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Dynamical systems

Infinite entropy is generic in Hölder and Sobolev spaces ☆

Propriétés génériques pour des systèmes dynamiques de faible régularité

Edson de Faria [a], Peter Hazard [a], Charles Tresser [b]

[a] *Instituto de Matemática e Estatística, USP, São Paulo, SP, Brazil*
[b] *Aperio, MATAM Scientific Industrial Ctr., 9 A. Sakharov St., Haïfa, 3508409, Israel*

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ABSTRACT

In 1980, Yano showed that on smooth compact manifolds, for endomorphisms in dimension one or above and homeomorphisms in dimensions greater than one, topological entropy is generically infinite. It had earlier been shown that, for Lipschitz endomorphisms on such spaces, topological entropy is always finite. In this article, we investigate what occurs between C^0-regularity and Lipschitz regularity, focussing on two cases: Hölder mappings and Sobolev mappings.

RÉSUMÉ

En 1980, Yano a montré que, sur une variété différentielle compacte, pour les endomorphismes en toutes dimensions et les homéomorphismes en dimension plus grande que un, l'entropie topologique est génériquement infinie. Il avait été auparavant montré que, pour les endomorphismes Lipschitz continus, l'entropie est toujours finie. Dans cette note, nous étudions ce qui se passe entre la régularité C^0 et la continuité de type Lipschitz, en nous concentrant sur deux cas, endomorphismes et homéomorphismes de classes de Hölder et de Sobolev.

1. Introduction

For a Lipschitz self-map f of a compact metric space with finite Hausdorff dimension, the topological entropy $h_{\rm top}(f)$ is always finite (see [6, Theorem 3.2.9]). By contrast, Yano [7] showed that, in the space of homeomorphisms of a smooth compact manifold of dimension $d \geq 2$, and in the space of endomorphisms of a smooth compact manifold of dimension $d \geq 1$ (both spaces endowed with the C^0-topology), infinite topological entropy is a generic property. A natural question for Euclidean spaces is:

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E-mail addresses: edson@ime.usp.br (E. de Faria), pete@ime.usp.br (P. Hazard), charles@aperio-system.com (C. Tresser).

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What happens for intermediate regularity between Lipschitz and C^0?

The term 'intermediate' can be understood in several different ways. Here we investigate what happens in the Hölder class C^α, $0 \le \alpha < 1$, of maps satisfying the Hölder condition of exponent α, and in the Sobolev class $W^{1,p} \cap C^0$, $1 \le p < \infty$, of continuous maps with weak derivatives lying in L^p. For fixed α or p, neither class is closed under composition (so the space of homeomorphisms of each class does not form a group). However, they are closed under pre- and post-composition by Lipschitz maps, and the union, over α and p respectively, of C^α and $W^{1,p} \cap C^0$ *is* closed under composition.

Before stating our main results from [2], we construct a one-parameter family of interval endomorphisms f_a, $a \in (0,1]$, with $h_{\text{top}}(f_a) = +\infty$ for all a, and where the regularity of f_a varies with the parameter. Such maps are toy-models of the example, constructed in [2, Appendix A], of a planar homeomorphism of infinite entropy with regularity close to bi-Lipschitz. It will be useful to first consider an auxiliary family $g_{a,b}$ of interval maps defined as follows. Fix $a \in (0,1]$ and a positive integer b. Given an interval J, let A_J denote the unique increasing affine bijection from J to $[0,1]$. Subdivide the interval $[0,1]$ into b closed intervals $J_{b,0}, J_{b,1}, \dots, J_{b,b-1}$ of equal length, ordered from left to right. Let $A_{b,k} = A_{J_{b,k}}$ for each $k = 0, 1, \dots, b-1$. Let ν denote the unique decreasing affine bijection of $[0,1]$ to itself, and let $q_a(x) = x^a$. For each $k = 0, 1, \dots, b-1$, define

$$g_{a,b}(x) = q_a \circ \nu^k \circ A_{b,k}(x), \qquad \forall x \in J_{b,k}. \tag{1}$$

Observe that $g_{a,b}$ is continuous on $[0,1]$. Also, $[0,1]$ contains a $g_{a,b}$-invariant subset on which $g_{a,b}$ acts as the unilateral b-shift. Thus, for all positive integers b, $h_{\text{top}}(g_{a,b}) = \log b$ (see, *e.g.*, [6, Section 3.2.c]).

We now define the family f_a as follows. For each positive integer n, define the interval $I_n = (2^{-n}, 2^{-n+1}]$ and define f_a by

$$f_a(x) = \begin{cases} A_{I_n}^{-1} \circ g_{a,2n+1} \circ A_{I_n}(x) & \forall x \in I_n, n = 1, 2 \dots \\ 0 & x = 0. \end{cases} \tag{2}$$

Observe that, since $g_{a,2n+1}$ fixes the endpoints of $[0,1]$ and is continuous, the map f_a is also continuous. Moreover, the closure of each interval I_n is totally invariant. Since the topological entropy of a map is the supremum of the topological entropy of its restrictions to all closed invariant subsets, and since topological entropy is invariant under topological conjugacy (see, *e.g.*, [6, Section 3.1.b]) it follows that

$$h_{\text{top}}(f_a) \ge \sup_n h_{\text{top}}(f_a|_{I_n}) = \sup_n h_{\text{top}}(g_{a,2n+1}) = \sup_n \log(2n+1) = +\infty. \tag{3}$$

Theorem 1.1. *For $f = f_1$, the following holds.*

(i) *f has modulus of continuity $\omega(t) = t\log(1/t)$.*
(ii) *f is in the Sobolev class $W^{1,p}$ for each $1 \le p < \infty$.*
(iii) $h_{\text{top}}(f) = +\infty$.

We have already seen in (3) that property (iii) holds. Properties (i)–(ii) are shown in [5]. We recall that maps with modulus of continuity $t\log(1/t)$ are in the Hölder class C^α for every $\alpha \in [0,1)$. Moreover, the map f_1 is a C^α-limit of piecewise-affine maps. Hence it lies in the C^α-boundary of the space of Lipschitz maps. Also, f_1 is not in the Zygmund class. (However, in [5], continuous examples with infinite entropy satisfying the little Zygmund condition are also constructed.)

When $a \ne 1$, the map f_a does not satisfy this property. More precisely, we have the following result.

Theorem 1.2. *For $f = f_a$, $a \in (0,1)$, the following holds.*

(i) *f is C^α if and only if $\alpha \le a$.*
(ii) *f is $W^{1,p}$ if and only if $p < (1-a)^{-1}$.*
(iii) $h_{\text{top}}(f) = +\infty$.

The proof of Theorem 1.2 could be made using the argument presented in [5] for Theorem 1.1. However, we give a more direct proof for this special case below. For that we need the following gluing principle, which is a simple exercise using Jensen's Inequality.

Proposition 1.1. *Let $a \in (0,1)$. Let f be continuous self-mapping of the compact interval I. Let $I_1, I_2, \dots$ denote a collection of closed intervals with pairwise disjoint interiors, covering I, and with the property that $f|_{I_k}$ is C^a, for all k. Denote the C^a-semi-norm of $f|_{I_k}$ by C_k. If $\sup_k C_k < \infty$ and $f|_{\partial I_k} = \text{id}$ for all k, then f is C^a-continuous with C^a-semi-norm bounded by $C = \text{diam}(I)^{1-a} + 2\sup_k C_k$.*

Proof of Theorem 1.2. We adopt the following notation. Given an interval J, denote by $[f]_{C^\alpha,J}$ and $[f]_{W^{1,p},J}$ the C^α- and $W^{1,p}$-semi-norms of $f|_J$, respectively. (For the more general case of compact Euclidean domains, the definitions are given at the start of Section 2 below.) Define the subintervals $I_{n,k} = A_{I_n}^{-1}(J_{2n+1,k})$ of I_n for $k = 0, 1, \dots, 2n$.

(i) Since $g_{a,b}$ cannot be C^α for any $\alpha > a$, it follows that f_a also cannot be C^α for any $\alpha > a$. To show that f is C^a we use the following:

Observation: For all non-negative integers n and k with $k \le n-1$, there exists a non-negative integer $\ell \le n-2$, and a subinterval $K_{n-1,\ell}$ of $I_{n-1,\ell}$ such that the graph of $f|_{I_{n,k}}$ is isometric to the graph of $f|_{K_{n-1,\ell}}$. Hence $[f]_{C^a, I_{n,k}} \le [f]_{C^a, I_{n-1,\ell}}$.

(The first part follows directly from the scale-invariance of q_a.) Therefore, $[f]_{C^a, I_{n,k}}$ is uniformly bounded. Applying Proposition 1.1 twice, we find that $[f]_{C^a,[0,1]}$ is bounded and thus f is C^a.

(ii) Observe that f is differentiable on each $I_{n,k}$. In fact, on $I_{n,k}$ we have $f(x) = A_{I_n}^{-1} \circ q_a \circ \nu^k \circ A_{I_{n,k}}(x)$. Differentiating f and applying the change of variables formula gives

$$[f]^p_{W^{1,p}, I_{n,k}} = (2n+1)^p \int_{[0,1]} |q_a'(y)|^p \frac{2^{-n}}{2n+1} \, \mathrm{d}y = \begin{cases} \dfrac{2^{-n} a^p (2n+1)^{p-1}}{(a-1)p+1} & \text{if } (a-1)p+1 > 0\,, \\ \infty & \text{otherwise}\,. \end{cases}$$

Hence $[f]_{W^{1,p},[0,1]} = \infty$ if $p \ge (1-a)^{-1}$ and otherwise

$$[f]^p_{W^{1,p},[0,1]} = \sum_{n=1}^{\infty} \sum_{k=0}^{2n} \int_{I_{n,k}} |f'|^p \, \mathrm{d}x = \frac{a^p}{(a-1)p-1} \sum_{j=1}^{\infty} (2n+1)^p 2^{-n}\,.$$

The right-hand side is a convergent series. Thus, f is $W^{1,p}$ for all $1 \le p < (1-a)^{-1}$, as required.

Property (iii) follows directly from inequality (3) above. Thus properties (i)–(iii) have been shown and the proof is complete. □

2. Main results

Let us now state the main results proved in [2]. By a *compact Euclidean domain* in $\mathbb{R}^d$, we will mean the closure of a bounded open subset of $\mathbb{R}^d$. Given a compact Euclidean domain Ω in $\mathbb{R}^d$, let d_Ω denote the distance induced by the usual Euclidean metric $d_{\mathbb{R}^d}$ on $\mathbb{R}^d$. Given a mapping $f : \Omega \to \mathbb{R}^k$, for some positive integer k, let $[f]_{C^\alpha,\Omega}$, $0 \le \alpha < 1$, denote the Hölder C^α-semi-norm, *i.e.*

$$[f]_{C^\alpha,\Omega} = \sup_{x,y\in\Omega, x\neq y} \frac{d_{\mathbb{R}^k}(f(x), f(y))}{d_\Omega(x,y)^\alpha} \tag{4}$$

and let $[f]_{W^{1,p},\Omega}$, $1 \le p \le \infty$, denote the Sobolev $W^{1,p}$-semi-norm, *i.e.*

$$[f]_{W^{1,p},\Omega} = \left(\int_\Omega |Df|^p \mathrm{d}\mu \right)^{1/p} \tag{5}$$

where μ denotes the Lebesgue measure on $\mathbb{R}^d$ and, given a matrix $A = (a_{ij})$, we use the matrix norm $|A| = \sum_{i,j} |a_{ij}|$. We then define the norms

$$\|f\|_{C^\alpha(\Omega)} = \|f\|_{C^0(\Omega)} + [f]_{C^\alpha,\Omega}\,, \qquad \|f\|_{W^{1,p}(\Omega)} = \|f\|_{C^0(\Omega)} + [f]_{W^{1,p},\Omega}\,. \tag{6}$$

In the Lipschitz case, we define the norm $\|f\|_{\mathrm{Lip}(\Omega)}$ in a similar fashion. Let $\mathcal{H}^\alpha(\Omega)$ denote the space of bi-α-Hölder homeomorphisms of Ω for $\alpha \in (0,1)$ and $\mathcal{H}^1(\Omega)$ denote the space of bi-Lipschitz homeomorphisms of Ω. For $\alpha < 1$, define the distance

$$d_\alpha(f,g) = \max\left\{ \|f-g\|_{C^\alpha(\Omega)}, \|f^{-1} - g^{-1}\|_{C^\alpha(\Omega)} \right\}\,. \tag{7}$$

For $\alpha = 1$, define $d_1(f,g)$ in similar fashion, replacing the C^α-norm by the Lipschitz norm. Endow $\mathcal{H}^\alpha(\Omega)$ with the topology induced by the metric d_α. Finally, given $\alpha < \beta \le 1$, denote by $\mathcal{H}^\beta_\alpha(\Omega)$ the closure of $\mathcal{H}^\beta(\Omega)$ as a subspace of $\mathcal{H}^\alpha(\Omega)$.

In the Sobolev case, for $1 \le p, p^* < \infty$, let $\mathcal{S}^{p,p^*}(\Omega)$ denote the space of homeomorphisms f of Ω such that $f \in W^{1,p}(\Omega)$ and $f^{-1} \in W^{1,p^*}(\Omega)$. We endow $\mathcal{S}^{p,p^*}(\Omega)$ with the topology induced by the distance

$$d_{p,p^*}(f,g) = \max\left\{ \|f-g\|_{W^{1,p}(\Omega)}, \|f^{-1} - g^{-1}\|_{W^{1,p^*}(\Omega)} \right\}\,. \tag{8}$$

Theorem 2.1. *Let Ω be a compact d-dimensional Euclidean domain with piecewise-smooth boundary.*

- *For $d \geq 2$ and $0 \leq \alpha < 1$; $\mathcal{H}^1_\alpha(\Omega)$ contains a residual subset of homeomorphisms with infinite topological entropy.*
- *For $d = 2$ and $1 \leq p, p^* < \infty$; or $d > 2$ and $d-1 < p, p^* < \infty$; $\mathcal{S}^{p,p^*}(\Omega)$ contains a residual subset of homeomorphisms with infinite topological entropy.*

Strategy of Proof of Theorem 2.1. The basic geometric idea behind the proof is similar to, but not entirely the same as, the one in Yano's Theorem [7]. (Our strategy is to the C^1-Closing Lemma, as Yano's strategy is to the C^0-Closing Lemma.)

(a) Take a homeomorphism $f: \Omega \to \Omega$ in the appropriate Hölder or Sobolev space. Given $k \in \mathbb{N}$ and $\epsilon > 0$, we consider a finite segment $x, f(x), \ldots, f^k(x)$, of a *recurrent* orbit such that x and $f^k(x)$ are less than a distance ϵ apart. We take pairwise disjoint ϵ-small neighbourhoods U_i, $0 \leq i \leq k-1$, with $f^i(x) \in U_i$, $0 \leq i \leq k-1$, and $f^k(x) \in U_0$.
(b) In each U_i we take a (solid) cylinder C_i. We perturb f by post-composing with a bi-Lipschitz homeomorphism having support in U_i, so that C_i *maps across* C_{i+1} (where addition is taken modulo k) like an N-branched horseshoe.
(c) Since the supports of these perturbations have size less that ϵ, and since the Lipschitz norm dominates either the Hölder or Sobolev norms, we get a new homeomorphism $g: \Omega \to \Omega$, which is ϵ-close to f in either the Hölder or Sobolev metrics, and for which there exists a cylinder C_0 with the property that $g^k|_{C_0}$ is a horseshoe with N^k branches. This implies that g has entropy at least $\log N$.

Remark 1. Another strategy of proof in the homeomorphism case is given in [4]. This uses the Closing Lemma and the Annulus Theorem, both in the C^0-category. (See, *e.g.*, [3] for more on the Annulus Theorem.) Unfortunately, we do not know whether the Annulus Theorem is valid in either the Hölder or Sobolev categories.

Remark 2. In the specific case of dimension $d = 2$, an alternative proof is given in the main paper [2], for Sobolev mappings, which relies on a surgery technique that uses p-harmonic maps and a generalization of the Radó–Kneser–Choquet theorem [1].

Remark 3. In the α-Hölder case, we must consider homeomorphisms in the closure of bi-Lipschitz mappings, rather than the full space of bi-α-Hölder homeomorphisms, as our perturbations are via composition with Lipschitz homeomorphisms. The size of such perturbations, in the α-Hölder topology, supported on an r-ball is of the order $Lr^{1-\alpha}$, where L denotes the Lipschitz constant of the perturbation.

As a corollary of the proof of Theorem 2.1, we get the following result.

Corollary 2.2. *Let Ω be a compact d-dimensional Euclidean domain with piecewise-smooth boundary.*

- *For $d \geq 2$ and $0 \leq \alpha < 1$; topological entropy is continuous on a residual subset of $\mathcal{H}^1_\alpha(\Omega)$.*
- *For $d = 2$ and $1 \leq p, p^* < \infty$; or $d > 2$ and $d-1 < p, p^* < \infty$; topological entropy is continuous on a residual subset of $\mathcal{S}^{p,p^*}(\Omega)$.*

As topological entropy is invariant under topological conjugacy, we also find the following.

Corollary 2.3. *Let Ω be a compact d-dimensional Euclidean domain with piecewise-smooth boundary.*

- *For $d \geq 2$ and $0 \leq \alpha < 1$; a generic homeomorphism in $\mathcal{H}^1_\alpha(\Omega)$ is not conjugate to any bi-Lipschitz homeomorphism.*
- *For $d = 2$ and $1 \leq p, p^* < \infty$; or $d > 2$ and $d-1 < p, p^* < \infty$; a generic homeomorphism in $\mathcal{S}^{p,p^*}(\Omega)$ is not conjugate to any bi-Lipschitz homeomorphism.*

In [2], we prove these results in more generality for homeomorphisms on smooth compact manifolds. However, the construction of the topology in the Sobolev and Hölder classes is more involved (analogous to the construction of the weak C^k-Whitney topology).

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