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Functional analysis

Almost uniform convergence in the noncommutative Dunford–Schwartz ergodic theorem

Convergence presque uniforme dans le théorème ergodique de Dunford–Schwartz non commutatif

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A R T I C L E   I N F O

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A B S T R A C T

This article gives an affirmative solution to the problem whether the ergodic Cesáro averages generated by a positive Dunford–Schwartz operator in a noncommutative space $L^p(\mathcal{M},\tau)$, $1 \le p < \infty$, converge almost uniformly (in Egorov's sense). This problem goes back to the original paper of Yeadon [21], published in 1977, where bilaterally almost uniform convergence of these averages was established for $p = 1$.

R É S U M É

Cette Note donne une réponse positive à la question suivante : les moyennes de Cesáro ergodiques engendrées par un opérateur de Dunford–Schwartz dans un espace non commutatif $L^p(\mathcal{M},\tau)$, $1 \le p < \infty$, convergent-elles presque uniformément (au sens d'Egorov) ? Ce problème remonte au texte original de Yeadon [21], publié en 1977, dans lequel la convergence presque uniforme bilatérale de ces moyennes est établie pour $p = 1$.

1. Introduction

Almost uniform (a.u.) convergence in Egorov's sense in a von Neumann algebra $\mathcal{M}$ was considered by Lance [9], where a breakthrough noncommutative individual ergodic theorem was established for a positive state preserving the automorphism of $\mathcal{M}$. Later, Lance's result was generalized, while the proofs were simplified; see [8,4,6].

For a semifinite von Neumann algebra $\mathcal{M}$ with a faithful normal semifinite trace τ, Yeadon [21] introduced the so-called bilaterally almost uniform (b.a.u.) convergence in Egorov's sense to prove a noncommutative individual ergodic theorem for a positive Dunford–Schwartz operator in the space $L^1(\mathcal{M},\tau)$ of τ-integrable operators affiliated with $\mathcal{M}$.

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Since b.a.u. convergence is generally weaker than a.u. convergence, serious attempts have been made to show that there is a.u. convergence in Yeadon's seminal result. But the problem persisted, and a significant number of noncommutative individual ergodic theorems concerning the b.a.u. convergence of ergodic averages have been established; see, for example, [15,5,3,11,7,18,12,2].

It was derived in [7] that if $1 < p < 2$ ($2 \le p < \infty$), then for a positive Dunford–Schwartz operator in a noncommutative space $L^p(\mathcal{M}, \tau)$, the corresponding ergodic Cesáro averages converge b.a.u. (respectively, a.u.). Later, in [10] (see also [2]), it was shown that this result can be obtained directly from Yeadon's maximal inequality for $L^1(\mathcal{M}, \tau)$ established in [21]. In particular, it was shown that a.u. convergence for $p \ge 2$ follows easily due to Kadison's inequality. But the case $1 \le p < 2$ still remained open.

The aim of this article is to prove that there is a.u. convergence for all $1 \le p < \infty$, which is given in Theorem 2.3. Note that this result was not known even for a finite trace. The main finding of the article is Proposition 3.2, where the matrix $\{e_{k,n}\}$ of projections in $\mathcal{M}$ is constructed. Also, the notion of (bilaterally) uniform equicontinuity in measure at zero of a family of maps from a normed space into the space of τ-measurable operators (see [1,10]) plays an important role.

2. Preliminaries

Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ. Let $\mathcal{P}(\mathcal{M})$ stand for the set of projections in $\mathcal{M}$. If $\mathbf{1}$ is the identity of $\mathcal{M}$ and $e \in \mathcal{P}(\mathcal{M})$, we write $e^\perp = \mathbf{1} - e$. Denote by $L^0 = L^0(\mathcal{M}, \tau)$ the $*$-algebra of τ-measurable operators affiliated with $\mathcal{M}$. Let $\|\cdot\|_\infty$ be the uniform norm in $\mathcal{M}$. Equipped with the *measure topology* given by the system

$$V(\epsilon, \delta) = \{x \in L^0 : \ \|xe\|_\infty \le \delta \ \text{ for some } \ e \in \mathcal{P}(\mathcal{M}) \ \text{ with } \ \tau(e^\perp) \le \epsilon\},$$

$\epsilon > 0$, $\delta > 0$, L^0 is a complete metrizable topological $*$-algebra [14].

Let $L^p = L^p(\mathcal{M}, \tau)$, $1 \le p \le \infty$, ($L^\infty(\mathcal{M}, \tau) = \mathcal{M}$) be the noncommutative L^p-space associated with $(\mathcal{M}, \tau)$.

For detailed accounts on the spaces $L^p(\mathcal{M}, \tau)$, $p \in \{0\} \cup [1, \infty)$, see [17,20,16].

Denote by $\|\cdot\|_p$ the standard norm in the space L^p, $1 \le p \le \infty$. A linear operator $T : L^1 + \mathcal{M} \to L^1 + \mathcal{M}$ is called a *Dunford–Schwartz operator* if

$$\|T(x)\|_1 \le \|x\|_1 \ \ \forall\, x \in L^1 \ \text{ and } \ \|T(x)\|_\infty \le \|x\|_\infty \ \ \forall\, x \in \mathcal{M}.$$

If a Dunford–Schwartz operator T is positive, that is, $T(x) \ge 0$ whenever $x \ge 0$, we will write $T \in DS^+$.

Note that, by [7, Lemma 1.1], any $T \in DS^+$ can be uniquely extended to a positive linear contraction (also denoted by T) in L^p, $1 \le p < \infty$.

Given $T \in DS^+$ and $x \in L^1 + \mathcal{M}$, denote

$$A_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k(x), \ \ n = 1, 2, \ldots, \tag{1}$$

the corresponding Cesáro ergodic averages of the operator x.

Definition 2.1. A sequence $\{x_n\} \subset L^0$ is said to converge *almost uniformly (a.u.) (bilaterally almost uniformly (b.a.u.))* to $\widehat{x} \in L^0$ if, for any given $\epsilon > 0$, there is a projection $e \in \mathcal{P}(\mathcal{M})$ such that $\tau(e^\perp) \le \epsilon$ and $\|(\widehat{x} - x_n)e\|_\infty \to 0$ (respectively, $\|e(\widehat{x} - x_n)e\|_\infty \to 0$).

Remark 2.1. A.u. convergence clearly implies b.a.u. convergence. Moreover, unless $\mathcal{M}$ is of type I, a.u. convergence is strictly stronger than b.a.u. convergence; see [13, Theorems 3.3.7, 3.3.17].

The following groundbreaking result was established in [21] as a corollary of a noncommutative maximal ergodic inequality given there in Theorem 1.

Theorem 2.1. *Let $T \in DS^+$ and $x \in L^1$. Then the averages (1) converge b.a.u. to some $\widehat{x} \in L^1$.*

Remark 2.2. As it was noticed in [2, Remark 1.2] (see also [7, Lemma 1.1]), the class of iterating operators α that was considered in [21] coincides, modulo unique extensions, with the class of positive Dunford–Schwartz operators T that was dealt with in [7].

In [7, Corollary 6.4], Theorem 2.1 was extended to the noncommutative L^p-spaces, $1 < p < \infty$, as follows (see also [10, Theorems 4.3, 4.4] and [2, proof of Theorem 1.5]).

Theorem 2.2. *Let $T \in DS^+$ and $x \in L^p$, $1 \le p < \infty$. Then the averages (1) converge to some $\widehat{x} \in L^p$ b.a.u. for $1 < p < 2$ and a.u. for $2 \le p < \infty$.*

Our goal is to prove that the averages (1) converge almost uniformly for all $1 \le p < \infty$:

Theorem 2.3. *Let $T \in DS^+$ and $1 \le p < \infty$. Given $x \in L^p$, the averages (1) converge a.u. to some $\widehat{x} \in L^p$.*

3. Proof of Theorem 2.3

Let $\{e_i\}_{i\in I} \subset \mathcal{P}(\mathcal{M})$. Denote by $\bigvee_{i\in I} e_i$ the projection on the subspace $\overline{\sum_{i\in I} e_i\mathcal{H}}$, and let $\bigwedge_{i\in I} e_i$ stand for the projection on the subspace $\bigcap_{i\in I} e_i\mathcal{H}$. $\mathcal{P}(\mathcal{M})$ is a complete lattice since $l.u.b.\ \{e_i\}_{i\in I} = \bigvee_{i\in I} e_i \in \mathcal{P}(\mathcal{M})$ whenever $\{e_i\}_{i\in I} \subset \mathcal{P}(\mathcal{M})$. Besides, a normal trace τ on $\mathcal{M}$ is countably subadditive, that is, given $\{e_n\}_{n=1}^\infty \subset \mathcal{P}(\mathcal{M})$, we have $\tau\left(\bigvee_{n=1}^\infty e_n\right) \le \sum_{n=1}^\infty \tau(e_n)$.

Definition 3.1. A sequence of maps $M_n : L^p \to L^0$ is called *bilaterally uniformly equicontinuous in measure (b.u.e.m.) at zero* if for every $\epsilon > 0$ and $\delta > 0$ there exists $\gamma > 0$ such that, given $x \in L^p$ with $\|x\|_p < \gamma$, there is a projection $e \in \mathcal{P}(\mathcal{M})$ satisfying conditions

$$\tau(e^\perp) \le \epsilon \quad \text{and} \quad \sup_n \|eM_n(x)e\|_\infty \le \delta.$$

Remark 3.1. It is easy to see [10, Proposition 1.1] that, in the commutative case, bilaterally uniform equicontinuity in measure at zero of a sequence $M_n : X \to L^0$ is equivalent to the continuity in measure at zero of the maximal operator

$$M^*(f) = \sup_n |M_n(f)|, \ \ f \in X.$$

The next property was noticed in [10, Corollary 2.1, Proposition 4.2].

Proposition 3.1. *The sequence $\{A_n\}$ given by (1) is b.u.e.m. at zero on L^p for every $1 \le p < \infty$.*

Remark 3.2. Proposition 3.1 can be easily seen from the maximal inequalities given in [21, Theorem 1] (for $p = 1$) and [2, Remark 2.2] (note [10, Lemma 4.1]) (for $1 < p < \infty$).

A proof of the following technical lemma can be found in [1, Lemma 1.6].

Lemma 3.1. *Let $(X, +)$ be a semigroup, and let $M_n : X \to L^0$ be a sequence of additive maps. Assume that $x \in X$ is such that, for every $\epsilon > 0$, there exist a sequence $\{x_k\} \subset X$ and a projection $e \in \mathcal{P}(\mathcal{M})$ satisfying conditions*
(i) $\{M_n(x + x_k)\}$ converges a.u. as $n \to \infty$ for each k;
(ii) $\tau(e^\perp) \le \epsilon$;
(iii) $\sup_n \|M_n(x_k)e\|_\infty \to 0$ as $k \to \infty$.
Then the sequence $\{M_n(x)\}$ converges a.u.

Proposition 3.2. *Let $1 \le p < \infty$, and let $\{A_n\}$ be given by (1). Then the set*

$$\mathcal{C} = \{x \in L^p : \ \{A_n(x)\} \text{ converges } a.u.\}$$

is closed in L^p.

Proof. Let a sequence $\{z_m\} \subset \mathcal{C}$ and $x \in X$ be such that $\|z_m - x\|_p \to 0$. Denote $y_m = z_m - x$ and fix $\epsilon > 0$.

Show first that for any positive integers n and k, there are projections $e_{n,k} \in \mathcal{P}(\mathcal{M})$ and a sequence $\{x_k\} \subset \{y_m\}$ such that

$$\tau(e_{n,k}^\perp) \le \frac{\epsilon}{2^{n+k}} \quad \text{and} \quad \|A_n(x_k)e_{n,k}\|_\infty \le \frac{1}{k} \quad \text{for all } \ n, k.$$

Fix n and k. Since $\|y_m\|_p \to 0$ and $\{A_n\}$, by Proposition 3.1, is b.u.e.m. at zero in L^p, there exist $x_k \in \{y_m\}$ and $g_{n,k} \in \mathcal{P}(\mathcal{M})$ such that

$$\tau(g_{n,k}^\perp) \le \frac{\epsilon}{2^{n+k+1}} \quad \text{and} \quad \sup_m \|g_{n,k}A_m(x_k)g_{n,k}\|_\infty \le \frac{1}{k}.$$

Let $\mathbf{l}(y)$ ($\mathbf{r}(y)$) be the left (respectively, right) support of an operator $y \in L^0$. Set $q_{n,k} = \mathbf{1} - \mathbf{r}(g_{n,k}^\perp A_n(x_k))$. Since for any $y \in L^0$ the projections $\mathbf{l}(y) \in \mathcal{P}(\mathcal{M})$ and $\mathbf{r}(y) \in \mathcal{P}(\mathcal{M})$ are equivalent [19, 9.29], it follows that

$$\tau(q_{n,k}^{\perp}) = \tau(\mathbf{r}(g_{n,k}^{\perp} A_n(x_k))) = \tau(\mathbf{l}(g_{n,k}^{\perp} A_n(x_k))) \le \tau(g_{n,k}^{\perp}) \le \frac{\epsilon}{2^{n+k+1}}.$$

Also,

$$A_n(x_k)q_{n,k} = g_{n,k}A_n(x_k)q_{n,k} + g_{n,k}^{\perp}A_n(x_k)q_{n,k} = g_{n,k}A_n(x_k)q_{n,k}.$$

Therefore, letting $e_{n,k} = g_{n,k} \wedge q_{n,k}$, we obtain $\tau(e_{n,k}^{\perp}) \le \frac{\epsilon}{2^{n+k}}$ and

$$A_n(x_k)e_{n,k} = A_n(x_k)q_{n,k}e_{n,k} = g_{n,k}A_n(x_k)q_{n,k}e_{n,k} = g_{n,k}A_n(x_k)g_{n,k}e_{n,k},$$

implying

$$\|A_n(x_k)e_{n,k}\|_\infty \le \|g_{n,k}A_n(x_k)g_{n,k}\|_\infty \le \frac{1}{k}$$

for all positive integers n, k.

If we put $e_k = \bigwedge_n e_{n,k}$, then we have

$$\tau(e_k^{\perp}) \le \frac{\epsilon}{2^k} \quad \text{and} \quad \sup_n \|A_n(x_k)e_k\|_\infty \le \frac{1}{k} \quad \text{for all } k.$$

Since $x + x_k \in \mathcal{C}$, it follows that the sequence $\{A_n(x + x_k)\}$ converges a.u. for each k. In addition, if $e = \bigwedge_k e_k$, then $\tau(e^{\perp}) \le \epsilon$ and $\sup_n \|A_n(x_k)e\|_\infty \le \frac{1}{k} \to 0$. This, by Lemma 3.1, implies that $x \in \mathcal{C}$, and we conclude that $\mathcal{C}$ is closed in L^p. □

Now we can finish proof of Theorem 2.3.

Proof. It is well known (see, for example, [2, Proof of Theorem 1.5]) that the sequence $\{A_n(x)\}$ converges a.u. whenever $x \in L^2$. Therefore, since the set $L^p \cap L^2$ is dense in L^p, Proposition 3.2 guarantees that the averages $A_n(x)$ converge a.u. for each $x \in L^p$ (to some $\widehat{x} \in L^0$), hence we also have $A_n(x) \to \widehat{x}$ in measure. As each A_n is a contraction in L^p the unit ball of which is closed in measure topology, we conclude that $\widehat{x} \in L^p$. □

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