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Mathematical problems in mechanics

Nonlinear estimates for hypersurfaces in terms of their fundamental forms

Estimations non linéaires pour des hypersurfaces en fonction de leurs formes fondamentales

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ABSTRACT

A sufficiently regular hypersurface immersed in the $(n+1)$-dimensional Euclidean space is determined up to a proper isometry of $\mathbb{R}^{n+1}$ by its first and second fundamental forms. As a consequence, a sufficiently regular hypersurface with boundary, whose position and positively-oriented unit normal vectors are given on a non-empty portion of its boundary, is uniquely determined by its first and second fundamental forms. We establish here stronger versions of these uniqueness results by means of inequalities showing that an appropriate distance between two immersions from a domain ω of $\mathbb{R}^n$ into $\mathbb{R}^{n+1}$ is bounded by the L^p-norm of the difference between matrix fields defined in terms of the first and second fundamental forms associated with each immersion.

RÉSUMÉ

Une hypersuface suffisamment régulière immergée dans l'espace euclidien $(n+1)$-dimensionnel est determinée à une isométrie propre de $\mathbb{R}^{n+1}$ près par ses deux premières formes fondamentales. Par conséquent, une hypersuface suffisamment régulière avec frontière, dont la position et les vecteurs unitaires normaux et orientés positivement sont donnés sur une partie non vide de sa frontière, est déterminée uniquement par ses deux premières formes fondamentales. Nous établissons ici des versions plus fortes de ce résultat, en établissant des inégalités montrant qu'une distance appropriée entre deux immersions d'un domaine ω de $\mathbb{R}^n$ dans $\mathbb{R}^{n+1}$ est majorée par la norme L^p de la difference entre des champs de matrices définies en fonction des deux premières formes fondamentales associées à chaque immersion.

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1. Introduction

Throughout this article, n denotes an integer > 1, p denotes a real number > 1, Greek indices vary in the set $\{1, 2, \ldots, n\}$, Latin indices vary in the set $\{1, 2, \ldots, n+1\}$ (save when they are used for indexing sequences), and the summation convention for repeated indices is used in conjunction with these rules.

$\mathbb{E} := \mathbb{R}^{n+1}$ and $\mathbb{F} := \mathbb{R}^{(n+1)\times n}$ respectively denote the $(n+1)$-dimensional Euclidean space and the space of all $(n+1)\times n$ real matrices. The set of all of all *proper isometries* of $\mathbb{E}$ is the set

$$\mathrm{Isom}_+(\mathbb{E}) := \left\{\boldsymbol{r} : \mathbb{E} \to \mathbb{E},\ \boldsymbol{r}(x) = \boldsymbol{a} + \boldsymbol{R}\,x \text{ for all } x \in \mathbb{E};\ \boldsymbol{a} \in \mathbb{E},\ \boldsymbol{R} \in \mathbb{O}^{n+1}_+\right\},$$

where $\mathbb{O}^{n+1}_+$ denotes the set of all $(n+1)\times(n+1)$ real proper orthogonal matrices.

$\mathbb{M}^n$ and $\mathbb{S}^n$ respectively denote the space of all $n\times n$ real matrices and the space of all symmetric $n\times n$ matrices. $\mathbb{M}^{n+1}$ denotes the space of $(n+1)\times(n+1)$ real matrices.

Let ω be a *domain in* $\mathbb{R}^n$; this means that ω is bounded, connected, open, with a Lipschitz-continuous boundary, and the set ω is locally on the same side of its boundary (see, e.g., Adams [1] or Nečas [8]). A generic point in ω is denoted $y = (y_\alpha)$ and partial derivatives with respect to y_α are denoted by $\partial_\alpha := \partial/\partial y_\alpha$.

The usual Lebesgue and Sobolev spaces of (real-valued) functions defined over ω are respectively denoted by $L^p(\omega)$ and $W^{1,p}(\omega)$. Given any finite-dimensional vector space $\mathbb{X}$, the spaces of $\mathbb{X}$-valued fields with components in $L^p(\omega)$, resp. in $W^{1,p}(\omega)$, defined over ω are denoted by $L^p(\omega;\mathbb{X})$, resp. $W^{1,p}(\omega;\mathbb{X})$. Likewise, for each integer $k \geqslant 0$, the notation $\mathcal{C}^k(\overline{\omega})$, resp. $\mathcal{C}^k(\overline{\omega};\mathbb{X})$, denotes the space of continuously differentiable functions, resp. $\mathbb{X}$-valued fields, up to order k over ω that possess, together with all their partial derivatives up to order k, continuous extensions to $\overline{\omega}$.

The Euclidean norm of vectors in $\mathbb{E}$ and the Frobenius norm of matrices in $\mathbb{F}$, $\mathbb{M}^n$, or $\mathbb{S}^n$, are all denoted by $|\cdot|$.

Given any n vectors $\boldsymbol{v}_\alpha \in \mathbb{E}$, the notation $[\boldsymbol{v}_1\ \boldsymbol{v}_2\ \ldots\ \boldsymbol{v}_n]$, resp. $\boldsymbol{v}_1 \wedge \ldots \wedge \boldsymbol{v}_n$, denotes the matrix in $\mathbb{F}$ whose α-th column vector is $\boldsymbol{v}_\alpha$, resp. the vector in $\mathbb{E}$ whose i-th component is the determinant of the $n\times n$ matrix obtained by deleting the first column and the i-th row of the $(n+1)\times(n+1)$-matrix $[\ \cdot\ \boldsymbol{v}_1\ \boldsymbol{v}_2\ \ldots\ \boldsymbol{v}_n]$.

Let

$$\boldsymbol{\theta} : \overline{\omega} \to \mathbb{E}$$

be an *immersion of class* $\mathcal{C}^1$ with a positively-oriented unit normal vector field also of class $\mathcal{C}^1$; this means that $\boldsymbol{\theta} \in \mathcal{C}^1(\overline{\omega};\mathbb{E})$, the n vectors fields $\partial_\alpha\boldsymbol{\theta}$ are linearly independent at each point $y \in \overline{\omega}$, and the vector field

$$\boldsymbol{\nu}(\boldsymbol{\theta}) := \frac{\partial_1\boldsymbol{\theta} \wedge \ldots \wedge \partial_n\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta} \wedge \ldots \wedge \partial_n\boldsymbol{\theta}|} : \overline{\omega} \to \mathbb{E}$$

is also in the space $\mathcal{C}^1(\overline{\omega};\mathbb{E})$. The *covariant components* of the *first and second fundamental forms* associated with the immersion $\boldsymbol{\theta}$ are defined at each point $y \in \overline{\omega}$ by

$$a_{\alpha\beta}(\boldsymbol{\theta})(y) := \partial_\alpha\boldsymbol{\theta}(y)\cdot\partial_\beta\boldsymbol{\theta}(y) \ \text{ and } \ b_{\alpha\beta}(\boldsymbol{\theta})(y) := -\partial_\alpha\boldsymbol{\nu}(\boldsymbol{\theta})(y)\cdot\partial_\beta\boldsymbol{\theta}(y).$$

Let

$$\boldsymbol{\varphi} : \omega \to \mathbb{E}$$

be an *immersion of class* $W^{1,p}$ with a positively-oriented unit normal vector field also of class $W^{1,p}$; this means that $\boldsymbol{\theta} \in W^{1,p}(\omega;\mathbb{E})$, the n vectors fields $\partial_\alpha\boldsymbol{\varphi}$ are linearly independent at almost each point $y \in \omega$, and the vector field

$$\boldsymbol{\nu}(\boldsymbol{\varphi}) := \frac{\partial_1\boldsymbol{\varphi} \wedge \ldots \wedge \partial_n\boldsymbol{\varphi}}{|\partial_1\boldsymbol{\varphi} \wedge \ldots \wedge \partial_n\boldsymbol{\varphi}|} : \omega \to \mathbb{E}$$

is also in the space $W^{1,p}(\omega;\mathbb{E})$. The *covariant components* of the *first and second fundamental forms* associated with the immersion $\boldsymbol{\varphi}$ are defined at almost each point $y \in \omega$ by

$$a_{\alpha\beta}(\boldsymbol{\varphi})(y) := \partial_\alpha\boldsymbol{\varphi}(y)\cdot\partial_\beta\boldsymbol{\varphi}(y) \ \text{ and } \ b_{\alpha\beta}(\boldsymbol{\varphi})(y) := -\partial_\alpha\boldsymbol{\nu}(\boldsymbol{\varphi})(y)\cdot\partial_\beta\boldsymbol{\varphi}(y).$$

Note that

$$a_{\alpha\beta}(\boldsymbol{\varphi}) = a_{\alpha\beta}(\boldsymbol{r}\circ\boldsymbol{\varphi}) \text{ and } b_{\alpha\beta}(\boldsymbol{\varphi}) = b_{\alpha\beta}(\boldsymbol{r}\circ\boldsymbol{\varphi}) \text{ a.e. in } \omega \text{ for all } \boldsymbol{r} \in \mathrm{Isom}_+(\mathbb{E}).$$

Then one can prove that, if

$$a_{\alpha\beta}(\boldsymbol{\theta}) = a_{\alpha\beta}(\boldsymbol{\varphi}) \text{ and } b_{\alpha\beta}(\boldsymbol{\theta}) = b_{\alpha\beta}(\boldsymbol{\varphi}) \text{ a.e. in } \omega,$$

then there exists a proper isometry $\boldsymbol{r} \in \mathrm{Isom}_+(\mathbb{E})$ such that

$$\boldsymbol{\theta} = \boldsymbol{r}\circ\boldsymbol{\varphi} \text{ in } \omega$$

(a proof in the particular case $n = 2$ and $p = 2$ can be found in [3, Theorem 3]).

The first objective of this article is to improve this *uniqueness* result by showing that the distance in $W^{1,p}(\omega;\mathbb{E})$ between the immersion $\boldsymbol{\theta}$ and an immersion of the form $\boldsymbol{r}\circ\boldsymbol{\varphi}$, $\boldsymbol{r}\in \mathrm{Isom}_+(\mathbb{E})$, is bounded by the distance in $L^p(\omega)$ between specific functions of the coefficients $a_{\alpha\beta}(\boldsymbol{\theta})$, $b_{\alpha\beta}(\boldsymbol{\theta})$, $a_{\alpha\beta}(\boldsymbol{\varphi})$ and $b_{\alpha\beta}(\boldsymbol{\varphi})$ (which vanish if $a_{\alpha\beta}(\boldsymbol{\theta})=a_{\alpha\beta}(\boldsymbol{\varphi})$ and $b_{\alpha\beta}(\boldsymbol{\theta})=b_{\alpha\beta}(\boldsymbol{\varphi})$ a.e. in ω); cf. Theorem 1.

The second objective of this article is to prove that the above result holds with $\boldsymbol{r}=\boldsymbol{id}$, where $\boldsymbol{id}$ denotes the identity mapping from $\mathbb{E}$ onto $\mathbb{E}$, provided the immersions $\boldsymbol{\theta}$ and $\boldsymbol{\varphi}$ satisfy in addition one of the following two boundary conditions on a non-empty relatively open subset γ_0 of the boundary of ω:

(i) either $\boldsymbol{\theta}=\boldsymbol{\varphi}$ and $\boldsymbol{\nu}(\boldsymbol{\theta})=\boldsymbol{\nu}(\boldsymbol{\varphi})$ on γ_0,

(ii) or $\boldsymbol{\theta}=\boldsymbol{\varphi}$ on γ_0 and $\boldsymbol{\theta}(\gamma_0)$ is not contained in an affine subspace of dimension $<n$ of $\mathbb{E}$;

cf. Theorems 3 and 4.

Note that some of the results of this article generalize, and some are based on, previous results of P. G. Ciarlet and the authors [2,4], as will be indicated in the next section. The details of the proofs of these theorems, which are only briefly sketched below, will appear in a forthcoming paper [7].

2. Two key lemmas

We begin by stating two lemmas, which are key to the proofs of all the ensuing theorems.

The notations $\nabla\boldsymbol{\theta}$ and $\nabla\boldsymbol{\varphi}$ used below respectively denote the gradients of the mappings $\boldsymbol{\theta}$ and $\boldsymbol{\varphi}$, which are defined as the matrix fields

$$\nabla\boldsymbol{\theta} := [\partial_1\boldsymbol{\theta}\ \dots\ \partial_n\boldsymbol{\theta}] : \overline{\omega}\to\mathbb{F} \text{ and } \nabla\boldsymbol{\varphi} := [\partial_1\boldsymbol{\varphi}\ \dots\ \partial_n\boldsymbol{\varphi}] : \omega\to\mathbb{F}.$$

The notation $C_1(\boldsymbol{\theta},\omega,p)$ means that the constant C_1 depends on, and only on, $\boldsymbol{\theta}$, ω, and p.

The next lemma is in effect a generalization of a previous result of P. G. Ciarlet and the second author (see [4, Lemma 2]), which itself generalizes a *"geometric rigidity lemma"* due to Friesecke, James, and Müller [6] (for $p=2$), and to Conti [5] (for $1<p<\infty$).

Lemma 1. *Let ω, p, $\boldsymbol{\theta}$ and $\boldsymbol{\varphi}$ satisfy the assumptions of the Introduction. Then*

$$\inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_+}\Big(\|\nabla\boldsymbol{\theta}-\boldsymbol{R}\,\nabla\boldsymbol{\varphi}\|_{L^p(\omega;\mathbb{F})}+\|\nabla\boldsymbol{\nu}(\boldsymbol{\theta})-\boldsymbol{R}\,\nabla\boldsymbol{\nu}(\boldsymbol{\varphi})\|_{L^p(\omega;\mathbb{F})}+\|\boldsymbol{\nu}(\boldsymbol{\theta})-\boldsymbol{R}\,\boldsymbol{\nu}(\boldsymbol{\varphi})\|_{L^p(\omega;\mathbb{E})}\Big)$$
$$\leqslant C_1(\boldsymbol{\theta},\omega,p)\left\|\inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_+}\Big(|\nabla\boldsymbol{\theta}-\boldsymbol{R}\,\nabla\boldsymbol{\varphi}|+|\nabla\boldsymbol{\nu}(\boldsymbol{\theta})-\boldsymbol{R}\,\nabla\boldsymbol{\nu}(\boldsymbol{\varphi})|+|\boldsymbol{\nu}(\boldsymbol{\theta})-\boldsymbol{R}\,\boldsymbol{\nu}(\boldsymbol{\varphi})|\Big)\right\|_{L^p(\omega)},$$

for some constant $C_1(\boldsymbol{\theta},\omega,p)$.

Sketch of proof. Given $\boldsymbol{\theta}:\overline{\omega}\to\mathbb{E}$ satisfying the assumptions of the Introduction, there exists $\varepsilon=\varepsilon(\boldsymbol{\theta},\omega)>0$ such that the mapping $\boldsymbol{\Theta}:\overline{\Omega}\to\mathbb{E}$, $\Omega:=\omega\times]-\varepsilon,\varepsilon[$, defined by $\boldsymbol{\Theta}(y,t):=\boldsymbol{\theta}(y)+t\boldsymbol{\nu}(\boldsymbol{\theta})(y)$ for all $(y,t)\in\overline{\Omega}$, is an immersion of class $\mathcal{C}^1$ over $\overline{\Omega}$. Then Lemma 2 in [4] shows that there exists a constant $c(\boldsymbol{\Theta},\Omega,p)=c(\boldsymbol{\theta},\omega,p)$ such that

$$\inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_+}\|\nabla\tilde{\boldsymbol{\Theta}}-\boldsymbol{R}\,\nabla\boldsymbol{\Theta}\|_{L^p(\Omega;\mathbb{M}^{n+1})}\leqslant c(\boldsymbol{\theta},\omega,p)\left\|\inf_{\boldsymbol{R}\in\mathbb{O}^{n+1}_+}|\nabla\tilde{\boldsymbol{\Theta}}-\boldsymbol{R}\,\nabla\boldsymbol{\Theta}|\right\|_{L^p(\Omega)},$$

for all mappings $\tilde{\boldsymbol{\Theta}}\in W^{1,p}(\Omega;\mathbb{E})$.

Next, given $\boldsymbol{\varphi}:\omega\to\mathbb{E}$ satisfying the assumptions of the Introduction, define the mapping $\boldsymbol{\Phi}:\Omega\to\mathbb{E}$ by $\boldsymbol{\Phi}(y,t):=\boldsymbol{\varphi}(y)+t\,\boldsymbol{\nu}(\boldsymbol{\varphi})(y)$ for almost all $(y,t)\in\Omega$. Note that $\boldsymbol{\Phi}\in W^{1,p}(\Omega;\mathbb{E})$.

The inequality of the lemma is obtained, after a series of straightforward computations, by letting $\tilde{\boldsymbol{\Theta}}=\boldsymbol{\Phi}$ in the above inequality. □

With an immersion $\boldsymbol{\theta}$ satisfying the assumptions of the Introduction, we associate the matrix fields

$$\boldsymbol{U}(\boldsymbol{\theta}):=(a_{\alpha\beta}(\boldsymbol{\theta}))^{1/2}:\overline{\omega}\to\mathbb{S}^n \text{ and } \boldsymbol{V}(\boldsymbol{\theta}):=(a_{\alpha\beta}(\boldsymbol{\theta}))^{-1/2}(b_{\alpha\beta}(\boldsymbol{\theta})):\overline{\omega}\to\mathbb{M}^n,$$

where $(a_{\alpha\beta}(\boldsymbol{\theta}))^{1/2}$ denotes the field of the square roots of the (positive-definite symmetric) $n\times n$ matrices with components $a_{\alpha\beta}(\boldsymbol{\theta})(y)$, $y\in\overline{\omega}$, $(a_{\alpha\beta}(\boldsymbol{\theta}))^{-1/2}$ denotes the field of the inverse square roots of the same matrices, and $(b_{\alpha\beta}(\boldsymbol{\theta}))$ denotes the field of matrices with components $b_{\alpha\beta}(\boldsymbol{\theta})(y)$, $y\in\overline{\omega}$.

With an immersion $\boldsymbol{\varphi}$ satisfying the assumptions of the Introduction, we likewise associate the matrix fields

$$\boldsymbol{U}(\boldsymbol{\varphi}):=(a_{\alpha\beta}(\boldsymbol{\varphi}))^{1/2}:\omega\to\mathbb{S}^n \text{ and } \boldsymbol{V}(\boldsymbol{\varphi}):=(a_{\alpha\beta}(\boldsymbol{\varphi}))^{-1/2}(b_{\alpha\beta}(\boldsymbol{\varphi})):\omega\to\mathbb{M}^n.$$

Note that these matrix fields satisfy $\boldsymbol{U}(\boldsymbol{\theta})\in\mathcal{C}^0(\overline{\omega};\mathbb{S}^n)$, $\boldsymbol{V}(\boldsymbol{\theta})\in\mathcal{C}^0(\overline{\omega};\mathbb{M}^n)$, $\boldsymbol{U}(\boldsymbol{\varphi})\in L^p(\omega;\mathbb{S}^n)$, $\boldsymbol{V}(\boldsymbol{\varphi})\in L^p(\omega;\mathbb{M}^n)$, and

$$\boldsymbol{U}(\boldsymbol{r}\circ\boldsymbol{\varphi})=\boldsymbol{U}(\boldsymbol{\varphi}) \text{ and } \boldsymbol{V}(\boldsymbol{r}\circ\boldsymbol{\varphi})=\boldsymbol{V}(\boldsymbol{\varphi}) \text{ a.e. in } \omega \text{ for all } \boldsymbol{r}\in\mathrm{Isom}_+(\mathbb{E}).$$

The next lemma establishes an estimate of the right-hand side of Lemma 1 in terms of the matrix fields defined above.

Lemma 2. *Let ω, p, $\boldsymbol{\theta}$ and $\boldsymbol{\varphi}$ satisfy the assumptions of the Introduction. Then*

$$\left\| \inf_{\boldsymbol{R} \in \mathbb{O}_+^{n+1}} \Big(|\nabla \boldsymbol{\theta} - \boldsymbol{R}\, \nabla \boldsymbol{\varphi}| + |\nabla \boldsymbol{\nu}(\boldsymbol{\theta}) - \boldsymbol{R}\, \nabla \boldsymbol{\nu}(\boldsymbol{\varphi})| + |\boldsymbol{\nu}(\boldsymbol{\theta}) - \boldsymbol{R}\, \boldsymbol{\nu}(\boldsymbol{\varphi})| \Big) \right\|_{L^p(\omega)}$$
$$\leqslant C_2(\boldsymbol{\theta}, \omega, p) \Big(\|\boldsymbol{U}(\boldsymbol{\theta}) - \boldsymbol{U}(\boldsymbol{\varphi})\|_{L^p(\omega;\mathbb{S}^n)} + \|\boldsymbol{V}(\boldsymbol{\theta}) - \boldsymbol{V}(\boldsymbol{\varphi})\|_{L^p(\omega;\mathbb{M}^n)} \Big),$$

for some constant $C_2(\boldsymbol{\theta}, \omega, p)$.

Sketch of proof. The proof consists in expressing the matrix field $\nabla \boldsymbol{\theta}$ and vector field $\boldsymbol{\nu}(\boldsymbol{\theta})$, resp. $\nabla \boldsymbol{\varphi}$ and $\boldsymbol{\nu}(\boldsymbol{\varphi})$, in terms of the matrix field $\boldsymbol{U}(\boldsymbol{\theta})$, resp. $\boldsymbol{U}(\boldsymbol{\varphi})$, by using the polar decomposition theorem applied to the matrix field $[\nabla \boldsymbol{\theta}\ \ \boldsymbol{\nu}(\boldsymbol{\theta})] : \overline{\omega} \to \mathbb{M}^{n+1}$, resp. $[\nabla \boldsymbol{\varphi}\ \ \boldsymbol{\nu}(\boldsymbol{\varphi})] : \omega \to \mathbb{M}^{n+1}$, and the matrix field $\nabla \boldsymbol{\nu}(\boldsymbol{\theta})$, resp. $\nabla \boldsymbol{\nu}(\boldsymbol{\varphi})$, in terms of the matrices $\boldsymbol{U}(\boldsymbol{\theta})$ and $\boldsymbol{V}(\boldsymbol{\theta})$, resp. $\boldsymbol{U}(\boldsymbol{\varphi})$ and $\boldsymbol{V}(\boldsymbol{\varphi})$, by using the Weingarten equations

$$\nabla \boldsymbol{\nu}(\boldsymbol{\theta}) = -(\nabla \boldsymbol{\theta}) \boldsymbol{S}(\boldsymbol{\theta}), \text{ where } \boldsymbol{S}(\boldsymbol{\theta}) = \boldsymbol{U}(\boldsymbol{\theta})^{-1} \boldsymbol{V}(\boldsymbol{\theta}),$$
$$\nabla \boldsymbol{\nu}(\boldsymbol{\varphi}) = -(\nabla \boldsymbol{\varphi}) \boldsymbol{S}(\boldsymbol{\varphi}), \text{ where } \boldsymbol{S}(\boldsymbol{\varphi}) = \boldsymbol{U}(\boldsymbol{\varphi})^{-1} \boldsymbol{V}(\boldsymbol{\varphi}). \qquad \square$$

3. Main results

We are now in a position to achieve our first objective announced in the Introduction, that is, to improve the uniqueness result according to which a sufficiently regular hypersurface immersed in $\mathbb{E}$ is determined up to a proper isometry of $\mathbb{E}$ by its first and second fundamental forms.

Theorem 1. *Let ω, p, $\boldsymbol{\theta}$ and $\boldsymbol{\varphi}$ satisfy the assumptions of the Introduction. Then*

$$\inf_{\boldsymbol{r} \in \mathrm{Isom}_+(\mathbb{E})} \Big(\|\boldsymbol{\theta} - \boldsymbol{r} \circ \boldsymbol{\varphi}\|_{W^{1,p}(\omega;\mathbb{E})} + \|\boldsymbol{\nu}(\boldsymbol{\theta}) - \boldsymbol{\nu}(\boldsymbol{r} \circ \boldsymbol{\varphi})\|_{W^{1,p}(\omega;\mathbb{E})} \Big)$$
$$\leqslant C_3(\boldsymbol{\theta}, \omega, p) \Big(\|\boldsymbol{U}(\boldsymbol{\theta}) - \boldsymbol{U}(\boldsymbol{\varphi})\|_{L^p(\omega;\mathbb{S}^n)} + \|\boldsymbol{V}(\boldsymbol{\theta}) - \boldsymbol{V}(\boldsymbol{\varphi})\|_{L^p(\omega;\mathbb{M}^n)} \Big),$$

for some constant $C_3(\boldsymbol{\theta}, \omega, p)$.

Proof. The proof is a simple consequence of Lemma 1 and Lemma 2 above, combined with the Poincaré–Wirtinger inequality, which ensures that, for all $\boldsymbol{R} \in \mathbb{O}_+^{n+1}$, there exists a proper isometry $\boldsymbol{r} \in \mathrm{Isom}_+(\mathbb{E})$, such that $\nabla \boldsymbol{r} = \boldsymbol{R}$ in ω and

$$\|\boldsymbol{\theta} - \boldsymbol{r} \circ \boldsymbol{\varphi}\|_{L^p(\omega;\mathbb{E})} \leqslant c(\omega, p) \|\nabla(\boldsymbol{\theta} - \boldsymbol{r} \circ \boldsymbol{\varphi})\|_{L^p(\omega;\mathbb{E})} = c(\omega, p) \|\nabla \boldsymbol{\theta} - \boldsymbol{R}\, \nabla \boldsymbol{\varphi}\|_{L^p(\omega;\mathbb{F})},$$

for some constant $c = c(\omega, p)$. $\square$

It is worth mentioning that it is possible to get rid of the infimum in the left-hand side of the inequality of Theorem 1 provided a "lower norm" (than that appearing in the left-hand side) is added to the right-hand side. Note that the theorem below generalizes Theorem 2 in [4] (the latter is obtained from the first by letting the last components $(i = n+1)$ of $\boldsymbol{\theta} = (\theta^i)$ and $\boldsymbol{\varphi} = (\varphi^i)$ vanish in ω).

Theorem 2. *Let ω, p, $\boldsymbol{\theta}$ and $\boldsymbol{\varphi}$ satisfy the assumptions of the Introduction. Then*

$$\|\boldsymbol{\theta} - \boldsymbol{\varphi}\|_{W^{1,p}(\omega;\mathbb{E})} + \|\boldsymbol{\nu}(\boldsymbol{\theta}) - \boldsymbol{\nu}(\boldsymbol{\varphi})\|_{W^{1,p}(\omega;\mathbb{E})} \leqslant C_4(\boldsymbol{\theta}, \omega, p) \Big(\|\boldsymbol{\theta} - \boldsymbol{\varphi}\|_{L^p(\omega;\mathbb{E})} + \|\boldsymbol{\nu}(\boldsymbol{\theta}) - \boldsymbol{\nu}(\boldsymbol{\varphi})\|_{L^p(\omega;\mathbb{E})}$$
$$+ \|\boldsymbol{U}(\boldsymbol{\theta}) - \boldsymbol{U}(\boldsymbol{\varphi})\|_{L^p(\omega;\mathbb{S}^n)} + \|\boldsymbol{V}(\boldsymbol{\theta}) - \boldsymbol{V}(\boldsymbol{\varphi})\|_{L^p(\omega;\mathbb{M}^n)} \Big),$$

for some constant $C_4(\boldsymbol{\theta}, \omega, p)$.

Sketch of proof. The above inequality is a consequence of the inequality of Theorem 1 combined with the following estimate, whose proof is based on a contradiction argument, of its left-hand side:

$$\|\boldsymbol{\theta} - \boldsymbol{\varphi}\|_{W^{1,p}(\omega;\mathbb{E})} \leqslant c(\boldsymbol{\theta}, \omega, p) \Big(\|\boldsymbol{\theta} - \boldsymbol{\varphi}\|_{L^p(\omega;\mathbb{E})} + \inf_{\boldsymbol{R} \in \mathbb{O}_+^{n+1}} \|\nabla \boldsymbol{\theta} - \boldsymbol{R}\, \nabla \boldsymbol{\varphi}\|_{L^p(\omega;\mathbb{F})} \Big),$$

for some constant $c(\boldsymbol{\theta}, \omega, p)$. $\square$

We are now in a position to achieve our second objective announced in the Introduction, that is, to improve the uniqueness result according to which a sufficiently regular hypersurface with boundary, whose position and positively-oriented unit normal vectors are given on a non-empty portion of its boundary, is uniquely determined by its first and second fundamental forms.

Theorem 3. *Let ω, p, $\boldsymbol{\theta}$ and $\boldsymbol{\varphi}$ satisfy the assumptions of the Introduction and let γ_0 be a non-empty relatively open subset of the boundary of ω. Assume in addition that $\boldsymbol{\theta} = \boldsymbol{\varphi}$ and $\boldsymbol{\nu}(\boldsymbol{\theta}) = \boldsymbol{\nu}(\boldsymbol{\varphi})$ on γ_0. Then*

$$\begin{aligned}&\|\boldsymbol{\theta} - \boldsymbol{\varphi}\|_{W^{1,p}(\omega;\mathbb{E})} + \|\boldsymbol{\nu}(\boldsymbol{\theta}) - \boldsymbol{\nu}(\boldsymbol{\varphi})\|_{W^{1,p}(\omega;\mathbb{E})}\\ &\quad \leqslant C_5(\boldsymbol{\theta}, \omega, p, \gamma_0)\Big(\|\boldsymbol{U}(\boldsymbol{\theta}) - \boldsymbol{U}(\boldsymbol{\varphi})\|_{L^p(\omega;\mathbb{S}^n)} + \|\boldsymbol{V}(\boldsymbol{\theta}) - \boldsymbol{V}(\boldsymbol{\varphi})\|_{L^p(\omega;\mathbb{M}^n)}\Big),\end{aligned}$$

for some constant $C_5(\boldsymbol{\theta}, \omega, p, \gamma_0)$.

Sketch of proof. The proof is a consequence of Lemma 1 and Lemma 2, combined with the following estimate of the left-hand side of the inequality of Lemma 1:

$$\begin{aligned}&\|\boldsymbol{\theta} - \boldsymbol{\varphi}\|_{W^{1,p}(\omega;\mathbb{E})} + \|\boldsymbol{\nu}(\boldsymbol{\theta}) - \boldsymbol{\nu}(\boldsymbol{\varphi})\|_{W^{1,p}(\omega;\mathbb{E})} \leqslant c_1(\boldsymbol{\theta}, \omega, p, \gamma_0) \inf_{\boldsymbol{R} \in \mathbb{O}^{n+1}_+}\Big(\|\nabla\boldsymbol{\theta} - \boldsymbol{R}\,\nabla\boldsymbol{\varphi}\|_{L^p(\omega;\mathbb{F})}\\ &\quad + \|\nabla\boldsymbol{\nu}(\boldsymbol{\theta}) - \boldsymbol{R}\,\nabla\boldsymbol{\nu}(\boldsymbol{\varphi})\|_{L^p(\omega;\mathbb{F})} + \|\boldsymbol{\nu}(\boldsymbol{\theta}) - \boldsymbol{R}\,\boldsymbol{\nu}(\boldsymbol{\varphi})\|_{L^p(\omega;\mathbb{E})}\Big),\end{aligned}$$

for some constant $c_1(\boldsymbol{\theta}, \omega, p, \gamma_0)$. The proof of the above estimate, which is long and somewhat technical, is based on a contradiction argument. □

We conclude this section by showing that, in many cases (e.g., when γ_0 is large enough) the assumption "$\boldsymbol{\nu}(\boldsymbol{\theta}) = \boldsymbol{\nu}(\boldsymbol{\varphi})$ on γ_0" can be dropped in the previous theorem. More specifically, the following theorem holds.

Theorem 4. *Let ω, p, $\boldsymbol{\theta}$ and $\boldsymbol{\varphi}$ satisfy the assumptions of the Introduction and let γ_0 be a non-empty relatively open subset of the boundary of ω such that $\boldsymbol{\theta}(\gamma_0)$ is not contained in any affine subspace of $\mathbb{E}$ of dimension $< n$. Assume in addition that $\boldsymbol{\theta} = \boldsymbol{\varphi}$ on γ_0. Then*

$$\begin{aligned}&\|\boldsymbol{\theta} - \boldsymbol{\varphi}\|_{W^{1,p}(\omega;\mathbb{E})} + \|\boldsymbol{\nu}(\boldsymbol{\theta}) - \boldsymbol{\nu}(\boldsymbol{\varphi})\|_{W^{1,p}(\omega;\mathbb{E})}\\ &\quad \leqslant C_6(\boldsymbol{\theta}, \omega, p, \gamma_0)\Big(\|\boldsymbol{U}(\boldsymbol{\theta}) - \boldsymbol{U}(\boldsymbol{\varphi})\|_{L^p(\omega;\mathbb{S}^n)} + \|\boldsymbol{V}(\boldsymbol{\theta}) - \boldsymbol{V}(\boldsymbol{\varphi})\|_{L^p(\omega;\mathbb{M}^n)}\Big),\end{aligned}$$

for some constant $C_6(\boldsymbol{\theta}, \omega, p, \gamma_0)$.

Sketch of proof. The proof consists in showing that the inequality mentioned in the proof of the previous theorem also holds, possibly with a different constant $c_1(\boldsymbol{\theta}, \omega, p, \gamma_0)$, under the assumptions of the present theorem. This is done with the help of a new inequality (which could be seen as a "nonlinear Poincaré inequality") showing that, if the mappings $\boldsymbol{\theta}$ and $\boldsymbol{\varphi}$ satisfy the assumptions of the theorem, then

$$\|\boldsymbol{\theta} - \boldsymbol{\varphi}\|_{W^{1,p}(\omega;\mathbb{E})} \leqslant c_2(\boldsymbol{\theta}, \omega, p, \gamma_0) \inf_{\boldsymbol{R} \in \mathbb{O}^{n+1}_+} \|\nabla\boldsymbol{\theta} - \boldsymbol{R}\,\nabla\boldsymbol{\varphi}\|_{\mathbb{L}^p(\omega;\mathbb{F})},$$

for some constant $c_2(\boldsymbol{\theta}, \omega, p, \gamma_0)$. The proof of the above inequality is also based on a contradiction argument. □

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