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Complex analysis

Second Hankel determinant for close-to-convex functions

Deuxième déterminant de Hankel pour les fonctions presque convexes

Dorina Răducanu [a], Paweł Zaprawa [b]

[a] *Faculty of Mathematics and Computer Science, Transilvania University of Braşov, Iuliu Maniu 50, 500091 Braşov, Romania*
[b] *Faculty of Mechanical Engineering, Department of Mathematics, Lublin University of Technology, Nadbystrzycka 38D, 20-618 Lublin, Poland*

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ABSTRACT

So far, the sharp bound of the expression $|a_2a_4 - a_3{}^2|$ for the class $\mathcal{C}$ of close-to-convex functions has remained unknown. In this paper, we obtain the estimation of this expression, called the second Hankel determinant, for $\mathcal{C}_0$, i.e. the subset of $\mathcal{C}$ consisting of functions f that satisfy in the unit disk the inequality $\operatorname{Re}\big(zf'(z)/g(z)\big) > 0$ with a starlike function g.
Moreover, some remarks on the second Hankel determinant for the class $\mathcal{S}$ of univalent functions are made. It is proven that $\max\{|a_2a_4 - a_3{}^2| : f \in \mathcal{S}\}$ is greater than 1.

RÉSUMÉ

Aucune estimation précise de l'expression $|a_2a_4 - a_3{}^2|$ pour la classe $\mathcal{C}$ des fonctions presque convexes n'était connue jusqu'à présent. Dans cette Note, nous présentons des estimations de cette expression, nommée deuxième déterminant de Hankel pour la classe $\mathcal{C}_0$, c'est-à-dire la sous-classe $\mathcal{C}$, composée des fonctions f qui vérifient, dans le disque unité, l'inégalité $\operatorname{Re}\big(zf'(z)/g(z)\big) > 0$ avec une fonction étoilée g.
De plus, nous formulons quelques remarques à propos du deuxième déterminant de Hankel pour la classe $\mathcal{S}$ des fonctions univalentes. Nous démontrons que $\max\{|a_2a_4 - a_3{}^2| : f \in \mathcal{S}\}$ est plus grand que 1.

1. Introduction

Let $\mathcal{A}$ denote the family of all analytic functions f in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0$, $f'(0) = 1$. Hence the functions in $\mathcal{A}$ are of the form

$$f(z) = z + a_2z^2 + a_3z^3 + \dots \, . \tag{1}$$

E-mail addresses: draducanu@unitbv.ro (D. Răducanu), p.zaprawa@pollub.pl (P. Zaprawa).

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The Hankel determinant for a given function f of the form (1) is defined as follows

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \dots & \dots & \dots & \dots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix},$$

where n, q are fixed positive integers.

The investigations of Hankel determinants for various classes of analytic functions started in the 1960s. It was Pommerenke [19], [20] who first studied Hankel's determinant for the class $\mathcal{S}$ of univalent functions given by (1). He proved for functions in $\mathcal{S}$ that $|H_q(n)| < Kn^{-(1/2+\beta)q+3/2}$, where $n, q \in \mathbb{N}$, $q \geq 2$, $\beta > 1/4000$ and K depends only on q. Similar findings, but for different classes, were reported by Hayman [6] and Noor [17], [18].

Many recent papers have been devoted to the problem of finding the exact bounds of $|H_q(n)|$ for various subfamilies of $\mathcal{A}$. The majority of the results were obtained for $H_2(2) = a_2a_4 - a_3{}^2$, which is called the second Hankel determinant (see, for example, [1], [7], [8], [13], [22], [23]). There are, however, few papers that discuss the third Hankel determinant $H_3(1)$ (see, for example: [2], [21], [24]). Although many estimates of $|H_2(2)|$ are sharp, for example for the classes $\mathcal{S}^*$ or $\mathcal{K}$ consisting of starlike or convex functions, respectively, the exact bound of $|H_2(2)|$ for $\mathcal{S}$ or for the class $\mathcal{C}$ of close-to-convex functions is still not known.

In this paper, we focus our discussion on $\mathcal{C}$. It is known (see [5]) that $f \in \mathcal{C}$ if there exist a starlike function g and a real number $\beta \in (-\pi/2, \pi/2)$ such that

$$\operatorname{Re}\left(e^{i\beta} zf'(z)/g(z)\right) > 0\,. \tag{2}$$

We distinguish subclasses of $\mathcal{C}$ according to a fixed number β. Namely, a function f of the form (1) is called close to convex with argument β if there exists $g \in \mathcal{S}^*$ such that the condition (2) holds. Let $\mathcal{C}_\beta$ denote the class of all such functions. It is obvious that

$$\mathcal{C} = \bigcup_{\beta \in (-\pi/2, \pi/2)} \mathcal{C}_\beta\,.$$

Taking into account (2), we can write

$$e^{i\beta} zf'(z)/g(z) = p(z)\cos\beta + i\sin\beta\,, \tag{3}$$

with $p \in \mathcal{P}$, where $\mathcal{P}$ is the well-known class of functions with positive real part that are normalized by $p(0) = 1$.

If $g \in \mathcal{S}^*$ and $p \in \mathcal{P}$ in (3) are given by

$$g(z) = z + b_2z^2 + b_3z^3 + \dots \tag{4}$$

and

$$p(z) = 1 + p_1z + p_2z^2 + \dots\,, \tag{5}$$

then

$$z + \sum_{n=2}^{\infty} na_nz^n = \left(z + \sum_{n=2}^{\infty} b_nz^n\right)\left(1 + e^{-i\beta}\cos\beta \sum_{n=1}^{\infty} p_nz^n\right). \tag{6}$$

Therefore,

$$na_n = b_n + e^{-i\beta}\cos\beta \left(p_{n-1} + \sum_{j=2}^{n-1} b_jp_{n-j}\right),\ n \geq 2\,. \tag{7}$$

If $n = 2$, then the sum in the parentheses vanishes.

It is clear that the maximum of $|H_2(2)|$ while f varies in the whole class $\mathcal{S}$ or $\mathcal{C}$ is greater than or equal to 1 because of the result of Janteng et al. [7]. The estimation of $|H_2(2)|$ for the functions f given by (1) belonging to $\mathcal{C}$ is difficult to obtain, because it involves the coefficients of both functions $g \in \mathcal{S}^*$, $p \in \mathcal{P}$ and a constant β (see, Remark 3 in [15]). For this reason, it is somewhat easier to estimate the second Hankel determinant if $\beta = 0$, i.e. in the class $\mathcal{C}_0$. Even for $\mathcal{C}_0$, the known bounds of $|H_2(2)|$ are not sharp. The best known result (excluding erroneous ones) was obtained by Prajapat et al. in [21]. They proved that $|H_2(2)| \leq 85/36 = 2.361\dots$ in $\mathcal{C}_0$. In Theorem 1, we essentially improve this result. Moreover, we discuss an example of univalent functions that shows that the maximum of $|H_2(2)|$ for $\mathcal{S}$ is actually greater than 1.

2. Preliminary results

At the beginning, let us discuss the invariance property of the class $\mathcal{C}$.

Let f be given by (1) and let

$$f_\varphi(z) = \mathrm{e}^{-\mathrm{i}\varphi} f(z\mathrm{e}^{\mathrm{i}\varphi}), \varphi \in \mathbb{R} . \tag{8}$$

Directly from the definition of a close-to-convex function, it follows that $f \in \mathcal{C}$ if and only if $f_\varphi \in \mathcal{C}$. The same remains true if we replace $\mathcal{C}$ by $\mathcal{S}^*$ or $\mathcal{S}$. Moreover, we can prove the following lemma.

Lemma 1. *The equivalence*

$$f \in \mathcal{C}_\beta \Leftrightarrow f_\varphi \in \mathcal{C}_\beta$$

holds for every $\varphi \in \mathbb{R}$ and a fixed $\beta \in (-\pi/2, \pi/2)$.

Proof. If f_φ is in $\mathcal{C}_\beta$ for every $\varphi \in \mathbb{R}$, so it is true also for $\varphi = 0$. For this reason, it is enough to prove only that $f \in \mathcal{C}_\beta \Rightarrow f_\varphi \in \mathcal{C}_\beta$. But for f in $\mathcal{C}_\beta$, there exists $g \in \mathcal{S}^*$ such that (2) holds. Writing $z\mathrm{e}^{\mathrm{i}\varphi}$ instead of z in (2), we obtain

$$\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\beta} z f'(z\mathrm{e}^{\mathrm{i}\varphi})/\mathrm{e}^{-\mathrm{i}\varphi} g(z\mathrm{e}^{\mathrm{i}\varphi})\right) > 0 , \tag{9}$$

which means that $f_\varphi \in \mathcal{C}_\beta$ with $\mathrm{e}^{-\mathrm{i}\varphi} g(z\mathrm{e}^{\mathrm{i}\varphi})$ as a starlike function. □

Suppose that a given class A of analytic functions is invariant under rotation. Let $f \in A$ be given by (1) and $f_\varphi(z) = z + \alpha_2 z^2 + \ldots$ is defined by (8). Hence,

$$|\alpha_2\alpha_4 - \mu{\alpha_3}^2| = \left|a_2\mathrm{e}^{\mathrm{i}\varphi} \cdot a_4\mathrm{e}^{3i\varphi} - \mu \cdot \left(a_3\mathrm{e}^{2i\varphi}\right)^2\right| = |a_2a_4 - \mu{a_3}^2| . \tag{10}$$

For this reason (or applying a similar argument), we have the following lemma.

Lemma 2. *If A is one of the classes: $\mathcal{C}, \mathcal{C}_\beta, \mathcal{S}^*, \mathcal{S}$ and $\Phi(f)$ is one of the following functionals: $|a_2a_4 - \mu{a_3}^2|$, $|a_4 - \mu a_2a_3|$, $|a_3 - \mu{a_2}^2|$ defined on $f \in A$ given by (1) with a fixed real number μ. Then $\Phi(f) = \Phi(f_\varphi)$ for every $\varphi \in \mathbb{R}$.*

To prove the main results, we need a few lemmas. The first one is by Libera and Złotkiewicz.

Lemma 3. [14] *Let $p_1 \in [0, 2]$. A function p given by (5) belongs to $\mathcal{P}$ if and only if*

$$2p_2 = {p_1}^2 + x(4 - {p_1}^2)$$

and

$$4p_3 = {p_1}^3 + 2p_1(4 - {p_1}^2)x - p_1(4 - {p_1}^2)x^2 + 2(4 - {p_1}^2)(1 - |x|^2)z$$

for some x and z such that $|x| \le 1$, $|z| \le 1$.

Let $g \in \mathcal{S}^*$ be given by (4). Applying the correspondence between functions in $\mathcal{S}^*$ and $\mathcal{P}$

$$\frac{zg'(z)}{g(z)} = q(z) \quad , \quad g \in \mathcal{S}^* , \, q \in \mathcal{P} \tag{11}$$

we get

$$(n-1)b_n = \sum_{j=1}^{n-1} b_j q_{n-j} \quad , \quad n = 2, 3, \ldots \tag{12}$$

where $q(z) = 1 + q_1 z + q_2 z^2 + \ldots$.

In particular,

$$b_2 = q_1 \, , \; b_3 = \tfrac{1}{2}(q_2 + {q_1}^2) \, , \; b_4 = \tfrac{1}{3}(q_3 + \tfrac{3}{2}q_1q_2 + \tfrac{1}{2}{q_1}^3) . \tag{13}$$

Lemma 4. *If $g \in \mathcal{S}^*$ is given by (4) and $\mu \in \mathbb{R}$, then*

$$|b_3 - \mu {b_2}^2| \le \begin{cases} 1 + (1/2 - \mu)|b_2|^2 & \text{for } \mu \le 3/4\,, \\ 1 + (\mu - 1)|b_2|^2 & \text{for } \mu \ge 3/4\,. \end{cases} \tag{14}$$

Proof. From (13) we get

$$b_3 - \mu {b_2}^2 = (1/2 - \mu){q_1}^2 + q_2/2\,.$$

By Lemma 2, we can assume that $q_1 \in [0, 2]$. Applying Lemma 3,

$$b_3 - \mu {b_2}^2 = (3/4 - \mu){q_1}^2 + (4 - {q_1}^2)y/4\,,\ \text{for some } y, |y| \le 1\,;$$

hence we obtain (14). □

As a simple consequence of Lemma 4, we get the well-known Fekete–Szegö inequality $|b_3 - \mu {b_2}^2| \le \max\{1, |4\mu - 3|\}$ for $\mathcal{S}^*$.

Lemma 5. *If $g \in \mathcal{S}^*$ is given by (4), then*

$$|b_4 - \tfrac{7}{9} b_2 b_3| \le H(|b_2|)\,, \tag{15}$$

where

$$H(b) = \begin{cases} \frac{1}{3}\left(2 + \frac{7}{18}b^2 + \frac{25}{36}b^3\right) & \text{for } b \in [0, 6/7]\,, \\ \frac{1}{9}\left(11b - 2b^3\right) & \text{for } b \in [6/7, 2]\,. \end{cases} \tag{16}$$

Proof. From (13), we have

$$b_4 - \tfrac{7}{9} b_2 b_3 = \tfrac{1}{3}\left(q_3 + \tfrac{1}{3} q_1 q_2 - \tfrac{2}{3} {q_1}^3\right)\,.$$

In view of Lemma 2, we write q instead of q_1, $q \in [0, 2]$. From Lemma 3,

$$b_4 - \tfrac{7}{9} b_2 b_3 = \tfrac{1}{36}\left[-3q^3 + 8q(4 - q^2)y - 3q(4 - q^2)y^2 + 6(4 - q^2)(1 - |y|^2)z\right]\,.$$

Denoting $|y| = r$ and applying the triangle inequality, we obtain

$$|b_4 - \tfrac{7}{9} b_2 b_3| \le \tfrac{1}{36}\left[3q^3 + 8q(4 - q^2)r + 3q(4 - q^2)r^2 + 6(4 - q^2)(1 - r^2)\right]\,.$$

Let us denote the expression in square brackets in the above inequality by $h(r)$. Since $h'(r) = 0$ only for $r_0 = \frac{4q}{3(2-q)}$, we conclude that $\max\{h(r) : r \in [0, 1]\}$ is equal to $h(r_0)$ if $q \in [0, 6/7]$ and is equal to $h(1)$ if $q \in [6/7, 2]$. This completes the proof. □

Lemma 6. *If $g \in \mathcal{S}^*$ is given by (4), then*

$$|b_2 b_4 - \tfrac{8}{9} {b_3}^2| \le \tfrac{1}{9}(4 - |b_2|^2)(2 + |b_2|^2)\,. \tag{17}$$

Proof. In view of Lemma 2, we assume $q = q_1 \in [0, 2]$. From (13) and from Lemma 3

$$b_2 b_4 - \tfrac{8}{9} {b_3}^2 = \tfrac{1}{36}(4 - q^2)\left[3q^2 y - (q^2 + 8)y^2 + 6q(1 - |y|^2)z\right]\,.$$

Hence, writing $r = |y|$,

$$|b_2 b_4 - \tfrac{8}{9} {b_3}^2| \le \tfrac{1}{36}(4 - q^2)\left[3q^2 r + (q^2 + 8)r^2 + 6q(1 - r^2)\right]\,.$$

The result follows if we take $r = 1$. □

It is easy to check that $\max\{\frac{1}{9}(4 - b^2)(2 + b^2) : b \in [0, 2]\} = 1$. Therefore, the result in Lemma 6 generalizes the result obtained in [25] (Theorem 3, for $\mu = 8/9$); according to this paper, if $g \in \mathcal{S}^*$, then $|b_2 b_4 - \frac{8}{9} {b_3}^2| \le 1$.

3. Main results

Taking into account Lemma 1, we can rotate $f \in \mathcal{C}_\beta$ in such a way that after this operation the second coefficient of f is real and non-negative. But, in this case, the coefficients b_2 and p_1 are not necessarily real. From now on, we proceed in a different manner. A function $f \in \mathcal{C}_\beta$ is rotated in such a way that p_1 in formula (5) is real and non-negative. Under this assumption, we cannot expect that a_2 and b_2 are real numbers.

Now, we are ready to prove the main theorem of this paper.

Theorem 1. *If $f \in \mathcal{C}_0$ is given by (1), then*

$$|a_2a_4 - a_3{}^2| \le 1.242\ldots . \tag{18}$$

Proof. From (7) it follows for $f \in \mathcal{C}_0$ that

$$2a_2 = b_2 + p_1 \tag{19}$$

$$3a_3 = b_3 + b_2p_1 + p_2 \tag{20}$$

$$4a_4 = b_4 + b_3p_1 + b_2p_2 + p_3 . \tag{21}$$

Hence,

$$\begin{aligned} a_2a_4 - a_3{}^2 &= \tfrac{1}{8}(b_2+p_1)(b_4+b_3p_1+b_2p_2+p_3) - \tfrac{1}{9}(b_3+b_2p_1+p_2)^2 \\ &= \tfrac{1}{8}(b_2b_4 - \tfrac{8}{9}b_3{}^2) + \tfrac{1}{8}p_1(b_4 - \tfrac{7}{9}b_2b_3) + \tfrac{1}{8}(p_1p_3 - \tfrac{8}{9}p_2{}^2) \\ &\quad + \tfrac{1}{8}p_1{}^2(b_3 - \tfrac{8}{9}b_2{}^2) - \tfrac{2}{9}p_2(b_3 - \tfrac{9}{16}b_2{}^2) + \tfrac{1}{8}b_2(p_3 - \tfrac{7}{9}p_1p_2). \end{aligned}$$

Taking into account Lemma 1 and formula (9), we can assume that p_1 is a non-negative real number; for this reason we write p instead of p_1. Applying Lemma 3, we get

$$\tfrac{1}{8}p^2(b_3 - \tfrac{8}{9}b_2{}^2) - \tfrac{2}{9}p_2(b_3 - \tfrac{9}{16}b_2{}^2) \;=\; \tfrac{1}{72}p^2(b_3 - \tfrac{7}{2}b_2{}^2) - \tfrac{1}{9}(4-p^2)(b_3 - \tfrac{9}{16}b_2{}^2)x ,$$

and

$$\begin{aligned} &\tfrac{1}{8}(pp_3 - \tfrac{8}{9}p_2{}^2) = \tfrac{1}{288}p^4 + \tfrac{1}{144}p^2(4-p^2)x - \tfrac{1}{288}(4-p^2)(32+p^2)x^2 + \tfrac{1}{16}p(4-p^2)(1-|x|^2)z , \\ &\tfrac{1}{8}b_2(p_3 - \tfrac{7}{9}pp_2) = \tfrac{1}{32}b_2\left[-\tfrac{5}{9}p^3 + \tfrac{4}{9}p(4-p^2)x - p(4-p^2)x^2 + 2(4-p^2)(1-|x|^2)z\right] , \end{aligned}$$

where $|x| \le 1$ and $|z| \le 1$. Therefore,

$$\begin{aligned} a_2a_4 - a_3{}^2 &= \tfrac{1}{8}(b_2b_4 - \tfrac{8}{9}b_3{}^2) + \tfrac{1}{8}p(b_4 - \tfrac{7}{9}b_2b_3) + \tfrac{1}{72}p^2(b_3 - \tfrac{7}{2}b_2{}^2 - \tfrac{5}{4}b_2p + \tfrac{1}{4}p^2) \\ &\quad - \tfrac{1}{9}(4-p^2)\left[b_3 - \tfrac{9}{16}b_2{}^2 - \tfrac{1}{8}b_2p - \tfrac{1}{16}p^2\right]x \\ &\quad - \tfrac{1}{288}(4-p^2)(32+9b_2p+p^2)x^2 + \tfrac{1}{16}(b_2+p)(4-p^2)(1-|x|^2)z . \end{aligned}$$

Let us denote $|b_2|$ by b and $|x|$ by ϱ; hence, $b \in [0,2]$, $\varrho \in [0,1]$. The triangle inequality leads to

$$\begin{aligned} |a_2a_4 - a_3{}^2| &\le \tfrac{1}{8}\left[|b_2b_4 - \tfrac{8}{9}b_3{}^2| + p|b_4 - \tfrac{7}{9}b_2b_3|\right] + \tfrac{1}{72}p^2|b_3 - \tfrac{7}{2}b_2{}^2 - \tfrac{5}{4}b_2p + \tfrac{1}{4}p^2| \\ &\quad + \tfrac{1}{9}(4-p^2)\left|b_3 - \tfrac{9}{16}b_2{}^2 - \tfrac{1}{8}b_2p - \tfrac{1}{16}p^2\right|\varrho \\ &\quad + \tfrac{1}{288}(4-p^2)(32+9bp+p^2)\varrho^2 + \tfrac{1}{16}(b+p)(4-p^2)(1-\varrho^2) . \end{aligned}$$

Applying Lemmas 4–6, we can write

$$|a_2a_4 - a_3{}^2| \le F(p,b,\varrho) ,$$

where

$$\begin{aligned} &F(p,b,\varrho) = A + B\varrho + C\varrho^2 \quad , \quad p,b \in [0,2] ,\ \varrho \in [0,1] , \\ &C = \tfrac{1}{288}(4-p^2)(2-p)(16-p-9b) \\ &B = \tfrac{1}{144}(4-p^2)(16-b^2+2bp+p^2) \\ &A = \tfrac{1}{72}(4-b^2)(2+b^2) + \tfrac{1}{8}pH(b) + \tfrac{1}{288}p^2(4+10b^2+5bp+p^2) + \tfrac{1}{16}(b+p)(4-p^2) , \end{aligned} \tag{22}$$

and $H(b)$ is defined by (16).

Now we shall show that F is an increasing function of $\varrho \in [0,1]$. We have

$$\frac{\partial F}{\partial \varrho} = \tfrac{1}{144}(4-p^2)\left[16-b^2+2bp+p^2+(2-p)(16-p-9b)\varrho\right] .$$

If $16-p-9b \geq 0$, then $\frac{\partial F}{\partial \varrho} \geq 0$. For $16-p-9b < 0$,

$$\frac{\partial F}{\partial \varrho} \geq \tfrac{1}{144}(4-p^2)h(p,b) ,$$

where

$$h(p,b) = 48+2p^2-18p-b^2-18b+11pb .$$

It is not a difficult task to prove that $h(p,b) \geq 0$ for all $(p,b) \in [0,2]\times[0,2]$. This proves that $\frac{\partial F}{\partial \varrho} \geq 0$ in $[0,2]\times[0,2]$. Therefore,

$$F(p,b,\varrho) \leq F(p,b,1) = A+B+C . \tag{23}$$

Let us denote $F(p,b,1)$ by $G(p,b)$. Hence,

$$G(p,b) = \tfrac{1}{288}\Big[(4-p^2)(64+3p^2+13pb-2b^2)+4(4-b^2)(2+b^2) \\ +36pH(b)+p^2(4+10b^2+5bp+p^2)\Big] \quad , \quad p,b\in[0,2] . \tag{24}$$

To obtain the declared result, we divide the set of variability of (p,b), i.e. $\Omega = [0,2]\times[0,2]$ into two subsets: $\Omega_1 = [0,2]\times[0,6/7]$ and $\Omega_2 = [0,2]\times[6/7,2]$.

I. First, assume that $(p,b)\in\Omega_2$. Then $G(p,b) = \frac{1}{288}G_2(p,b)$, where

$$G_2(p,b) = -2p^4-8p^3b+12p^2b^2-48p^2-8pb^3+96pb-4b^4+288 . \tag{25}$$

Our task is to find

$$\max\{G_2(p,b) : (p,b)\in\Omega_2\} . \tag{26}$$

Instead of (26), we shall derive

$$\max\{G_2(p,b) : (p,b)\in\Omega\} . \tag{27}$$

Observe that the critical points of G_2 satisfy the following system of equations

$$\begin{cases} -p^3-3p^2b+3pb^2-12p-b^3+12b=0 \\ -p^3+3p^2b-3pb^2+12p-2b^3=0 . \end{cases} \tag{28}$$

For the point $(0,0)$, (28) is fulfilled. Assume now that $b\neq 0$. Summing both equations in (28) we obtain

$$2p^3 = 3b(4-b^2) . \tag{29}$$

Applying it in one of the equations of (28), we get

$$6bp^2+6(4-b^2)p-b(12+b^2)=0 . \tag{30}$$

Hence,

$$p = \frac{1}{6b}\left(3(b^2-4)+\sqrt{15b^4+144}\right) \tag{31}$$

is the positive solution to (30).

Combining (29) with (31), and dividing the obtained equation by b^3, we get

$$2\left[\tfrac{1}{2}\left(1-\tfrac{4}{b^2}\right)+\tfrac{1}{6}\sqrt{15+\left(\tfrac{12}{b^2}\right)^2}\right]^3 = 3\left(\tfrac{4}{b^2}-1\right) . \tag{32}$$

Substituting $t = 3(4/b^2-1)$, $t\geq 0$, equation (32) takes the form

$$2\left(-\tfrac{1}{6}t+\tfrac{1}{6}\sqrt{24+6t+t^2}\right)^3 = t , \tag{33}$$

or equivalently,

$$\sqrt{24+6t+t^2}-t=3\sqrt[3]{4t}\,. \tag{34}$$

Now, it is not difficult to show that (34) has only one positive solution. Indeed, a function $f_1(t)=\sqrt{24+6t+t^2}-t$ is decreasing and a function $f_2(t)=3\sqrt[3]{4t}$ is increasing for $t\geq 0$. Moreover, $f_1(0)=2\sqrt{6}>0=f_2(0)$ and $f_1(2)=2(\sqrt{10}-1)<6=f_2(2)$. It means that the only positive solution to (34) belongs to $(0,2)$. Its numerical value is $t_0=0.899\ldots$.

For the reason presented above, we know that (28) has exactly one critical point such that $p>0$ and $b>0$; namely,

$$p_0=1.343\ldots\quad ,\quad b_0=1.754\ldots\,, \tag{35}$$

for which $G_2(p_0,b_0)=357.819\ldots$.

On the boundary of Ω, we discuss the following cases. For $b\in[0,2]$, $G_2(0,b)=288-4b^4\leq 288$. Similarly, for $p\in[0,2]$, $G_2(p,0)=288-48p^2-2p^4\leq 288$. If $p=2$, then $G_2(2,b)=64+128b+48b^2-16b^3-4b^4$ is an increasing function because its derivative $(4+b)(1+b)(2-b)$ is greater than or equal to 0. For $p\in[0,2]$, $G_2(p,2)=224+128p-16p^3-2p^4$. The derivative of this function is equal to $8(2+p)(8-4p-p^2)$. Now, we deduce that the greatest value of $G_2(p,2)$ for $p\in[0,2]$ is equal to $G_2(2(\sqrt{3}-1),2)=352$.

Summing up,

$$\max\{G_2(p,b):(p,b)\in\Omega\}=G_2(p_0,b_0)\,. \tag{36}$$

But $(p_0,b_0)\in\Omega_2$, so

$$\max\{G_2(p,b):(p,b)\in\Omega_2\}=G_2(p_0,b_0)\,. \tag{37}$$

II. Let $(p,b)\in\Omega_1$. Then $G(p,b)=\frac{1}{288}G_1(p,b)$, where

$$G_1(p,b)=-2p^4-8p^3b-12(4-b^2)p^2+\left(\tfrac{25}{3}b^3+\tfrac{14}{3}b^2+52b+24\right)p-4b^4+288\,. \tag{38}$$

Let us denote $f_3(p,b)=-2p^4-8p^3b-4b^4+288$ and $f_4(p,b)=\left(\frac{25}{3}b^3+\frac{14}{3}b^2+52b+24\right)p-12(4-b^2)p^2$. Hence, $f_3(p,b)\leq 288$ for all $(p,b)\in\Omega_1$.

The quadratic function f_4 of the variable p takes the greatest value for

$$p_*=\frac{\frac{25}{3}b^3+\frac{14}{3}b^2+52b+24}{24(4-b^2)}\,.$$

Since $p_*\in[0,2]$ for $b\in[0,6/7]$, so

$$f_4(p,b)\leq f_4(p_*,b)=\frac{\left(\frac{25}{3}b^3+\frac{14}{3}b^2+52b+24\right)^2}{48(4-b^2)}\,.$$

If $b\in[0,6/7]$, then the last expression is increasing; consequently

$$f_4(p_*,b)\leq f_4(p_*,6/7)=38.072\ldots\,.$$

Hence, for $(p,b)\in\Omega_1$,

$$G_1(p,b)\leq 326.072\ldots<G_2(p_0,b_0)\,. \tag{39}$$

Comparing the bounds obtained for Ω_1 and Ω_2, we deduce (18). □

The result obtained in Theorem 1 is not sharp. Under the additional assumption that b_2 is real, this bound can be improved. This improved value is a little bit greater than 1, but still it is not sharp. We can pose the natural conjecture that $|H_2(2)|\leq 1$ for all functions in $\mathcal{C}_0$. The same inequality likely holds also for $\mathcal{C}$.

4. Remarks on the second Hankel determinant for univalent functions

So far, we have not found any result concerning the estimates, even rough, of the expression $|a_2a_4-{a_3}^2|$ for the whole class $\mathcal{S}$ of univalent functions. Can it be true that $|H_2(2)|\leq 1$ for $\mathcal{S}$?

Regarding the results of some coefficients problems one can find cases when the solutions to problems in $\mathcal{C}$ and $\mathcal{S}$ are the same and those when the solutions are different. For example, the bounds of $|a_n|$ or $|a_3-{a_2}^2|$ are the same for both $\mathcal{C}$ and $\mathcal{S}$ (namely: n and 1, respectively). However, the Fekete–Szegö functional $|a_3-\mu {a_2}^2|$, $\mu\in[0,1]$ is bounded by $1+2\exp\left(-\frac{2\mu}{1-\mu}\right)$ in $\mathcal{S}$ (see, [4]) and by $3-4\mu$ for $0\leq\mu\leq 1/3$, $1/3+4/9\mu$ for $1/3\leq\mu\leq 2/3$ and 1 for $2/3\leq\mu\leq 1$ in $\mathcal{C}$. The latter was obtained at first by Keogh and Merkes in [10] for $\mathcal{C}_0$, and next, by Eenigenburg and Silvia in [3] (independently by

Koepf ([11])) for the whole $\mathcal{C}$. It is worth recalling another example of a problem that has two different solutions for the two discussed classes. Namely, $\max\{||a_3|-|a_2|| : f \in \mathcal{C}\} = 1$ (Koepf, [11]) and $\max\{||a_3|-|a_2|| : f \in \mathcal{S}\} = 1.029\dots$ (Jenkins, [9]).

Consider a family of functions f_ε, $\varepsilon \in (0,1)$ that map Δ onto the sets

$$\mathbb{C} \setminus \left((-\infty, -d_\varepsilon] \cup \{d_\varepsilon e^{i\theta}, \theta_\varepsilon \le |\theta| \le \pi\} \right) . \tag{40}$$

Krzyż and Reade proved that the functions f_ε determine the Koebe set for the class $\mathcal{Y}$ of circularly symmetric univalent functions, see [12]. In [16], Netanyahu showed that the maximum in $\mathcal{S}$ of an expression $|a_2| \cdot d_f$, where $d_f = \inf\{|\gamma| : f(z) \ne \gamma, z \in \Delta\}$, is achieved by f_ε with properly taken $\varepsilon \in (0,1)$.

Given $\varepsilon \in (0,1)$, the function f_ε is obtained as a composition of a function $s(z)$ satisfying

$$\frac{s}{(1+s)^2} = \frac{4\varepsilon}{(1+\varepsilon)^2} \cdot \frac{z}{(1+z)^2} , \tag{41}$$

and a function

$$w(s) = \frac{(1+\varepsilon)^2}{4} \cdot \frac{s(1-\varepsilon s)}{\varepsilon - s} . \tag{42}$$

We have $s(\Delta) = \Delta \setminus [\varepsilon, 1)$. The numbers that appear in (40) take values:

$$d_\varepsilon = \frac{(1+\varepsilon)^2}{4} \quad \text{and} \quad \theta_\varepsilon = 2 \arccos \varepsilon . \tag{43}$$

Observe that, in the limiting case, f_1 is the identity function. Since both $s(z)$ and $w(s)$ are univalent, f_ε is also univalent. From (40) we conclude that f_ε is not close to convex.

The function $s(z)$ can be written as $s(z) = k^{-1}(Ak(z))$, with $k(z) = \frac{z}{(1+z)^2}$ and $A = \frac{4\varepsilon}{(1+\varepsilon)^2}$. Since $k^{-1}(\zeta) = \zeta + 2\zeta^2 + 5\zeta^3 + 14\zeta^4 + 42\zeta^5 + \dots$, we have

$$\frac{s(z)}{A} = z - (2-2A)z^2 + (3 - 8A + 5A^2)z^3 - (4 - 20A + 30A^2 - 14A^3)z^4 \\ + (5 - 40A + 105A^2 - 112A^3 + 42A^4)z^5 + \dots .$$

In a small neighbourhood of the origin

$$w(s) = \frac{1}{A}\left[s + \varepsilon(1-\varepsilon^2) \sum_{k=2}^{\infty} \left(\frac{s}{\varepsilon}\right)^k \right] .$$

Therefore,

$$f_\varepsilon(z) = z + \frac{2(1-\varepsilon)(1+3\varepsilon)}{(1+\varepsilon)^2} z^2 + \frac{(1-\varepsilon)(3+15\varepsilon+33\varepsilon^2-19\varepsilon^3)}{(1+\varepsilon)^4} z^3 \\ + \frac{4(1-\varepsilon)(1+7\varepsilon+18\varepsilon^2+54\varepsilon^3-59\varepsilon^4+11\varepsilon^5)}{(1+\varepsilon)^6} z^4 + \dots . \tag{44}$$

From (44) it follows that $f_1(z) = z$. Moreover, taking $\varepsilon = 0$ in (44), we obtain $f_0(z) = z + 2z^2 + \dots = \frac{z}{(1+z)^2}$. In this case, the set (40) coincides with $\mathbb{C} \setminus (-\infty, -1/4]$.

For a function (44),

$$H_2(2) = -F(\varepsilon) ,$$

where

$$F(\varepsilon) = \frac{(1-\varepsilon)^4}{(1+\varepsilon)^8} (1 + 12\varepsilon + 134\varepsilon^2 + 268\varepsilon^3 + 97\varepsilon^4) , \ \varepsilon \in [0,1] . \tag{45}$$

Therefore, $H_2(2) \le 0$ and

$$\frac{F'(\varepsilon)}{F(\varepsilon)} = \frac{128\varepsilon(1 - 6\varepsilon - 20\varepsilon^2 - 7\varepsilon^3)}{(1-\varepsilon^2)(1+12\varepsilon+134\varepsilon^2+268\varepsilon^3+97\varepsilon^4)} .$$

Denoting by ε_0 the only solution to $1 - 6\varepsilon - 20\varepsilon^2 - 7\varepsilon^3 = 0$ in $(0,1)$, i.e. $\varepsilon_0 = 0.118\dots$, we can write

$$\max\{F(\varepsilon) : \varepsilon \in [0,1]\} = F(\varepsilon_0) = 1.175\dots \tag{46}$$

We have proved the following theorem.

Theorem 2. *If f is given by (1), then*

$$\max\{|a_2 a_4 - {a_3}^2| : f \in \mathcal{S}\} \ge 1.175\dots . \tag{47}$$

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