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Number theory

On the sum of reciprocals of least common multiples ☆

Sur les sommes des inverses de plus petits communs multiples

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ABSTRACT

Let $\{a_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of positive integers ($a_i < a_j$ if $i < j$). In 1978, Borwein showed that for any positive integer n, we have $\sum_{i=1}^{n} \frac{1}{\mathrm{lcm}(a_i, a_{i+1})} \leq 1 - \frac{1}{2^n}$, with equality occurring if and only if $a_i = 2^{i-1}$ for $1 \leq i \leq n+1$. Let $3 \leq r \leq 7$ be an integer. In this paper, we investigate the sum $\sum_{i=1}^{n} \frac{1}{\mathrm{lcm}(a_i, \ldots, a_{i+r-1})}$ and show that $\sum_{i=1}^{n} \frac{1}{\mathrm{lcm}(a_i, \ldots, a_{i+r-1})} \leq U_r(n)$ for any positive integer n, where $U_r(n)$ is a constant depending on r and n. Further, for any integer $n \geq 2$, we also give a characterization of the sequence $\{a_i\}_{i=1}^{\infty}$ such that the equality $\sum_{i=1}^{n} \frac{1}{\mathrm{lcm}(a_i, \ldots, a_{i+r-1})} = U_r(n)$ holds.

RÉSUMÉ

Soit $\{a_i\}_{i=1}^{\infty}$ une suite strictement croissante d'entiers positifs ($a_i < a_j$ pour $i < j$). En 1978, Borwein a montré que, pour tout entier positif n, on a $\sum_{i=1}^{n} \frac{1}{\mathrm{ppcm}(a_i, a_{i+1})} \leq 1 - \frac{1}{2^n}$, avec égalité si et seulement si $a_i = 2^{i-1}$ pour $1 \leq i \leq n+1$. Soit $3 \leq r \leq 7$ un entier. Dans cette Note, nous étudions les sommes $\sum_{i=1}^{n} \frac{1}{\mathrm{ppcm}(a_i, \ldots, a_{i+r-1})}$ et nous montrons qu'elles sont majorées, pour tout entier positif r, par une constante $U_r(n)$ dépendant de r et n. De plus, pour tout entier $n \geq 2$, nous caractérisons aussi les suites $\{a_i\}_{i=1}^{\infty}$ pour lesquelles l'égalité $\sum_{i=1}^{n} \frac{1}{\mathrm{ppcm}(a_i, \ldots, a_{i+r-1})} = U_r(n)$ est vérifiée.

1. Introduction

The least common multiple of consecutive terms in a given integer sequence was first investigated by Chebyshev [3], who introduced the function $\Psi(n) = \sum_{p^k \leq n} \log p = \log \mathrm{lcm}(1, \ldots, n)$ and made an important progress for the final proof of the prime number theorem. From Chebyshev's work, one can easily deduce that the prime number theorem is equivalent

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to the asymptotic estimate $\log \operatorname{lcm}(1, \ldots, n) \sim n$. Since then, the least common multiple of sequences of integers received a lot of attention from many authors. Heilbronn [9], Behrend [1], and Van der Corput [4] investigated inequalities involving a least common multiple of integer sequence. Hanson [8] and Nair [14] got the upper bound and the lower bound of $\operatorname{lcm}_{1\le i\le n}\{i\}$, respectively. In [7], Farhi studied the series of the reciprocals of least common multiples of sequences of positive integers. Recently, the topic has undergone important developments. For the detailed background information about the latest progress on the least common multiple of integer sequences, we refer the readers to [5,6,10–13].

Let $\mathcal{A} = \{a_i\}_{i=1}^{\infty}$ be a sequence of positive integers. We call it *a strictly increasing sequence* if $a_i < a_j$ holds for any two positive integers i, j satisfying $i < j$. In the 1970s, Erdös once posed a conjecture involving the upper bound on the sum of reciprocals of least common multiples of any two consecutive terms in a strictly increasing sequence of positive integers (see [2]). In 1978, D. Borwein [2] proved the conjecture posed by P. Erdös and obtained the following interesting result.

Theorem 1.1. ([2]) *Let $\{a_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of positive integers. Then for any positive integer n, we have*

$$\sum_{i=1}^{n} \frac{1}{\operatorname{lcm}(a_i, a_{i+1})} \le 1 - \frac{1}{2^n},$$

where the equality occurs if and only if $a_i = 2^{i-1}$ for $1 \le i \le n+1$.

In this paper, we mainly investigate the sum of reciprocals of least common multiples of consecutive terms in a strictly increasing sequence of positive integers. Let $\mathcal{A} = \{a_i\}_{i=1}^{\infty}$ be a given strictly increasing sequence of positive integers and let $r \ge 3$ be an integer. For any positive integer n, let

$$S_{\mathcal{A},r}(n) = \sum_{i=1}^{n} \frac{1}{\operatorname{lcm}(a_i, \ldots, a_{i+r-1})}. \tag{1.1}$$

One then naturally asks the following interesting problem.

Problem 1.2. *Is there a tight upper bound $T_r(n)$ for the sum $S_{\mathcal{A},r}(n)$? If so, determine the exact value of $T_r(n)$ and characterize the sequence $\mathcal{A}$ such that $S_{\mathcal{A},r}(n) = T_r(n)$ holds.*

For $3 \le r \le 7$, we let

$$U_r(n) = \begin{cases} \frac{1}{r-1}(1 - \frac{1}{2^n}), & \text{if } 3 \le r \le 4, \\ \frac{1}{4}(1 - \frac{1}{3\times 2^{n-2}}), & \text{if } r = 5, \\ \frac{1}{6}(1 - \frac{1}{2^n}), & \text{if } r = 6, \\ \frac{1}{8}(1 - \frac{1}{3\times 2^{n-2}}), & \text{if } r = 7, \end{cases} \tag{1.2}$$

and let

$$C_r(n) = \begin{cases} 2^{n+1}, & \text{if } r = 3, \\ 3 \times 2^{n+\lfloor \frac{r-4}{2} \rfloor}, & \text{if } 4 \le r \le 7. \end{cases} \tag{1.3}$$

Then for $n \ge 2$, we have the following equality

$$U_r(n) - U_r(n-1) = \frac{1}{C_r(n)}. \tag{1.4}$$

We can now state the main result of this paper.

Theorem 1.3. *Let r be an integer with $3 \le r \le 7$ and let $\mathcal{A} = \{a_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of positive integers. Then for any positive integer n,*

$$S_{\mathcal{A},r}(n) \le U_r(n). \tag{1.5}$$

Further, for any integer $n \ge 2$, the equality $S_{\mathcal{A},r}(n) = U_r(n)$ holds if and only if

$$a_i = \begin{cases} i, & \text{if } 3 \le r \le 5 \text{ and } 1 \le i \le r-1, \\ i, & \text{if } 6 \le r \le 7 \text{ and } 1 \le i \le 4, \\ 6, & \text{if } 6 \le r \le 7 \text{ and } i = 5, \\ 8, & \text{if } r = 7 \text{ and } i = 6, \\ C_r(i-r+1), & \text{if } 3 \le r \le 7 \text{ and } r \le i \le n+r-1. \end{cases} \tag{1.6}$$

Thus Theorem 1.3 answers Problem 1.2 for $3 \le r \le 7$.

2. Proof of Theorem 1.3

In this section, we show Theorem 1.3. We begin with the following lemma.

Lemma 2.1. *Let $r \geq 2$ be an integer and let $\mathcal{A} = \{a_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of positive integers. Then for any two positive integers n and m with $m \leq n$, we have*

$$\sum_{i=m}^{n} \frac{1}{\mathrm{lcm}(a_i, ..., a_{i+r-1})} \leq \frac{1}{a_{m+r-2}} - \frac{1}{a_{n+r-1}}. \tag{2.1}$$

In particular, we have

$$S_{\mathcal{A},r}(n) = \sum_{i=1}^{n} \frac{1}{\mathrm{lcm}(a_i, ..., a_{i+r-1})} \leq \frac{1}{a_{r-1}} - \frac{1}{a_{n+r-1}}. \tag{2.2}$$

Proof. Since $a_{i+r-2} < a_{i+r-1}$ for any positive integer i, we have

$$\frac{\mathrm{lcm}(a_{i+r-2}, a_{i+r-1})}{a_{i+r-2}} - \frac{\mathrm{lcm}(a_{i+r-2}, a_{i+r-1})}{a_{i+r-1}} \geq 1.$$

Hence for any positive integer i, multiplying $1/\mathrm{lcm}(a_{i+r-2}, a_{i+r-1})$ by both sides of the above inequality, we obtain that

$$\frac{1}{\mathrm{lcm}(a_{i+r-2}, a_{i+r-1})} \leq \frac{1}{a_{i+r-2}} - \frac{1}{a_{i+r-1}}. \tag{2.3}$$

Obviously, for any $1 \leq i \leq n$, we have

$$\frac{1}{\mathrm{lcm}(a_i, ..., a_{i+r-1})} \leq \frac{1}{\mathrm{lcm}(a_{i+r-2}, a_{i+r-1})}.$$

It then follows from (2.3) that

$$\sum_{i=m}^{n} \frac{1}{\mathrm{lcm}(a_i, ..., a_{i+r-1})} \leq \sum_{i=m}^{n} \left(\frac{1}{a_{i+r-2}} - \frac{1}{a_{i+r-1}}\right) = \frac{1}{a_{m+r-2}} - \frac{1}{a_{n+r-1}}$$

as desired. This completes the proof of Lemma 2.1. □

Lemma 2.2. *Let r be an integer with $3 \leq r \leq 7$ and let $\mathcal{A} = \{a_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of positive integers. Then we have*

$$S_{\mathcal{A},r}(1) \leq U_r(1). \tag{2.4}$$

Proof. It is easy to see that $S_{\mathcal{A},r}(1)$ can reach the maximal value only when $\mathrm{lcm}(a_1, ..., a_r)$ takes the minimal value. Since $\mathcal{A} = \{a_i\}_{i=1}^{\infty}$ is a strictly increasing sequence of positive integers, $\mathrm{lcm}(a_1, ..., a_r)$ has at least r distinct divisors. Obviously, $a_1, ..., a_r$ are r distinct divisors of $\mathrm{lcm}(a_1, ..., a_r)$. Then by classified discussion on the number of distinct prime divisors of the integer $\mathrm{lcm}(a_1, ..., a_r)$, one can easily derive that the smallest positive integers having at least r distinct divisors for $r = 3, 4, 5, 6, 7$ are respectively $2^2, 2 \times 3, 2^2 \times 3, 2^2 \times 3$ and $2^3 \times 3$. So we get that $S_{\mathcal{A},r}(1) \leq U_r(1)$ for $3 \leq r \leq 7$ as desired. □

Lemma 2.3. *Let $3 \leq r \leq 7$ be an integer and let $\mathcal{A} = \{a_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of positive integers. Let $n \geq 2$ be a positive integer. If $a_{n+r-1} < C_r(n)$, then we have*

$$S_{\mathcal{A},r}(n) < U_r(n). \tag{2.5}$$

Proof. If $3 \leq r \leq 5$, then we obtain by Lemma 2.1 that

$$S_{\mathcal{A},r}(n) \leq \frac{1}{a_{r-1}} - \frac{1}{a_{n+r-1}} < \frac{1}{r-1} - \frac{1}{C_r(n)} = U_r(n)$$

as required.

If $r = 6$ and $a_5 \geq 6$, then, by Lemma 2.1, we get:

$$S_{\mathcal{A},6}(n) \leq \frac{1}{a_5} - \frac{1}{a_{n+5}} < \frac{1}{6} - \frac{1}{C_6(n)} = U_6(n).$$

If $r=6, a_5=5$ and $a_6 \geq 7$, then we have $a_1=1, a_2=2, a_3=3, a_4=4$. Hence, by Lemma 2.1, we obtain:

$$\begin{aligned} S_{\mathcal{A},6}(n) &= \frac{1}{\operatorname{lcm}(1,2,3,4,5,a_6)} + \sum_{i=2}^{n} \frac{1}{\operatorname{lcm}(a_i,\ldots,a_{i+5})} \\ &< \frac{1}{6} - \frac{1}{7} + \frac{1}{a_6} - \frac{1}{a_{n+5}} < \frac{1}{6} - \frac{1}{C_6(n)} = U_6(n). \end{aligned}$$

If $r=6, a_5=5, a_6=6$ and $a_7=7$, then, by Lemma 2.1, we have

$$S_{\mathcal{A},6}(n) = \begin{cases} \frac{1}{60} + \frac{1}{420} < U_6(2), & \text{if } n=2, \\ \frac{1}{60} + \frac{1}{420} + \sum_{i=3}^{n} \frac{1}{\operatorname{lcm}(a_i,\ldots,a_{i+5})} < \frac{1}{6} - \frac{1}{7} + \frac{1}{7} - \frac{1}{a_{n+5}} < U_6(n), & \text{if } n \geq 3. \end{cases}$$

If $r=6, a_5=5, a_6=6$ and $a_7 \geq 8$, then by Lemma 2.1 we derive that

$$S_{\mathcal{A},6}(n) \leq \begin{cases} \frac{1}{60} + \frac{1}{60} < U_6(2), & \text{if } n=2, \\ \frac{1}{60} + \frac{1}{60} + \frac{1}{a_7} - \frac{1}{a_{n+5}} < \frac{1}{6} - \frac{1}{8} + \frac{1}{a_7} - \frac{1}{a_{n+5}} < \frac{1}{6} - \frac{1}{C_6(n)} = U_6(n), & \text{if } n \geq 3. \end{cases}$$

If $r=7$ and $a_6 \geq 8$, then we obtain by Lemma 2.1 that

$$S_{\mathcal{A},7}(n) \leq \frac{1}{a_6} - \frac{1}{a_{n+6}} < \frac{1}{8} - \frac{1}{C_7(n)} = U_7(n).$$

If $r=7, a_6=6$ and $a_7=7$, then $a_i=i$ for $1 \leq i \leq 5$. It follows from Lemma 2.1 that

$$S_{\mathcal{A},7}(n) \leq \begin{cases} \frac{1}{420} + \frac{1}{420} < U_7(2), & \text{if } n=2, \\ \frac{1}{420} + \frac{1}{420} + \frac{1}{420} < U_7(3), & \text{if } n=3, \\ \frac{3}{420} + \sum_{i=4}^{n} \frac{1}{\operatorname{lcm}(a_i,\ldots,a_{i+6})} < \frac{1}{8} - \frac{1}{9} + \frac{1}{a_9} - \frac{1}{a_{n+6}} < U_7(n), & \text{if } n \geq 4. \end{cases}$$

If $r=7, a_6=6, a_7=8$ and $a_8=9$, we can deduce by Lemma 2.1 that

$$S_{\mathcal{A},7}(n) = \begin{cases} \frac{1}{120} + \frac{1}{360} < U_7(2), & \text{if } n=2, \\ \frac{1}{90} + \sum_{i=3}^{n} \frac{1}{\operatorname{lcm}(a_i,\ldots,a_{i+6})} < \frac{1}{8} - \frac{1}{9} + \frac{1}{a_8} - \frac{1}{a_{n+6}} < U_7(n), & \text{if } n \geq 3. \end{cases}$$

If $r=7, a_6=6, a_7=8$ and $a_8 \geq 10$, we can derive from Lemma 2.1 that

$$S_{\mathcal{A},7}(n) \leq \begin{cases} \frac{1}{120} + \frac{1}{120} < U_7(2), & \text{if } n=2, \\ \frac{1}{60} + \sum_{i=3}^{n} \frac{1}{\operatorname{lcm}(a_i,\ldots,a_{i+6})} < \frac{1}{8} - \frac{1}{10} + \frac{1}{a_8} - \frac{1}{a_{n+6}} < U_7(n)), & \text{if } n \geq 3. \end{cases}$$

If $r=7, a_6=6$ and $a_7=9$, then one can derive from Lemma 2.1 that

$$S_{\mathcal{A},7}(n) = \frac{1}{\operatorname{lcm}(1,2,3,4,5,6,9)} + \sum_{i=2}^{n} \frac{1}{\operatorname{lcm}(a_i,\ldots,a_{i+6})} < \frac{1}{8} - \frac{1}{9} + \frac{1}{a_7} - \frac{1}{a_{n+6}} < U_7(n).$$

If $r=7, a_6=6$ and $a_7 \geq 10$, then we obtain by Lemma 2.1 that

$$S_{\mathcal{A},7}(n) = \frac{1}{\operatorname{lcm}(1,2,3,4,5,6,a_7)} + \sum_{i=2}^{n} \frac{1}{\operatorname{lcm}(a_i,\ldots,a_{i+6})} < \frac{1}{8} - \frac{1}{10} + \frac{1}{a_7} - \frac{1}{a_{n+6}} < U_7(n).$$

If $r=7, a_6=7$ and $a_7=8$, then $a_8 \geq 9$. Hence, we obtain by Lemma 2.1 that

$$S_{\mathcal{A},7}(n) \leq \begin{cases} \frac{1}{168} + \frac{1}{168} < U_7(2), & \text{if } n=2, \\ \frac{1}{84} + \sum_{i=3}^{n} \frac{1}{\operatorname{lcm}(a_i,\ldots,a_{i+6})} < \frac{1}{8} - \frac{1}{9} + \frac{1}{a_8} - \frac{1}{a_{n+6}} < U_7(n)), & \text{if } n \geq 3. \end{cases}$$

If $r=7, a_6=7$ and $a_7 \geq 9$, then for any positive integer $n \geq 2$, we obtain by Lemma 2.1 that

$$\begin{aligned} S_{\mathcal{A},7}(n) &= \frac{1}{\operatorname{lcm}(a_1,a_2,a_3,a_4,a_5,7,a_7)} + \sum_{i=2}^{n} \frac{1}{\operatorname{lcm}(a_i,\ldots,a_{i+6})} \\ &\leq \frac{1}{84} + \frac{1}{a_7} - \frac{1}{a_{n+6}} < \frac{1}{8} - \frac{1}{9} + \frac{1}{a_7} - \frac{1}{a_{n+6}} < U_7(n). \end{aligned}$$

This ends the proof of Lemma 2.3. □

Now we are in a position to give the proof of Theorem 1.3.

Proof of Theorem 1.3. First, we show the inequality (1.5). For $n=1$, by Lemma 2.2, we have $S_{\mathcal{A},r}(1) \le U_r(1)$.

Now we let $n \ge 2$ be an integer. By Lemma 2.3, we know that $S_{\mathcal{A},r}(n) \le U_r(n)$ also holds if $a_{n+r-1} < C_r(n)$. Thus, it remains to deal with the case that $a_{n+r-1} \ge C_r(n)$. We consider the following two cases.

Case 1. $a_{i+r-1} \ge C_r(i)$ for $2 \le i \le n$. Then by Lemma 2.2, we have

$$S_{\mathcal{A},r}(n) \le S_{\mathcal{A},r}(1) + \frac{1}{C_r(2)} + \cdots + \frac{1}{C_r(n)} \le U_r(1) + \frac{1}{C_r(2)} + \cdots + \frac{1}{C_r(n)} = U_r(n).$$

Case 2. $a_{j+r-1} < C_r(j)$ for some integer $2 \le j < n$ and $a_{i+r-1} \ge C_r(i)$ for $j+1 \le i \le n$. Then, we obtain by Lemma 2.3 that

$$S_{\mathcal{A},r}(n) \le S_{\mathcal{A},r}(j) + \frac{1}{C_r(j+1)} + \cdots + \frac{1}{C_r(n)} < U_r(j) + \frac{1}{C_r(j+1)} + \cdots + \frac{1}{C_r(n)} = U_r(n).$$

This completes the proof of inequality (1.5).

Further, it is immediate that $S_{\mathcal{A},r}(n) = U_r(n)$ holds for $n \ge 2$ if the first $n+r-1$ terms of the sequence $\mathcal{A}$ are of the form (1.6) for $3 \le r \le 7$.

Suppose next that $S_{\mathcal{A},r}(n) = U_r(n)$ for $n \ge 2$. Since the inequality $S_{\mathcal{A},r}(m) \le U_r(m)$ holds for each integer m with $1 \le m \le n$, we have by (1.4) that

$$\frac{1}{\operatorname{lcm}(a_n, \ldots, a_{n+r-1})} = S_{\mathcal{A},r}(n) - S_{\mathcal{A},r}(n-1) \ge U_r(n) - U_r(n-1) = \frac{1}{C_r(n)}.$$

It is easy to see that $a_{n+r-1} > C_r(n)$ is impossible since it will lead to the fact that $\frac{1}{\operatorname{lcm}(a_n,\ldots,a_{n+r-1})} < \frac{1}{C_r(n)}$. Hence, we have $a_{n+r-1} \le C_r(n)$. Lemma 2.3 tells us that $a_{n+r-1} < C_r(n)$ will lead to the fact that $S_{\mathcal{A},r}(n) < U_r(n)$. So $a_{n+r-1} < C_r(n)$ is also impossible. Consequently, we have

$$a_{n+r-1} = C_r(n).$$

By (1.5) with $n-1$ in place of n, we obtain by (1.4) that

$$U_r(n-1) \ge S_{\mathcal{A},r}(n-1) = S_{\mathcal{A},r}(n) - \frac{1}{\operatorname{lcm}(a_n, \ldots, a_{n+r-1})} \ge U_r(n) - \frac{1}{C_r(n)} = U_r(n-1).$$

So we have

$$S_{\mathcal{A},r}(n-1) = U_r(n-1).$$

By repeated discussion as above, one can easily derive that

$$S_{\mathcal{A},r}(i) = U_r(i) \text{ for each } 1 \le i \le n \text{ and } a_{i+r-1} = C_r(i) \text{ for } 2 \le i \le n.$$

Now we have

$$S_{\mathcal{A},r}(1) = U_r(1) = \begin{cases} \frac{1}{4}, & \text{if } r = 3, \\ \frac{1}{6}, & \text{if } r = 4, \\ \frac{1}{12}, & \text{if } 5 \le r \le 6, \\ \frac{1}{24}, & \text{if } r = 7. \end{cases}$$

So by the proof of Lemma 2.2, one can easily derive that $S_{\mathcal{A},r}(1)$ arrives at the maximal value if and only if $a_r = C_r(1)$ and $a_j = j$ for $1 \le j \le r-1$ when $3 \le r \le 4$, and $a_j = j$ for $1 \le j \le 4$, $a_5 = 6$ and $a_6 = 12 = C_6(1)$ when $r = 6$. It is easy to see from $S_{\mathcal{A},5}(1) = \frac{1}{12}$ that $\{a_1, a_2, a_3, a_4, a_5\}$ must be a 5-element subset of $\{1, 2, 3, 4, 6, 12\}$. But $a_6 = C_5(2) = 12$ when $r = 5$. So we get $\{a_1, a_2, a_3, a_4, a_5\} = \{1, 2, 3, 4, 6\}$ when $r = 5$. From $S_{\mathcal{A},7}(1) = \frac{1}{24}$, one can deduce that $\{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$ must be a 7-element subset of $\{1, 2, 3, 4, 6, 8, 12, 24\}$. But $a_8 = C_7(2) = 24$ when $r = 7$. Hence we obtain that $a_j = j$ for $1 \le j \le 4$, $a_5 = 6$, $a_6 = 8$ and $a_7 = 12 = C_7(1)$ when $r = 7$. This concludes the proof of Theorem 1.3. □

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