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Partial differential equations

On distributional solutions of local and nonlocal problems of porous medium type

Sur des solutions distributionnelles de problèmes locaux et non locaux de type milieux poreux

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ABSTRACT

We present a theory of well-posedness and a priori estimates for bounded distributional (or very weak) solutions of

$$\partial_t u - \mathfrak{L}^{\sigma,\mu}[\varphi(u)] = g(x,t) \qquad \text{in} \quad \mathbb{R}^N \times (0,T), \tag{0.1}$$

where φ is merely continuous and nondecreasing, and $\mathfrak{L}^{\sigma,\mu}$ is the generator of a general symmetric Lévy process. This means that $\mathfrak{L}^{\sigma,\mu}$ can have both local and nonlocal parts like, e.g., $\mathfrak{L}^{\sigma,\mu} = \Delta - (-\Delta)^{\frac{1}{2}}$. New uniqueness results for bounded distributional solutions to this problem and the corresponding elliptic equation are presented and proven. A key role is played by a new Liouville type result for $\mathfrak{L}^{\sigma,\mu}$. Existence and a priori estimates are deduced from a numerical approximation, and energy-type estimates are also obtained.

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RÉSUMÉ

Nous montrons l'unicité, l'existence, et des estimations a priori pour des solutions distributionnelles bornées de (0.1), où φ est continue et croissante et $\mathfrak{L}^{\sigma,\mu}$ est le générateur d'un processus de Lévy symétrique général. Cela veut dire que $\mathfrak{L}^{\sigma,\mu}$ peut avoir des parties locales et non locales, comme par exemple $\mathfrak{L}^{\sigma,\mu} = \Delta - (-\Delta)^{\frac{1}{2}}$. Nous présentons et montrons des nouveaux résultats d'unicité pour des solutions distributionnelles bornées de ce problème. Un nouveau résultat de type Liouville pour $\mathfrak{L}^{\sigma,\mu}$ joue un rôle clé. L'existence et des estimations a priori sont déduites d'une approximation numérique ; des inégalités de type énergie sont aussi obtenues.

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Version française abrégée

Nous étudions le problème de Cauchy pour l'équation de diffusion non linéaire de type Lévy (1.1). Ici u est la solution, u_0 la donnée initiale, $\varphi : \mathbb{R} \to \mathbb{R}$ une fonction continue croissante quelconque, g le terme du membre de droite de l'équation, et $T > 0$. L'opérateur de diffusion $\mathfrak{L}^{\sigma,\mu}$ est défini par (1.3), (1.4) et (1.5), et pourrait être le générateur d'un processus de Lévy quelconque, comme le laplacien ou le laplacian fractionaire.

Dans cette note, nous donnons des résultats d'existence, d'unicité, et des estimations a priori pour les solutions distributionelles de (1.1)–(1.2) dans $L^1 \cap L^\infty$, ainsi que pour son équation elliptique associée (1.6). Les preuves sont liée à l'article [1] et à des extentions récentes de [5].

Les résultats d'unicité de la première partie de cette note jouent un rôle clé dans les preuves de convergence des méthodes numériques de [3]. Dans la deuxième partie, nous annonçons quelques résultats de [3]. Nous obtenons l'existence des solutions distributionelles via une approximation numérique de (1.1)–(1.2), ainsi qu'un principe de contraction dans L^1, un principe de comparaison, la décroissance des normes L^1 et L^∞, et la continuité en temps pour la norme L^1. Ensuite, d'après les résultats de [4], nous héritons d'une famillie d'inegalitées d'énergie, ce qui implique, en particulier, la décroissance des normes L^p pour chaque $1 < p < \infty$.

1. Introduction

We study the Cauchy problem for the nonlinear Lévy-type diffusion equation

$$\partial_t u - \mathfrak{L}^{\sigma,\mu}[\varphi(u)] = g(x,t) \qquad \text{in} \quad Q_T := \mathbb{R}^N \times (0,T), \tag{1.1}$$

$$u(x,0) = u_0(x) \qquad \text{on} \quad \mathbb{R}^N, \tag{1.2}$$

where $u = u(x,t)$ is the solution, u_0 the initial data, $\varphi : \mathbb{R} \to \mathbb{R}$ an arbitrary continuous nondecreasing function, g the right-hand side, and $T > 0$. For smooth functions ψ, the diffusion operator $\mathfrak{L}^{\sigma,\mu}$ is defined as

$$\mathfrak{L}^{\sigma,\mu}[\psi] := L^\sigma[\psi] + \mathcal{L}^\mu[\psi], \tag{1.3}$$

where the local and nonlocal parts are given by

$$L^\sigma[\psi](x) := \operatorname{tr}\big(\sigma\sigma^{\mathrm{T}} D^2\psi(x)\big) = \sum_{i=1}^{P} \partial^2_{\sigma_i}\psi(x) \qquad \text{where} \qquad \partial_{\sigma_i} := \sigma_i \cdot D, \tag{1.4}$$

$$\mathcal{L}^\mu[\psi](x) := \int_{\mathbb{R}^N\setminus\{0\}} \big(\psi(x+z) - \psi(x) - z \cdot D\psi(x)\mathbf{1}_{|z|\le 1}\big)\,\mathrm{d}\mu(z), \tag{1.5}$$

and $\sigma = (\sigma_1, ..., \sigma_P) \in \mathbb{R}^{N\times P}$, $P \in \mathbb{N}$ and $\sigma_i \in \mathbb{R}^N$, and μ are nonnegative symmetric Radon measures. This class of diffusion operators coincides with the class of operators of symmetric Lévy processes. Examples are the classical Laplacian Δ, the fractional Laplacians $(-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$, the relativistic Schrödinger-type operators $m^\alpha I - (m^2 I - \Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$ and $m > 0$, the strongly degenerate operators, and, surprisingly, the numerical discretizations of $\mathfrak{L}^{\sigma,\mu}$. Due to the general assumptions on φ, (generalized) porous medium, fast diffusion, and Stefan-type problems are included in (1.1)–(1.2).

In this note, we present new existence and uniqueness results and a priori estimates for distributional solutions of (1.1)–(1.2) in $L^1 \cap L^\infty$. In particular, we present and prove new uniqueness results for bounded distributional solutions of both (1.1)–(1.2) and the related elliptic equation

$$w - \mathfrak{L}^{\sigma,\mu}[\varphi(w)] = f(x) \qquad \text{on} \quad \mathbb{R}^N. \tag{1.6}$$

The proofs are inspired by the seminal work [1] and the later extension to the nonlocal setting in [5]. Most of the other properties generalize well-known results both for the local case $\mathfrak{L}^{\sigma,\mu} = \Delta$ (cf. [6]) and for the nonlocal case $\mathfrak{L}^{\sigma,\mu} = -(-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$ (cf. [2]).

These uniqueness results will play a crucial role in the convergence proofs for numerical methods in [3]. In this note, we also announce some of the results of [3]. From a novel numerical approximation of (1.1)–(1.2), we obtain existence of distributional solutions, L^1 contraction, comparison principle, decay of the L^1 and L^∞ norms, and continuity in time of the L^1 norm. Moreover, by adapting the results of [4], we also inherit a family of energy estimates that, in particular, allow us to show decay of any L^p norm for $1 < p < \infty$.

2. Main results

We use the following assumptions:

$$\varphi : \mathbb{R} \to \mathbb{R} \text{ is nondecreasing and continuous.} \tag{A_φ}$$

$$g \in L^1(Q_T) \cap L^1(0,T; L^\infty(\mathbb{R}^N)). \tag{A_g}$$

$$u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \tag{A_{u_0}}$$

$$\mu \text{ is a nonnegative symmetric Radon measure on } \mathbb{R}^N \setminus \{0\} \text{ satisfying } \int_{|z|>0} \min\{|z|^2, 1\}\,\mathrm{d}\mu(z) < \infty. \tag{A_μ}$$

The notation $(f, g) := \int_{\mathbb{R}^N} fg\,\mathrm{d}x$ is used whenever the integral is well defined. If $f, g \in L^2$, we write $(f, g)_{L^2}$.

Definition 2.1. Let $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and $g \in L^1_{\mathrm{loc}}(Q_T)$. We say that a function $u \in L^\infty(Q_T)$ is a distributional (or very weak) solution of (1.1)–(1.2) if

$$\int_0^T \int_{\mathbb{R}^N} \left(u\partial_t\psi + \varphi(u)\mathfrak{L}^{\sigma,\mu}[\psi] + g\psi\right) \mathrm{d}x\,\mathrm{d}t = 0 \qquad \text{for all} \qquad \psi \in C_c^\infty(Q_T), \tag{2.1}$$

and $\operatorname{ess\,lim}_{t\to 0^+} \int_{\mathbb{R}^N} u(x,t)\psi(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^N} u_0(x)\psi(x,0)\,\mathrm{d}x$ for all $\psi \in C_c^\infty(\mathbb{R}^N \times [0,T))$.

Under our assumptions $\|\mathfrak{L}^{\sigma,\mu}[\psi]\|_{L^1} \le C\|\psi\|_{W^{2,1}}$, see Lemma 3.5 in [5], so (2.1) is well defined for $u \in L^\infty$.

Remark 2.2.

(a) Associated with the operator $\mathfrak{L}^{\sigma,\mu}$ is a bilinear form defining an energy: for $\phi, \psi \in C_c^\infty(\mathbb{R}^N)$, $\mathcal{E}_{\sigma,\mu}[\phi,\psi] := -(\phi, \mathfrak{L}^{\sigma,\mu}[\psi])$. Equivalently (cf. [4, Section 4]),

$$\mathcal{E}_{\sigma,\mu}[\phi,\psi] = \sum_{i=1}^P \int_{\mathbb{R}^N} \partial_{\sigma_i}\phi(x)\partial_{\sigma_i}\psi(x)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N}\int_{|z|>0} (\phi(x+z)-\phi(x))\,(\psi(x+z)-\psi(x))\,\mathrm{d}\mu(z)\,\mathrm{d}x.$$

The energy of a function ϕ is then defined as $\overline{\mathcal{E}}_{\sigma,\mu}[\phi] := \mathcal{E}_{\sigma,\mu}[\phi,\phi]$.

(b) $\mathfrak{L}^{\sigma,\mu}$ is a Fourier multiplier operator, $\mathcal{F}(\mathfrak{L}^{\sigma,\mu}[\psi])(\xi) = -\widehat{\mathfrak{L}}^{\sigma,\mu}(\xi)\mathcal{F}(\psi)(\xi)$, where

$$\widehat{\mathfrak{L}}^{\sigma,\mu}(\xi) := \widehat{L}^\sigma(\xi) + \widehat{\mathcal{L}}^\mu(\xi) = \sum_{i=1}^P (\sigma_i\cdot\xi)^2 + \int_{|z|>0} (1-\cos(z\cdot\xi))\,\mathrm{d}\mu(z).$$

The square-root operator $(\mathfrak{L}^{\sigma,\mu})^{\frac{1}{2}}$ is defined as the operator with Fourier symbol $-(\widehat{\mathfrak{L}}^{\sigma,\mu}(\xi))^{\frac{1}{2}}$.

Theorem 2.3 *(Well-posedness). Assume* (A_φ), (A_g), (A_{u_0}), *and* (A_μ).

(a) *There exists a unique distributional solution* $u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ *of* (1.1)–(1.2).

(b) *If* u, v *are solutions with data* u_0, v_0 *and* g, h *satisfying resp.* (A_{u_0}) *and* (A_g), *then, for every* $t \in [0,T]$,

(i) (L^1 contraction) $\int_{\mathbb{R}^N}(u(x,t)-v(x,t))^+\,\mathrm{d}x \le \int_{\mathbb{R}^N}(u_0(x)-v_0(x))^+\,\mathrm{d}x + \int_0^t\int_{\mathbb{R}^N}(g(x,\tau)-h(x,\tau))^+\,\mathrm{d}x\,\mathrm{d}\tau$;

(ii) (Comparison) *if* $u_0 \le v_0$ *a.e. and* $g \le h$ *a.e., then* $u \le v$ *a.e.;*

(iii) (L^p estimate 1) *for* $1 \le p \le \infty$, $\|u(\cdot,t)\|_{L^p(\mathbb{R}^N)} \le \|u_0\|_{L^p(\mathbb{R}^N)} + \int_0^t \|g(\cdot,\tau)\|_{L^p(\mathbb{R}^N)}\,\mathrm{d}\tau$;

(iv) (L^p estimate 2) *for* $1 < p < \infty$, $\|u(\cdot,t)\|^p_{L^p(\mathbb{R}^N)} \le \|u_0\|^p_{L^p(\mathbb{R}^N)} + p\int_0^t\int_{\mathbb{R}^N}|u(x,\tau)|^{p-2}u(x,\tau)g(x,\tau)\,\mathrm{d}x\,\mathrm{d}t$;

(v) (Energy estimate) *if* $\Phi:\mathbb{R}\to\mathbb{R}$ *is defined by* $\Phi(\xi) := \int_0^\xi \varphi(\eta)\,\mathrm{d}\eta$, *then*

$$\int_{\mathbb{R}^N}\Phi(u(x,t))\,\mathrm{d}x + \int_0^t \overline{\mathcal{E}}_{\sigma,\mu}[\varphi(u(\cdot,\tau))]\,\mathrm{d}\tau \le \int_{\mathbb{R}^N}\Phi(u_0(x))\,\mathrm{d}x + \int_0^t\int_{\mathbb{R}^N} g(x,\tau)\varphi(u(x,\tau))\,\mathrm{d}x\,\mathrm{d}\tau;$$

(vi) (Time regularity) *for every* $t, s \in [0,T]$ *and every compact set* $K \subset \mathbb{R}^N$,

$$\|u(\cdot,t)-u(\cdot,s)\|_{L^1(K)} \le 2\lambda\left(|t-s|^{\frac{1}{3}}\right) + C\left(|t-s|^{\frac{1}{3}} + |t-s|\right) + |K|\int_s^t \|g(\cdot,\tau)\|_{L^\infty(\mathbb{R}^N)}\,\mathrm{d}\tau,$$

where $\lambda(\delta) = \max_{|h|\le\delta}\|u_0 - u_0(\cdot+h)\|_{L^1(\mathbb{R}^N)}$ *and* $C = C(K, u_0, \varphi) > 0$*;*

(vii) (Conservation of mass) *if, in addition, there exist* $L, \delta > 0$ *such that* $|\varphi(r)| \le L|r|$ *for* $|r| \le \delta$, *then*

$$\int_{\mathbb{R}^N} u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^N} u_0(x)\,\mathrm{d}x + \int_0^t\int_{\mathbb{R}^N} g(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau.$$

3. Uniqueness of distributional solutions

We obtain uniqueness for a class of bounded distributional solutions of (1.1)–(1.2) and (1.6). One of the key tools in the proof of these results is the Liouville-type result given by Theorem 3.3.

Theorem 3.1 *(Uniqueness 1). Assume* (A_φ), (A_μ), $g \in L^1_{\mathrm{loc}}(Q_T)$, *and* $u_0 \in L^\infty(\mathbb{R}^N)$. *Then there is at most one distributional solution* u *of* (1.1)–(1.2) *such that* $u \in L^\infty(Q_T)$ *and* $u - u_0 \in L^1(Q_T)$.

Theorem 3.2 *(Uniqueness 2). Assume* (A_φ), (A_μ), *and* $f \in L^\infty(\mathbb{R}^N)$. *Then there is at most one distributional solution* w *of* (1.6) *such that* $w \in L^\infty(\mathbb{R}^N)$ *and* $w - f \in L^1(\mathbb{R}^N)$.

Theorem 3.3 *("Liouville"). Assume* (A_μ) *and that either* $\sigma \not\equiv 0$ *or* $\operatorname{supp}\mu \neq \emptyset$. *If* $v \in C_0(\mathbb{R}^N)$ *solves* $\mathfrak{L}^{\sigma,\mu}[v] = 0$ *in* $\mathcal{D}'(\mathbb{R}^N)$, *then* $v \equiv 0$ *in* $\mathbb{R}^N$.

Proof. If $\sigma \equiv 0$, then $\mathfrak{L}^{\sigma,\mu} = \mathcal{L}^\mu$ and the result follows by Theorem 3.9 in [5]. Assume that $\sigma \not\equiv 0$, and note that by a change of coordinates we may also assume that $L^\sigma = \Delta_l := \sum_{i=1}^l \partial^2_{x_i}$ for some $1 \le l \le N$.

Let ω_δ be a standard mollifier in $\mathbb{R}^N$ and define $v_\delta := v * \omega_\delta \in C_0(\mathbb{R}^N) \cap C_b^\infty(\mathbb{R}^N)$. As shown in the proof of Theorem 3.9 in [5], $\int_{\mathbb{R}^N} v(y)\mathcal{L}^\mu[\omega_\delta(x-\cdot)](y)\,\mathrm{d}y = \mathcal{L}^\mu[v_\delta](x)$. We also have that $\int_{\mathbb{R}^N} v(y)\Delta_l[\omega_\delta(x-\cdot)](y)\,\mathrm{d}y = \Delta_l[v_\delta](x)$. In this way, taking $w_\delta(x-y)$ as a test function in the distributional formulation we get that

$$\Delta_l[v_\delta](x) + \mathcal{L}^\mu[v_\delta](x) = 0 \qquad \text{for every} \qquad x \in \mathbb{R}^N. \tag{3.1}$$

Now we multiply (3.1) by v_δ, integrate over $\mathbb{R}^N$, integrate by parts, and use Plancherel's theorem to get

$$0 = -\sum_{i=1}^{l} \int_{\mathbb{R}^N} v_\delta(x)\partial^2_{x_i} v_\delta(x)\,\mathrm{d}x - \int_{\mathbb{R}^N} v_\delta(x)\mathcal{L}^\mu[v_\delta](x)\,\mathrm{d}x = \sum_{i=1}^{l} \int_{\mathbb{R}^N} \left|\partial_{x_i} v_\delta(x)\right|^2 \mathrm{d}x + \|(\mathcal{L}^\mu)^{\frac{1}{2}}[v_\delta]\|^2_{L^2(\mathbb{R}^N)}.$$

Since all the terms in the last expression are nonnegative, they are all zero. In particular $\int_{\mathbb{R}^N} \left|\partial_{x_1} v_\delta(x)\right|^2 \mathrm{d}x = 0$, and then $\partial_{x_1} v_\delta(x) = 0$ for every $x \in \mathbb{R}^N$. Hence $0 = \int_{x_1}^b \partial_{x_1} v_\delta(s,x')\,\mathrm{d}s = v_\delta(b,x') - v_\delta(x_1,x')$ for every $x_1 < b$ and every $x' = (x_2,\cdots,x_N) \in \mathbb{R}^{N-1}$. Since $v_\delta \in C_0(\mathbb{R}^N)$, we send $b \to \infty$ in the previous expression to see that $v_\delta(x_1,x') = v_\delta(b,x') \to 0$ as $b \to \infty$. Hence $v_\delta(x) = 0$ for every $x \in \mathbb{R}^N$. By properties of mollifiers, $v_\delta \to v$ locally uniformly in $\mathbb{R}^N$ as $\delta \to 0^+$, which means that also $v(x) = 0$ for every $x \in \mathbb{R}^N$. □

Proof of Theorem 3.1. Step 1: The resolvent $B_\varepsilon^{\sigma,\mu}$ of $\mathfrak{L}^{\sigma,\mu}$. Formally, the resolvent of $\mathfrak{L}^{\sigma,\mu}$ is given as $B_\varepsilon^{\sigma,\mu} = (\varepsilon I - \mathfrak{L}^{\sigma,\mu})^{-1}$ for $\varepsilon > 0$. But to give a rigorous meaning to this operator even when $\mathfrak{L}^{\sigma,\mu}$ is strongly degenerate, we define it as $B_\varepsilon^{\sigma,\mu}[\gamma](x) := v_\varepsilon(x)$ where v_ε is the solution of the linear elliptic equation

$$\varepsilon v_\varepsilon(x) - \mathfrak{L}^{\sigma,\mu}[v_\varepsilon](x) = \gamma(x) \qquad \text{in} \qquad \mathbb{R}^N. \tag{3.2}$$

To be able to apply $B_\varepsilon^{\sigma,\mu}$ to L^1, L^∞, and smooth γ, we need to prove existence and uniqueness for L^1 and L^∞ distributional and C_b^∞ classical solutions of (3.2) along with the following estimates

$$\varepsilon\|B_\varepsilon^{\sigma,\mu}[\gamma]\|_{L^1} \le \|\gamma\|_{L^1}, \quad \varepsilon\|B_\varepsilon^{\sigma,\mu}[\gamma]\|_{L^\infty} \le \|\gamma\|_{L^\infty}, \quad \text{and} \quad \varepsilon\|D^\beta B_\varepsilon^{\sigma,\mu}[\gamma]\|_{L^\infty} \le \|D^\beta\gamma\|_{L^\infty}\ \forall \beta \in \mathbb{N}^N. \tag{3.3}$$

The proof can be deduced by following the ideas of the proof of Theorem 3.1 in [5]. The idea is to approximate $\mathfrak{L}^{\sigma,\mu}$ by a bounded nonlocal operator $\mathcal{L}^{\nu^h}$, and then approximate (3.2) by the equation

$$\varepsilon v_{h,\varepsilon}(x) - \mathcal{L}^{\nu^h}[v_{h,\varepsilon}](x) = \gamma(x) \qquad \text{in} \qquad \mathbb{R}^N. \tag{3.4}$$

Because of the local terms, we have to modify the choice of ν^h from [5] and take

$$\nu^h(z) := \nu^h_\sigma(z) + \nu^h_\mu(z) = \frac{1}{h^2}\sum_{i=1}^{P}\left(\delta_{h\sigma_i}(z) + \delta_{-h\sigma_i}(z)\right) + \mu(z)\mathbf{1}_{|z|>h}, \tag{3.5}$$

where δ_a is the delta-measure supported at a. By a similar argument as in Lemma 5.2 in [5], ν^h is a nonnegative symmetric Radon measure satisfying $\nu^h(\mathbb{R}^N) < \infty$ and $\|\mathcal{L}^{\nu^h}[\psi] - \mathfrak{L}^{\sigma,\mu}[\psi]\|_{L^p(\mathbb{R}^N)} \to 0$ as $h \to 0^+$ for all $\psi \in C_c^\infty(\mathbb{R}^N)$ and $p = \{1,\infty\}$. Note that $\mathcal{L}^{\nu^h}$ is in the class of operators (1.5) with $\mu = \nu^h$ satisfying (A_μ), and thus, (3.4) has already been studied in [5]. In particular, we have existence, uniqueness and estimates (3.3) for solutions of (3.4) by Theorem 3.1 in [5]. The corresponding results for equation (3.2) then follow using compactness arguments to pass to the limit as $h \to 0^+$ and then verifying that the limit satisfies equation (3.2). There are three different cases, L^1, L^∞, and smooth, but all arguments follow as in [5] with only easy modifications. To give an idea, we investigate the case of smooth solutions when $\gamma \in C_b^\infty$ (cf. Proposition 6.12 in [5]). The Arzelà–Ascoli theorem and the third estimate in (3.3) ensure that there is a function $\overline{v}_\varepsilon$ such that $(v_{h,\varepsilon}, Dv_{h,\varepsilon}, D^2 v_{h,\varepsilon}) \to (\overline{v}_\varepsilon, D\overline{v}_\varepsilon, D^2\overline{v}_\varepsilon)$ locally uniformly as $h \to 0^+$. To see that $\overline{v}_\varepsilon$ is a classical solution of (3.2), it remains to show that $\mathcal{L}^{\nu^h}[v_{h,\varepsilon}](x) \to \mathfrak{L}^{\sigma,\mu}[\overline{v}_\varepsilon](x)$ in $\mathbb{R}^N$. Indeed,

$$|\mathcal{L}^{\nu^h}[v_{h,\varepsilon}](x) - \mathfrak{L}^{\sigma,\mu}[\bar{v}_\varepsilon](x)| \le |\mathcal{L}^{\nu^h_\sigma}[v_{h,\varepsilon}](x) - L^\sigma[\bar{v}_\varepsilon](x)| + |\mathcal{L}^{\nu^h_\mu}[v_{h,\varepsilon}](x) - \mathcal{L}^\mu[\bar{v}_\varepsilon](x)|.$$

The second term on the right-hand side converges to zero as in the proof of Proposition 6.12 in [5], while for the remaining one we have:

$$\begin{aligned}
&|\mathcal{L}^{\nu^h_\sigma}[v_{h,\varepsilon}](x) - L^\sigma[\bar{v}_\varepsilon](x)| \le |\mathcal{L}^{\nu^h_\sigma}[\bar{v}_\varepsilon](x) - L^\sigma[\bar{v}_\varepsilon](x)| + |\mathcal{L}^{\nu^h_\sigma}[v_{h,\varepsilon} - \bar{v}_\varepsilon]| \\
&\le h^2 \|D^4\bar{v}_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \sum_{i=1}^P \sum_{|\alpha|=4} \frac{2}{\alpha!}|\sigma_i|^\alpha + \sum_{i=1}^P \max_{|\xi|\le h} |D^2(v_{h,\varepsilon} - \bar{v}_\varepsilon)(x+\xi\sigma_i)| \sum_{|\alpha|=2} \frac{2}{\alpha!}|\sigma_i|^\alpha.
\end{aligned}$$

This concludes the proof of existence since $D^2 v_{h,\varepsilon} \to D^2\overline{v}_\varepsilon$ locally uniformly as $h \to 0^+$. Repeating the compactness argument for higher derivatives and passing to the limit, we find that $\overline{v}_\varepsilon$ also satisfies the third estimate in (3.3). Uniqueness is a trivial consequence of the linearity of (3.2) and the estimates in (3.3).

STEP 2: $\varepsilon B_\varepsilon^{\sigma,\mu}[q] \to 0$ A.E. AS $\varepsilon \to 0^+$ FOR $q \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Let $\gamma \in C_c^\infty(\mathbb{R}^N)$ and $\Gamma_\varepsilon := \varepsilon B_\varepsilon^{\sigma,\mu}[\gamma]$. We first show that all subsequences $\{\Gamma_{\varepsilon_j}\}_{j\in\mathbb{N}}$ converging in L^∞_{loc} as $\varepsilon_j \to 0^+$ converge to $\Gamma \equiv 0$. Indeed, by (3.2)

$$\varepsilon_j \int_{\mathbb{R}^N} \Gamma_{\varepsilon_j}\psi \,\mathrm{d}x - \int_{\mathbb{R}^N} \Gamma_{\varepsilon_j}\mathfrak{L}^{\sigma,\mu}[\psi]\,\mathrm{d}x = \varepsilon_j \int_{\mathbb{R}^N} \gamma\psi \,\mathrm{d}x \qquad \text{for all} \qquad \psi \in C_c^\infty(\mathbb{R}^N),$$

and we send $\varepsilon_j \to 0^+$ to find that $\mathfrak{L}^{\sigma,\mu}[\Gamma] = 0$ in $\mathcal{D}'$. Since Γ is Lipschitz and in L^1 by (3.3), $\lim_{|x|\to\infty}\Gamma(x) = 0$, and then $\Gamma \equiv 0$ by the Liouville-type result in Theorem 3.3. The next step is to observe that Γ_ε is equibounded and equi-Lipschitz by (3.3), and use the first part and the Arzelà–Ascoli theorem to conclude that any subsequence of $\{\Gamma_\varepsilon\}_{\varepsilon>0}$ has a further subsequence converging to zero in L^∞_{loc}. This implies that the whole sequence converges to zero in L^∞_{loc}. Now we study $Q_\varepsilon := \varepsilon B_\varepsilon^{\sigma,\mu}[q]$. By self-adjointness of $B_\varepsilon^{\sigma,\mu}$ (cf. Lemma 3.4 in [5]), the properties of Γ_ε, and the dominated convergence theorem, $\int_{\mathbb{R}^N} Q_\varepsilon\gamma\,\mathrm{d}x = \int_{\mathbb{R}^N} q\Gamma_\varepsilon\,\mathrm{d}x \to \int_{\mathbb{R}^N} q\Gamma\,\mathrm{d}x = 0$, i.e. $Q_\varepsilon \to 0$ in $\mathcal{D}'$ as $\varepsilon \to 0^+$. Then since $\mathcal{D}'$ and L^1_{loc} limits coincide and $\{Q_\varepsilon\}_{\varepsilon>0}$ is precompact in L^1_{loc} by (3.3) and Kolmogorov's compactness theorem, all subsequences of $\{Q_\varepsilon\}_{\varepsilon>0}$ have further subsequences converging to zero in L^1_{loc} and a.e. The full sequence thus converges to zero a.e.

STEP 3: THE DIFFERENCE $\mathcal{U}$ OF TWO SOLUTIONS OF (1.1)–(1.2) AND "ENERGY" FROM $B_\varepsilon^{\sigma,\mu}$. Let $u, \hat{u} \in L^\infty(Q_T)$ be two distributional solutions of (1.1)–(1.2) with initial data u_0 such that $u - u_0, \hat{u} - u_0 \in L^1(Q_T)$. Define $\mathcal{U} := u - \hat{u}$ and $\mathcal{Z} := \varphi(u) - \varphi(\hat{u}) \in L^\infty(Q_T)$. Note that $\|u - \hat{u}\|_{L^1(Q_T)} \le \|u - u_0\|_{L^1(Q_T)} + \|u - u_0\|_{L^1(Q_T)} < \infty$, and thus, $\mathcal{U} \in L^1(Q_T) \cap L^\infty(Q_T)$. We subtract the equations for u and $\hat{u}$ (distributional formulation of (1.1)), and take $\psi = B_\varepsilon^{\sigma,\mu}[\gamma]$ for $\gamma \in C_c^\infty(\mathbb{R}^N)$ as a test function. By the properties of the solutions of (3.2), we get $\int_0^T \int_{\mathbb{R}^N} \left(\mathcal{U}B_\varepsilon^{\sigma,\mu}[\partial_t\gamma] + \mathcal{Z}(\varepsilon B_\varepsilon^{\sigma,\mu}[\gamma] - \gamma)\right)\mathrm{d}x\,\mathrm{d}t = 0$. Thus, by the self-adjointness of $B_\varepsilon^{\sigma,\mu}$,

$$\partial_t B_\varepsilon^{\sigma,\mu}[\mathcal{U}] = \varepsilon B_\varepsilon^{\sigma,\mu}[\mathcal{Z}] - \mathcal{Z} \qquad \text{in} \qquad \mathcal{D}'(Q_T). \tag{3.6}$$

Now consider the "energy"-like function $h_\varepsilon(t) = \int_{\mathbb{R}^N} B_\varepsilon^{\sigma,\mu}[\mathcal{U}](x,t)\mathcal{U}(x,t)\,\mathrm{d}x$. Note that by (3.3), $h_\varepsilon \in L^1(0,T)$ since $\|h_\varepsilon\|_{L^1(0,T)} \le \frac{1}{\varepsilon}\|\mathcal{U}\|_{L^\infty(Q_T)}\|\mathcal{U}\|_{L^1(Q_T)}$. As in Proposition 3.11 in [5], we get that h_ε is absolutely continuous and $h'_\varepsilon(t) = 2\left(\partial_t B_\varepsilon^{\sigma,\mu}[\mathcal{U}](\cdot,t), \mathcal{U}(\cdot,t)\right)$ in $\mathcal{D}'(0,T)$. By (3.6) and (3.9) below, and since $\mathcal{Z}\mathcal{U} \ge 0$,

$$0 \le h_\varepsilon(t) = h_\varepsilon(0^+) + \int_0^t h'_\varepsilon(s)\mathrm{d}s \le 0 + 2\int_0^t \left(\varepsilon B_\varepsilon^{\sigma,\mu}[\mathcal{Z}](\cdot,s), \mathcal{U}(\cdot,s)\right)\mathrm{d}s. \tag{3.7}$$

Let now $\xi > 0$. By the self-adjointness of $B_\varepsilon^{\sigma,\mu}$, we have for a.e. $t \in [0,T]$

$$\left(\varepsilon B_\varepsilon^{\sigma,\mu}[\mathcal{Z}](\cdot,t), \mathcal{U}(\cdot,t)\right) \le \|\mathcal{Z}\|_{L^\infty(Q_T)} \int_{\mathbb{R}^N} \left|\varepsilon B_\varepsilon^{\sigma,\mu}[\mathcal{U}](x,t)\right| \mathbf{1}_{|\mathcal{Z}(x,t)|>\xi}\,\mathrm{d}x + \xi\|\mathcal{U}(\cdot,t)\|_{L^1(\mathbb{R}^N)}. \tag{3.8}$$

Note that $\left|\varepsilon B_\varepsilon^{\sigma,\mu}[\mathcal{U}](x,t)\right| \mathbf{1}_{|\mathcal{Z}(x,t)|>\xi} \le \|\mathcal{U}\|_{L^\infty(Q_T)}\mathbf{1}_{|\mathcal{Z}(x,t)|>\xi} \in L^1(\mathbb{R}^N)$ (see [1] and also Lemma 3.13 in [5]), and hence by Step 2 with $q = \mathcal{U}(\cdot,t) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, the first integral on the right-hand side of (3.8) goes to zero as $\varepsilon \to 0^+$. Then sending $\xi \to 0^+$ in the above estimate and using Lebesgue's dominated convergence theorem in (3.7), we conclude that $h_\varepsilon(t) \to 0$ as $\varepsilon \to 0^+$ for a.e. $t \in [0,T]$.

Step 4: Deducing that $\mathcal{U} \equiv 0$. Since all terms in (3.2) are in L^2, for a.e. $t \in [0, T]$,

$$\begin{aligned} h_\varepsilon(t) &= \left(B_\varepsilon^{\sigma,\mu}[\mathcal{U}](\cdot,t), \varepsilon B_\varepsilon^{\sigma,\mu}[\mathcal{U}](\cdot,t) - \mathfrak{L}^{\sigma,\mu}[B_\varepsilon^{\sigma,\mu}[\mathcal{U}]](\cdot,t)\right)_{L^2(\mathbb{R}^N)} \\ &= \varepsilon \left\| B_\varepsilon^{\sigma,\mu}[\mathcal{U}](\cdot,t)\right\|^2_{L^2(\mathbb{R}^N)} + \|(\mathfrak{L}^{\sigma,\mu})^{\frac{1}{2}}[B_\varepsilon^{\sigma,\mu}[\mathcal{U}]](\cdot,t)\|^2_{L^2(\mathbb{R}^N)}. \end{aligned} \tag{3.9}$$

By the conclusion of Step 3 and since all terms in the last equality of (3.9) are nonnegative, they must all converge to zero as $\varepsilon \to 0^+$. Hence, the following integrals also converge to zero for all $\psi \in C_c^\infty(\mathbb{R}^N)$,

$$\left|\int_{\mathbb{R}^N} B_\varepsilon^{\sigma,\mu}[\mathcal{U}]\mathfrak{L}^{\sigma,\mu}[\psi]\,\mathrm{d}x\right| = \left|\int_{\mathbb{R}^N} (\mathfrak{L}^{\sigma,\mu})^{\frac{1}{2}}[B_\varepsilon^{\sigma,\mu}[\mathcal{U}]](\mathfrak{L}^{\sigma,\mu})^{\frac{1}{2}}[\psi]\,\mathrm{d}x\right| \le \|(\mathfrak{L}^{\sigma,\mu})^{\frac{1}{2}}[B_\varepsilon^{\sigma,\mu}[\mathcal{U}]]\|_{L^2}\|(\mathfrak{L}^{\sigma,\mu})^{\frac{1}{2}}[\psi]\|_{L^2},$$

and $\left|\int_{\mathbb{R}^N} \varepsilon B_\varepsilon^{\sigma,\mu}[\mathcal{U}]\psi\,\mathrm{d}x\right| \le \|\varepsilon B_\varepsilon^{\sigma,\mu}[\mathcal{U}]\|_{L^2}\|\psi\|_{L^2}$. We thus conclude the proof by noting that $\mathcal{U} = \varepsilon B_\varepsilon^{\sigma,\mu}[\mathcal{U}] - \mathfrak{L}^{\sigma,\mu}[B_\varepsilon^{\sigma,\mu}[\mathcal{U}]] \to 0$ in $\mathcal{D}'(\mathbb{R}^N)$ as $\varepsilon \to 0^+$ in $\mathcal{D}'(\mathbb{R}^N)$ for a.e. $t \in [0, T]$, that is $u - \hat{u} = \mathcal{U} = 0$ a.e. in Q_T. $\square$

Proof of Theorem 3.2. Steps 1 and 2 from the proof of Theorem 3.1 are independent of the equation itself and remain true in this case since the operator is the same. Let $w, \hat{w} \in L^\infty(\mathbb{R}^N)$ be two distributional solutions of (1.6) with right-hand side f such that both $w - f$ and $\hat{w} - f$ belong to $L^1(\mathbb{R}^N)$. Define $\mathcal{W} := w - \hat{w} \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\mathcal{Z} := \varphi(w) - \varphi(\hat{w}) \in L^\infty(\mathbb{R}^N)$. As before, we also define the quantity $h_\varepsilon = (\mathcal{W}, B_\varepsilon^{\sigma,\mu}[\mathcal{W}])$. Since w and $\hat{w}$ are distributional solutions of (1.6), we have that (see Step 3 in the proof of Theorem 3.1)

$$\int_{\mathbb{R}^N} \mathcal{W} B_\varepsilon^{\sigma,\mu}[\gamma]\,\mathrm{d}x = \int_{\mathbb{R}^N} \mathcal{Z}\mathfrak{L}^{\sigma,\mu}[B_\varepsilon^{\sigma,\mu}[\gamma]]\,\mathrm{d}x = \int_{\mathbb{R}^N} \mathcal{Z}\left(\varepsilon B_\varepsilon^{\sigma,\mu}[\gamma] - \gamma\right)\mathrm{d}x \qquad \text{for all} \qquad \gamma \in C_c^\infty(\mathbb{R}^N). \tag{3.10}$$

In fact, γ can be replaced by $\mathcal{W}$ in (3.10) by the density of $C_c^\infty(\mathbb{R}^N)$ in $L^1(\mathbb{R}^N)$ and the estimate $\varepsilon\|B_\varepsilon^{\sigma,\mu}[\gamma] - B_\varepsilon^{\sigma,\mu}[\mathcal{W}]\|_{L^1(\mathbb{R}^N)} = \varepsilon\|B_\varepsilon^{\sigma,\mu}[\gamma - \mathcal{W}]\|_{L^1(\mathbb{R}^N)} \le \|\gamma - \mathcal{W}\|_{L^1(\mathbb{R}^N)}$. Then $h_\varepsilon = \int_{\mathbb{R}^N} \mathcal{Z}\left(\varepsilon B_\varepsilon^{\sigma,\mu}[\mathcal{W}] - \mathcal{W}\right)\mathrm{d}x$ goes to zero as $\varepsilon \to 0^+$, like in (3.8). The rest of the proof follows as in the proof of Theorem 3.1, by replacing $\mathcal{U}$ by $\mathcal{W}$ and dropping the t dependence of h_ε. $\square$

4. Ideas on how to prove Theorem 2.3

4.1. Existence and a priori estimates via numerical approximations

Once the uniqueness given by Theorem 3.1 is available, it is possible to provide (1.1)–(1.2) with existence and suitable a priori estimates for initial data $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ – see (a), (b)(i), (b)(ii), (b)(iii) with $p = \{1, \infty\}$, and (b)(vi) of Theorem 2.3. This task is one of the objectives of [3]. A crucial idea is the fact that the class of operators given by (1.5) with μ satisfying (A_μ) is so general that it includes many monotone discretizations of the more general operator $\mathfrak{L}^{\sigma,\mu}$. In this way, we can formulate a numerical method for (1.1)–(1.2): Choose $x_\beta = h\beta, t_j = kj$ for $\beta \in \mathbb{Z}^N$, $j \in \mathbb{N}$, and $h, k > 0$, and consider

$$U^j(x_\beta) = U^{j-1}(x_\beta) + k\Big(\mathcal{L}^{\nu_1^h}[\varphi(U^j)](x_\beta) + \mathcal{L}^{\nu_2^h}[\varphi_2^h(U^{j-1})](x_\beta) + G^j(x_\beta)\Big), \tag{4.1}$$

where $\mathcal{L}^{\nu_1^h}$ and $\mathcal{L}^{\nu_2^h}$ are discretizations of $\mathfrak{L}^{\sigma,\mu}$, $\nu_1^h(\mathbb{R}^N), \nu_2^h(\mathbb{R}^N) < \infty$, φ_2^h approximate φ, G^j is a time average of g, and U^0 is defined as a space average of u_0. In fact, if we extend (4.1) to all $\mathbb{R}^N$, the numerical method can be seen, at every time step, as a nonlinear and nonlocal elliptic equation of the form (1.6) with $w = U^j$, $\mathfrak{L}^{\sigma,\mu} = k\mathcal{L}^{\nu_1^h}$ and $f = U^{j-1} + k\big(\mathcal{L}^{\nu_2^h}[\varphi_2^h(U^{j-1})] + G^j\big)$. In this way, we can study the properties of the numerical scheme (4.1) by studying the nonlinear equation (1.6) and iterating in time. This leads to the corresponding discrete time version of the above-mentioned estimates. Since approximation, stability and compactness will be used to deduce such results, uniqueness of distributional solutions of (1.6) – that is, Theorem 3.2 – plays a crucial role. By passing to the limit (up to subsequences) as $h, k \to 0^+$, we get the continuous time estimates and also the existence of $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ distributional solutions to the parabolic problem. Furthermore, the uniqueness result given by Theorem 3.1 ensures that the full sequence of numerical solutions converges to the unique distributional solution of (1.1)–(1.2).

4.2. Energy estimates and conservation of mass

A trivial adaptation of the results and proofs presented by Corollary 2.18 and Theorems 2.19 and 2.21 in [4] (where the case $\mathfrak{L}^{\sigma,\mu} = \mathcal{L}^\mu$ is covered) shows that for solutions $u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0, T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$ of (1.1)–(1.2) the concepts of distributional and energy solutions are equivalent, and the estimates (b)(iv) and (b)(v) of Theorem 2.3 hold. As a consequence of Theorem 2.3 (b)(iv), we also obtain (b)(iii) with $p \in (1, \infty)$ by Hölder and Grönwall inequalities. In the present setting, we must ensure the convergence of the local part of the energy, which is done using the discretization (3.5), summation by parts, and Theorem 2.3 (b)(v):

$$\overline{\mathcal{E}}_{0,\nu_\sigma^h}[\varphi(u(\cdot,t))] = -\int_{\mathbb{R}^N} \varphi(u)\mathcal{L}^{\nu_\sigma^h}[\varphi(u)]\,\mathrm{d}x = \sum_{i=1}^{P}\int_{\mathbb{R}^N} \left|\frac{\varphi(u(x+h\sigma_i,t))-\varphi(u(x,t))}{h}\right|^2 \mathrm{d}x \leq K,$$

where $K = K(\varphi, u_0, g)$ is a constant. Since the difference quotients of $\varphi(u)$ are uniformly bounded, the weak derivative $\partial_{\sigma_i}\varphi(u)$ exists in L^2, and a standard argument (like in Section 4 in [4]) shows the convergence of the local part of the energy). To conclude, we obtain conservation of mass by following the proof of Theorem 2.10 in [5]. Note that neither the local term nor the right-hand side g add any extra difficulty to the proof. See Remark 2.11 in [5] for the optimality of the condition on φ.

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