e., if the quotient of periods of these functions is not rational, the overall time dependence will not be periodic, but almost periodic in the sense of Bohr. In reality, the parameters $a(t), b(t), c(t)$ of Eq. (1.1) may be outputs of other almost periodic dynamical systems. However, it is well known in general that almost periodic systems do not carry necessarily almost periodic dynamics [6,9,11]. Although these systems may have bounded oscillating solutions, these oscillations belong to a class larger than the class of almost periodic functions: we are talking about *almost automorphic* functions. Bochner introduced the concept of almost automorphy in the literature in [1] as a generalization of almost periodicity. This concept was then deeply investigated by Veech [12] and many other authors. That is why it is natural to assume that the coefficients $a(t), b(t), c(t)$ in Eq. (1.1) are almost automorphic. To the best of our knowledge, no authors have considered the problems of positive almost automorphic solutions to logistic-type delay equations.

The purpose of this work is to give sufficient conditions for the existence and uniqueness of a positive compact almost automorphic solution to Eq. (1.1) when the coefficients are almost automorphic. In the case where the coefficients are periodic, a variant of Eq. (1.1) has been studied extensively in [4]. The almost periodic case was treated by Seifert in an article [10] where he gave a result for the existence and uniqueness of an almost periodic solution to

$$N'(t) = N(t) \left(a(t) - b(t) \int_0^\infty k(s)N(t-s) ds \right) \quad \text{for } t \in \mathbb{R} \tag{1.3}$$

provided that

$$\frac{b_0^2}{b_1^2} > a_1 \frac{\int_0^\infty sk(s) ds}{\int_0^\infty e^{-a_1s}k(s) ds}. \tag{1.4}$$

In many works in the literature such as [7], [8] and [13], the authors claimed that the delay logistic equation (1.3) and other variants have a unique positive almost periodic solution without assuming any assumption such as (1.4). Unfortunately, those results seem to be incorrect. We will explain this in Section 3 through a counterexample.

2. Main results

The following lemma gives an *a priori* lower and upper bounds for positive solutions to Eq. (1.1).

Lemma 2.1. *Let N be a positive solution to Equation (1.1) on \mathbb{R}^+ . Then,*

$$\limsup_{t \rightarrow \infty} N(t) \leq \frac{a_1}{b} := m_1 \tag{2.1}$$

and

$$\liminf_{t \rightarrow \infty} N(t) \geq \frac{a_0}{b_1 e^{-d_0 \tau_1} + c_1 \int_0^{\tau_2} k(s) e^{-d_0 s} ds} := m_0, \tag{2.2}$$

where $\tilde{b} = b_0 e^{-a_1 \tau_1} + c_0 \int_0^{\tau_2} k(s) e^{-a_1 s} ds$ and $d_0 = a_0 - 2m_1 b_1 - 2c_1 m_1 \int_0^{\tau_2} k(s) ds$.

Proof. We have for each $t \geq 0$

$$N'(t) \leq a_1 N(t) - b_0 N(t)N(t - \tau_1) - c_0 N(t) \int_0^{\tau_2} k(s)N(t-s) ds, \tag{2.3}$$

which implies, by the positivity of N , that $N'(t) < a_1 N(t)$. It follows that $N(t) < e^{a_1 \tau_1} N(t - \tau_1)$ and $N(t) < e^{a_1 s} N(t - s)$ for all $s \in [0, \tau_2]$. We deduce from (2.3) that

$$N'(t) < a_1 N(t) - \tilde{b} N^2(t),$$

where $\tilde{b} = b_0 e^{-a_1 \tau_1} + c_0 \int_0^{\tau_2} k(s) e^{-a_1 s} ds$. Put $y(t) = \frac{1}{N(t)}$, then we have

$$y'(t) > e^{-a_1 t} y(0) + \frac{\tilde{b}}{a_1} (1 - e^{-a_1 t}).$$

Thus

$$N(t) < \frac{1}{\frac{e^{-a_1 t}}{N(0)} + \frac{\tilde{b}}{a_1} (1 - e^{-a_1 t})},$$

which implies that

$$\limsup_{t \rightarrow \infty} N(t) \leq \frac{a_1}{\tilde{b}} := m_1. \quad (2.4)$$

On the other hand, we have

$$N'(t) = d(t)N(t),$$

where $d(t) = a(t) - b(t)N(t - \tau_1) - c(t) \int_0^{\tau_2} k(s)N(t-s) ds$. It follows that $N(t) = e^{\int_{t-\tau_1}^t d(\theta) d\theta} N(t - \tau_1)$ and $N(t) = e^{\int_{t-s}^t d(\theta) d\theta} N(t - s)$ for all $s \in [0, \tau_2]$. For sufficiently large t we have by (2.4) $N(\theta) \leq 2m_1$ for all $\theta \in [t - \max(\tau_1, \tau_2), t]$ and thus

$$d(\theta) \geq a_0 - 2m_1 b_1 - 2c_1 m_1 \int_0^{\tau_2} k(s) ds := d_0.$$

This implies that $N(t - \tau_1) \leq e^{-d_0 \tau_1} N(t)$ and $N(t - s) \leq e^{-d_0 s} N(t)$ for all $s \in [0, \tau_2]$. We get the following differential inequality

$$N'(t) \geq a_0 N(t) - \left(b_1 e^{-d_0 \tau_1} + c_1 \int_0^{\tau_2} k(s) e^{-d_0 s} ds \right) N^2(t).$$

By derivating again the function $y(t) = \frac{1}{N(t)}$, we deduce that

$$\liminf_{t \rightarrow \infty} N(t) \geq \frac{a}{b_1 e^{-d_0 \tau_1} + c_1 \int_0^{\tau_2} k(s) e^{-d_0 s} ds} := m_0.$$

This ends the proof of the lemma. \square

Consider the following Cauchy problem

$$\begin{cases} N'(t) = N(t) \left(a(t) - b(t)N(t - \tau_1) - c(t) \int_0^{\tau_2} k(s)N(t-s) ds \right) & \text{for } t \geq 0 \\ N(t) = \varphi(t) & \text{for } -\tau \leq t \leq 0 \end{cases} \quad (2.5)$$

where $\tau = \max\{\tau_1, \tau_2\}$ and φ is a continuous function from $[-\tau, 0]$ to \mathbb{R} .

Proposition 2.2. For each nonnegative initial data φ with $\varphi(0) > 0$, there exists a unique positive global solution to Eq. (2.5).

Proof. The local existence of a solution is guaranteed by [5, Theorem 2.3, Chapter 2], where

$$f(t, \phi) = \phi(0) \left(a(t) - b(t)\phi(-\tau_1) - c(t) \int_0^{\tau_2} k(s)\phi(-s) ds \right).$$

The positivity of this solution can be proved using the same arguments as in [3, Lemma 1]. This local solution is global, since otherwise the solution must blow up at the maximal time of existence and thus will contradict the *a priori* estimate in Lemma 2.1. \square

Definition 2.3. [1] A continuous function $f : \mathbb{R} \mapsto X$ is said to be *almost automorphic* if, for every sequence of real numbers $(s_n)_n$, there exist a subsequence $(s'_n)_n \subset (s_n)_n$ and a function \tilde{f} , such that, for each $t \in \mathbb{R}$

$$f(t + s'_n) \rightarrow \tilde{f}(t)$$

and

$$\tilde{f}(t - s'_n) \rightarrow f(t)$$

as $n \rightarrow \infty$. If the above limits hold uniformly in compact subsets of \mathbb{R} , then f is said to be *compact almost automorphic*.

The following proposition gives a characterization of compact almost automorphic functions.

Proposition 2.4. [2] *A function f is compact almost automorphic if and only if it is almost automorphic and uniformly continuous.*

Theorem 2.5. *There exists a positive solution S to Eq. (1.1) on \mathbb{R} such that*

$$m_0 \leq S(t) \leq m_1, \tag{2.6}$$

where m_0 and m_1 are the positive constants defined by (2.1) and (2.2). Moreover, this solution is unique provided that

$$\left(b_1 \tau_1 + c_1 \int_0^{\tau_2} s k(s) ds \right) m_1^2 < \frac{b_0 + c_0 \int_0^{\tau_2} k(s) ds}{b_1 + c_1 \int_0^{\tau_2} k(s) ds} m_0. \tag{2.7}$$

Proof. Let N be a positive solution to Eq. (1.1) on \mathbb{R}^+ . Using (2.1), one can see that N and its derivative are bounded, thus N is uniformly continuous. Let $(t_n)_n$ be a sequence of real numbers such that $\lim_{t \rightarrow \infty} t_n = \infty$. Then, for sufficiently large n , the sequence of functions $N_n : t \mapsto N(t + t_n)$ is well defined on $[-1, 1]$ and is equicontinuous. It follows by Arzelà–Ascoli’s theorem that there exist a function Q and a subsequence $(t_n^1)_n \subset (t_n)_n$ such that

$$N(t + t_n^1) \rightarrow Q(t) \text{ as } n \rightarrow \infty$$

uniformly on $[-1, 1]$. By applying the same argument to the subsequence $(t_n^1)_n$, we extract a subsequence $(t_n^2)_n \subset (t_n^1)_n \subset (t_n)_n$ such that

$$N(t + t_n^2) \rightarrow Q(t) \text{ as } n \rightarrow \infty$$

uniformly on $[-2, 2]$. By proceeding inductively, we obtain for each $m \in \mathbb{N}^*$ a subsequence $(t_n^m)_n \subset \dots \subset (t_n^1)_n \subset (t_n)_n$ such that

$$N(t + t_n^m) \rightarrow Q(t) \text{ as } n \rightarrow \infty$$

uniformly on $[-m, m]$. Let $(t_n^n)_n := (t_n^n)_n$ be the diagonal sequence, then we have

$$N(t + t_n^n) \rightarrow Q(t) \text{ as } n \rightarrow \infty \tag{2.8}$$

uniformly on each compact subset of \mathbb{R} . Note that

$$m_0 \leq Q(t) \leq m_1.$$

Since $a(\cdot)$, $b(\cdot)$ and $c(\cdot)$ are almost automorphic functions, we can extract a subsequence $(t_n^n)_n \subset (t_n^n)_n$ such that $a(t + t_n^n) \rightarrow \tilde{a}(t)$, $\tilde{a}(t - t_n^n) \rightarrow a(t)$, $b(t + t_n^n) \rightarrow \tilde{b}(t)$, $\tilde{b}(t - t_n^n) \rightarrow b(t)$, $c(t + t_n^n) \rightarrow \tilde{c}(t)$, and $\tilde{c}(t - t_n^n) \rightarrow c(t)$ as $n \rightarrow \infty$. For each $t \geq s$ and for $n \in \mathbb{N}$ sufficiently large, we have

$$N(t + t_n^n) = N(s + t_n^n) + \int_s^t N(u + t_n^n) \left(a(u + t_n^n) - b(u + t_n^n)N(u + t_n^n - \tau_1) - c(u + t_n^n) \int_0^{\tau_2} k(\theta)N(u + t_n^n - \theta) d\theta \right) du. \tag{2.9}$$

By taking $n \rightarrow \infty$, we get for each $t \geq s$

$$Q(t) = Q(s) + \int_s^t Q(u) \left(\tilde{a}(u) - \tilde{b}(u)Q(u - \tau_1) - \tilde{c}(u) \int_0^{\tau_2} k(\theta)Q(u - \theta) d\theta \right) du.$$

By applying the above argument to the returning sequence $(-t_n^n)_n$, we obtain a subsequence $(t_n^n)_n \subset (t_n^n)_n$ and a function S such that

$$Q(t - t_n^n) \rightarrow S(t) \text{ as } n \rightarrow \infty \tag{2.10}$$

uniformly on each compact subset of \mathbb{R} . One can see that S is a solution to (1.1) on \mathbb{R} that satisfies

$$m_0 \leq S(t) \leq m_1.$$

Now, for the uniqueness, assume that (2.7) holds. Let \tilde{S} be another solution on \mathbb{R} that satisfies (2.6). Let $x(t) = \log S(t)$, $y(t) = \log \tilde{S}(t)$ and $L(t) = x(t) - y(t)$. For each fixed $t_0 \in \mathbb{R}$, we set $L_0(t) := L(t + t_0)$. One can see that there exists a continuous function $\theta(t)$ such that, for all $t \in \mathbb{R}$,

$$S(t) - \tilde{S}(t) = \theta(t)L(t) \quad \text{and} \quad m_0 \leq \theta(t) \leq m_1.$$

Then L_0 satisfies the following differential equation

$$L_0'(t) = -b_0(t)\theta_0(t - \tau_1)L_0(t - \tau_1) - c_0(t) \int_0^{\tau_2} k(s)\theta_0(t - s)L_0(t - s) ds.$$

Consider the following Lyapunov function

$$V(t) = V_1(t) + V_2(t) + V_3(t)$$

where

$$V_1(t) = \left[L_0(t) - \int_{t-\tau_1}^t b_0(s + \tau_1)\theta_0(s)L_0(s) ds - \int_0^{\tau_2} \int_{t-s}^t k(s)c_0(u + s)\theta_0(u)L_0(u) du ds \right]^2,$$

$$V_2(t) = \left(b_1 + c_1 \int_0^{\tau_2} k(s) ds \right) b_1 m_1^2 \int_{t-\tau_1}^t \int_s^t L_0^2(u) du ds,$$

and

$$V_3(t) = \left(b_1 + c_1 \int_0^{\tau_2} k(s) ds \right) c_1 m_1^2 \int_0^{\tau_2} k(s) \int_{t-s}^t \int_u^t L_0^2(\xi) d\xi du ds.$$

By derivating V using $2xy \leq x^2 + y^2$, we get the following inequality

$$V'(t) \leq -CL_0^2(t),$$

where

$$C = 2 \left(m_0 \left[b_0 + c_0 \int_0^{\tau_2} k(s) ds \right] - \left(b_1 + c_1 \int_0^{\tau_2} k(s) ds \right) \left(b_1 \tau_1 m_1^2 + c_1 m_1^2 \int_0^{\tau_2} sk(s) ds \right) \right).$$

Notice that $C > 0$ by (2.7). By integrating the above inequality and using the positivity of $V(t)$, we obtain

$$\int_0^t L_0^2(s) ds \leq \frac{V(0)}{C}. \tag{2.11}$$

Notice also that there exists a constant M independent of t_0 such that $V(0) \leq M$. Thus,

$$\int_{-\infty}^{\infty} L^2(s) ds \leq \frac{M}{C}.$$

Since the function $s \mapsto L^2(s)$ is uniformly continuous ($\frac{d}{ds}L^2(s)$ is bounded), we obtain by Barbalat's Lemma [4, Lemma 1.2.2] that $\lim_{t \rightarrow \pm\infty} L^2(t) = 0$. Let $\varepsilon > 0$, then there exists $T > 0$ such that, for all $t \in \mathbb{R}$ with $|t| > T$, we have $|L(t)| < \varepsilon$. We fix $t_0 \in \mathbb{R}$ such that $t_0 < -T$. Thus we have from (2.11)

$$\int_{t_0}^{\infty} L^2(s) ds \leq \frac{V(0)}{C}.$$

On the other hand, since $t_0 < -T$, one can see that there exists a constant $M_1 > 0$ independent of t_0 such that $V(0) \leq M_1 \varepsilon^2$. Therefore, we get

$$\int_{-\infty}^{\infty} L^2(s) ds \leq \frac{M_1}{C} \varepsilon^2.$$

Since $\varepsilon > 0$ is arbitrary, $L(t) = 0$ for all $t \in \mathbb{R}$, and thus $S(t) = \tilde{S}(t)$ for all $t \in \mathbb{R}$. \square

We are now in a position to present our main result.

Theorem 2.6. Assume that

$$\left(b_1 \tau_1 + c_1 \int_0^{\tau_2} sk(s) ds \right) m_1^2 < \frac{b_0 + c_0 \int_0^{\tau_2} k(s) ds}{b_1 + c_1 \int_0^{\tau_2} k(s) ds} m_0. \tag{2.12}$$

Then Equation (1.1) has a unique compact almost automorphic solution S on \mathbb{R} such that $m_0 \leq S(t) \leq m_1$ for $t \in \mathbb{R}$. Furthermore, S attracts all positive solutions on $(0, \infty)$.

Proof. From Theorem 2.5, Equation (1.1) has a unique solution S on \mathbb{R} such that

$$m_0 \leq S(t) \leq m_1 \quad \text{for } t \in \mathbb{R}. \tag{2.13}$$

We claim that S is compact almost automorphic. In fact, S is uniformly continuous as it has a bounded derivative. Let $(t_n)_n$ be a sequence of real numbers. Using the equicontinuity of the family of functions $S_n : t \mapsto S(t + t_n)$ and Arzelà–Ascoli’s theorem, there exist a function Q and a subsequence $(t'_n)_n \subset (t_n)_n$ such that

$$S(t + t'_n) \rightarrow Q(t) \quad \text{as } n \rightarrow \infty \tag{2.14}$$

uniformly on each compact subset of \mathbb{R} . On the other hand, we can extract a subsequence $(t''_n)_n \subset (t'_n)_n$ such that $a(t + t''_n) \rightarrow \tilde{a}(t)$, $\tilde{a}(t - t''_n) \rightarrow a(t)$, $b(t + t''_n) \rightarrow \tilde{b}(t)$, $\tilde{b}(t - t''_n) \rightarrow b(t)$, $c(t + t''_n) \rightarrow \tilde{c}(t)$, and $\tilde{c}(t - t''_n) \rightarrow c(t)$ as $n \rightarrow \infty$. Thus, Q satisfies the following differential equation:

$$Q'(t) = Q(t) \left(\tilde{a}(t) - \tilde{b}(t)Q(t) - \tilde{c}(t) \int_0^{\tau_2} k(s)Q(t-s) ds \right) \quad \text{for } t \in \mathbb{R}.$$

By applying the above argument to the returning sequence $(-t''_n)_n$, we obtain a subsequence $(t'''_n)_n \subset (t'_n)_n$ such that

$$Q(t - t'''_n) \rightarrow R(t) \quad \text{as } n \rightarrow \infty \tag{2.15}$$

uniformly on each compact subset of \mathbb{R} , where R is a solution to (1.1) on \mathbb{R} . In addition, it follows from (2.13), (2.14) and (2.15) that

$$m_0 \leq R(t) \leq m_1 \quad \text{for } t \in \mathbb{R}.$$

We deduce using Theorem 2.5 that $R(t) = S(t)$ for all $t \in \mathbb{R}$, and thus by (2.14) and (2.15) that S is compact almost automorphic. The attractiveness of the solution S follows from the proof of Theorem 2.5. \square

3. A counterexample

In many works in the literature such as [7], [8], and [13], the authors claimed that delay logistic equations similar to (1.1) have a unique positive almost periodic solution when the coefficients are almost periodic, without assuming any assumption such as (2.12). These results seem, however, to be incorrect. In fact, if we consider, for example, the following autonomous delay logistic equation

$$u'(t) = hu(t) \left(1 - \frac{1}{c} \int_0^{\infty} k(s)u(t-s) ds \right) \quad \text{for } t \in \mathbb{R} \tag{3.1}$$

where c is a positive constant and $k(t) = te^{-t}$ for $t \geq 0$. Then it is known (see [4, Chapter 2]) that Eq. (3.1) exhibits a Hopf bifurcation with respect to the parameter h . More specifically, when $0 < h < 2$, then the steady state $u(t) = c$ is locally asymptotically stable. But when $h > 2$ and $h - 2$ is small, the steady state loses its asymptotic stability and at the same time a nonconstant periodic solution arises. Therefore, in this case, Eq. (3.1) has two positive almost periodic solutions: the steady state $u_1(t) = c$ and the bifurcating nonconstant periodic solution u_2 , which contradicts the main results in [7], [8], and [13]. The flaw in all the proofs given in [7], [8] and [13] seems to be the use of the estimation

$$\int_0^r L_0(t) (L_0(t) - L_0(t-s)) dt \leq \sup_t |L_0(t)| \left| \int_0^r (L_0(t) - L_0(t-s)) dt \right|$$

which does not hold when the function $L_0(t) - L_0(t-s)$ is not nonnegative, where $L_0(t) = \log S(t + t_0) - \log \tilde{S}(t + t_0)$ as in the proof of Theorem 2.5.

This phenomenon happens even for simpler logistic equations. For instance, consider the following logistic equation

$$u'(t) = u(t) (1 - u(t-r)). \tag{3.2}$$

When $r = 1.2$, there is a unique positive periodic solution, which is the asymptotically stable steady state $u(t) = 1$ (Fig. 3.1). But when $r = 1.62$, there are two positive periodic solutions: the steady state $u_1(t) = 1$ and the bifurcating nonconstant periodic solution u_2 (Fig. 3.2).

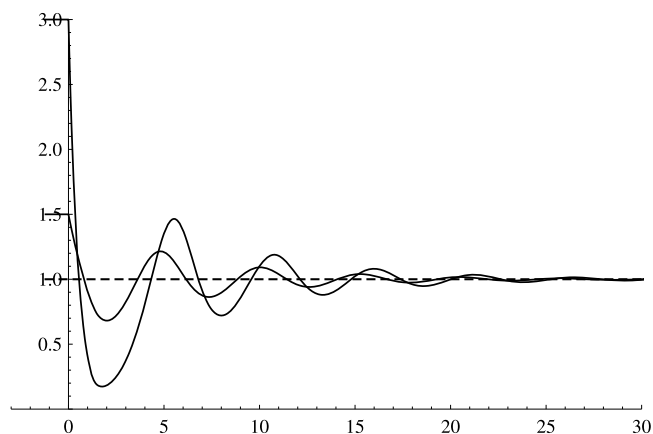


Fig. 3.1. $r = 1.2$, the steady state (in dashed line) is the only positive periodic solution.

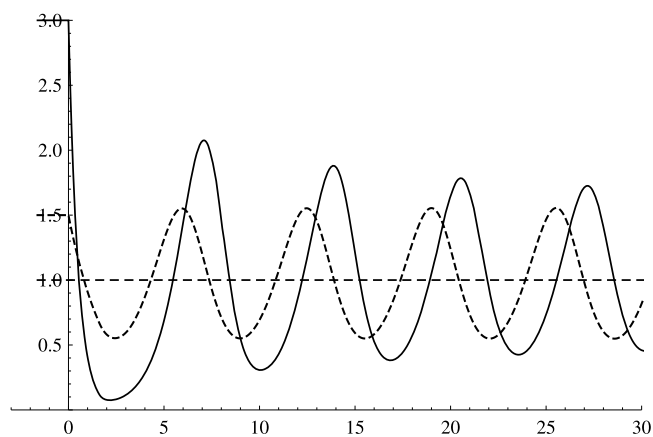


Fig. 3.2. $r = 1.62$, there are two positive periodic solutions (in dashed line).

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