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Topology/Differential topology

On maps which are the identity on the boundary

Sur les applications qui sont l'identité sur la frontière

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ABSTRACT

The following fact seems to have been unnoticed until now:
Let F be a closed subset of the (finite-dimensional) connected manifold M. If $f : F \to M$ is a proper continuous map which is the identity on the boundary ∂F of F in M, then either $f(F) \supset F$ or $f(F) \supset M \setminus F$.
The proof is elementary and simple using degree theory.
The statement has many deep consequences.

RÉSUMÉ

Le fait suivant ne semble pas être connu :
Soit F un sous-ensemble fermé de la variété connexe M (de dimension finie). Si $f : F \to M$ est une application continue et propre qui est l'identité sur la frontière ∂F de F dans M, alors, on a, soit $f(F) \supset F$, soit $f(F) \supset M \setminus F$.
La preuve, qui utilise la théorie du degré, est élémentaire et simple.
Ce fait a des conséquences profondes.

Version française abrégée

Le fait suivant ne semble pas être connu :

Théorème 0.1. *Soit F un sous-ensemble fermé de la variété connexe (de dimension finie) M. Si $f : F \to M$ est une application continue qui est l'identité sur la frontière ∂F de F dans M, alors, soit $f(F) \supset F$, soit $f(F) \supset M \setminus F$.*

De plus, si f est proprement homotope à l'inclusion $F \subset M$ relativement à ∂F, on a nécessairement $f(F) \supset F$.

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Rappelons que f est proprement homotope à l'inclusion $F \subset M$ relativement à ∂F, s'il existe une application $H : F \times [0,1] \to M$ continue et propre telle que $H(x,0) = x$, $H(x,1) = f(x)$ pour tout $x \in F$ et $H(x,t) = x$ pour tout $(x,t) \in \partial F \times [0,1]$.

Le résultat ci-dessus a des conséquences profondes. Nous en mentionnons quelques-unes ici.

Corollaire 0.2. *Soit F un sous-ensemble fermé, d'intérieur $\mathring{F}$ non vide, de la variété connexe M. Il n'existe pas de rétraction continue et propre de F sur sa frontière ∂F.*

Corollaire 0.3. *Soit F un sous-ensemble fermé de la variété connexe M. Si $H : F \times [0,1] \to M$ est une homotopie continue telle que $H(x,0) = x$, pour tout $x \in F$, et $H \mid F \times \{1\}$ est constante, alors on a obligatoirement une des deux possibilités suivantes :*

(i) *pour toute composante connexe relativement compacte C de $M \setminus F$, on a $H(\partial C \times [0,1]) \supset C$;*
(ii) *il existe une composante connexe relativement compacte C de $M \setminus F$ avec $H(\partial C \times [0,1]) \supset M \setminus C$.*

En particulier, on a :

(a) *si M est compacte, alors $H(F \times [0,1])$ contient toutes les composantes connexes de $M \setminus F$, sauf peut-être au plus une ;*
(b) *si M n'est pas compacte, alors $H(F \times [0,1])$ contient toutes les composantes connexes relativement compactes de $M \setminus F$.*

L'homotopie H du corollaire ci-dessus ne peut pas être propre, sauf si F est compact. Une version propre de ce corollaire est donnée par le résultat ci-dessous.

Théorème 0.4. *Soit F un sous-ensemble fermé de la variété connexe M. Si $H : F \times [0,1[\to M$ est une homotopie continue et propre telle que $H(x,0) = x$, pour tout $x \in F$, alors $H(F \times [0,1[)$ contient toutes les composantes connexes de $M \setminus F$, sauf peut-être au plus une.*

Théorème 0.5. *Soit $h : M \to M$ un homéomorphisme de la variété connexe M (de dimension finie). Si on note par $\mathrm{Fix}(h)$ l'ensemble des points fixes de h, on a :*

(i) *soit $h(C) = C$ pour toute composante connexe de $M \setminus \mathrm{Fix}(h)$;*
(ii) *soit $M \setminus \mathrm{Fix}(h)$ a exactement deux composantes connexes qui sont permutées par h.*

De plus, si M est orientable et h préserve l'orientation, alors $h(C) = C$ pour toute composante connexe de $M \setminus \mathrm{Fix}(h)$.

La plupart des énoncés ci-dessus peuvent être améliorés, parfois substantiellement. Une version détaillée de ce travail contiendra ces améliorations.

1. Statements

The following fact seems to have been unnoticed until now.

Theorem 1.1. *Let F be a closed subset of the (finite-dimensional) connected manifold M. If $f : F \to M$ is a proper continuous map which is the identity on the boundary ∂F of F in M, then either $f(F) \supset F$ or $f(F) \supset M \setminus F$.*

Moreover, if f is properly homotopic to the inclusion $F \subset M$ modulo ∂F, we must have $f(F) \supset F$.

We recall that f is properly homotopic to the inclusion $F \subset M$ modulo ∂F if there exists a proper continuous map $H : F \times [0,1] \to M$ such that $H(x,0) = x$, $H(x,1) = f(x)$, for all $x \in F$, and $H(x,t) = x$, for all $(x,t) \in \partial F \times [0,1]$.

Theorem 1.1 has several deep consequences; we mention here some of them.

Corollary 1.2. *Let F be a closed subset with non-empty interior $\mathring{F}$ of the connected manifold M. There does not exist a continuous proper retraction of F on its boundary ∂F.*

The corollary above is well known for the unit ball of an Euclidean space: it is one of the equivalent forms of Brouwer's fixed point theorem. It is also known when the boundary of F is a smooth submanifold of M [3, Proposition 7.1, page 133]. Moreover, there is a simple proof in [1, 3.6, page 12] when F is a general compact subset of the Euclidean space M.

This shows of course that the proof of the theorem must use some deep topological fact.

Corollary 1.3. *Let F be a closed subset of the connected manifold M. If $H : F \times [0,1] \to M$ is a continuous homotopy such that $H(x,0) = x$, for all $x \in F$, and $H \mid F \times \{1\}$ is constant, then one of the following two possibilities holds:*

(i) *for every relatively compact connected component C of $M \setminus F$, we have $H(\partial C \times [0,1]) \supset C$;*
(ii) *there exists a relatively compact connected component C of $M \setminus F$ with $H(\partial C \times [0,1]) \supset M \setminus C$.*

In particular, we have:

(a) *if M is compact, then $H(F \times [0,1])$ contains all the connected components of $M \setminus F$ except at most one;*
(b) *if M is not compact, then $H(F \times [0,1])$ contains all the relatively compact connected components of $M \setminus F$.*

Note that the homotopy H in Corollary 1.3 above cannot be proper unless F is compact. A proper version of this Corollary 1.3 is given by the following result.

Theorem 1.4. *Let F be a closed subset of the connected manifold M. If $H : F \times [0,1[\to M$ is a continuous proper homotopy such that $H(x,0) = x$, for all $x \in F$, then $H(F \times [0,1[)$ contains all the connected components of $M \setminus F$, except at most one.*

Theorem 1.5. *Let $h : M \to M$ be a homeomorphism of the connected (finite-dimensional) manifold M. If* $\mathrm{Fix}(h)$*, denotes the set of fixed points of h, we have:*

(i) *either $h(C) = C$ for every connected component of $M \setminus \mathrm{Fix}(h)$;*
(ii) *or $M \setminus \mathrm{Fix}(h)$ has exactly two connected components, which are permuted by h.*

Moreover, if M is orientable and h preserves the orientation, then $h(C) = C$ for every connected component of $M \setminus \mathrm{Fix}(h)$.

Most of the statements can be improved. There are even more consequences of Theorem 1.1. We postpone these facts to a more detailed version of this work.

2. Proofs

Proof of Theorem 1.1. Since f is proper, $f(F)$ is a closed subset. Assume that $f(F)$ does not contain $M \setminus F$. Then the open set $U = M \setminus (F \cup f(F))$ is not empty. We extend f to a map $\hat{f} : M \to M$ by the identity on $M \setminus F$. This extension $\hat{f}$ is continuous and proper, and has therefore a well-defined degree mod 2, [3, 6.1, page 124]. We now compute this degree using the points in $U = M \setminus (F \cup f(F))$. In fact, since U is an open non-empty subset of M with $\hat{f}^{-1}(U) = U$, and $\hat{f}$ is the identity on U, the degree must be 1. This in turn implies that $\hat{f}$ is surjective, hence $f(F) \supset F$.

If H is a proper continuous homotopy from the inclusion $F \subset M$ to f modulo ∂F, we can extend it to a proper continuous homotopy $\hat{H} : M \times [0,1] \to M$ by $\hat{H}(t,x) = x$, for $x \notin F$, $t \in [0,1]$. This gives a proper homotopy from the identity of M to $\hat{f}$. Therefore, the degree of $\hat{f}$ must be 1, the degree of the identity map of M, [3, 6.1, page 124]. This implies again that $\hat{f}$ is surjective, and $f(F) \supset F$. □

Proof of Corollary 1.2. Assume $r : F \to \partial F$ is such a proper continuous retraction. Since $\mathring{F}$ is not empty, the image $r(F) = \partial F$ is also non-empty. This implies that the interior of $M \setminus F$ is also non-empty. By Theorem 1.1, the image $r(F)$ contains either the interior of F or the interior of $M \setminus F$, which are both non-empty and disjoint from $\partial F = r(F)$. This is a contradiction. □

Proof of Corollary 1.3. We consider a relatively compact component C of the open set $M \setminus F$. Arguing by contradiction, suppose that $H(\partial C \times [0,1])$ does not contain either C or $M \setminus C$. Fix $\hat{x} \in C \setminus H(\partial C \times [0,1])$ and $\hat{y} \in M \setminus (C \cup H(\partial C \times [0,1]))$. Note that $\hat{x} \neq x_0$ and $\hat{y} \neq x_0$, where $\{x_0\} = H(F \times \{1\})$. Since $\partial C \subset F$, the homotopy H is defined on $\partial C \times [0,1]$, and we can extend it to a continuous map $\hat{H} : \partial C \times [0,1] \cup \bar{C} \times \{1\} \to M$ by $\hat{H}(x,1) = x_0$, for $x \in \bar{C}$. Since M is a manifold, we can extend $\hat{H}$ to a map $\hat{H} : W \to M$, where W is a neighborhood of the closed subset $\partial C \times [0,1] \cup \cup \bar{C} \times \{1\}$ in $\bar{C} \times [0,1]$. Since neither $\hat{x}$ nor $\hat{y}$ are in $H(\partial C \times [0,1])$, we can find a open neighborhood V of ∂C in $\bar{C}$ such that $V \times [0,1] \subset W$, and neither $\hat{x}$ nor $\hat{y}$ are in $\hat{H}(V \times [0,1])$. We then choose a continuous function $\rho : \bar{C} \to [0,1]$, with $\rho = 0$ on ∂C and $\rho = 1$ outside V. We can define the continuous map $f : \bar{C} \to M$ by $f(x) = \hat{H}(x, \rho(x))$, for $x \in \bar{C}$. The map f is the identity on $\partial \bar{C} = \partial C$, and $f(\bar{C})$ contains neither $\hat{x} \in C$ nor $\hat{y} \in M \setminus \bar{C}$. Since C is compact, this contradicts Theorem 1.1. □

The proof of Theorem 1.4 is a more involved version of the proof given above. We postpone it to a more detailed version of this work.

Proof of Theorem 1.5. Since h is a homeomorphism, it permutes the connected components of $M \setminus \mathrm{Fix}(h)$. Therefore, if C is a connected component of $M \setminus \mathrm{Fix}(h)$ such that $C \neq h(C)$ then $h(C) \cap C = \emptyset$. We can apply Theorem 1.1 to $h \mid \bar{C}$ to conclude that $M \setminus \bar{C} \subset h(C)$. Since $h(C) \cap \bar{C} = \emptyset$, we get $h(C) = M \setminus \bar{C}$. It is not difficult to conclude that $\mathrm{Fix}(h) = \partial C$ and $M \setminus \mathrm{Fix}(h)$ has exactly two connected components C and $h(C)$ which are permuted by C.

To prove the last part, we argue by contradiction. If C is a component of $M \setminus \mathrm{Fix}(h)$ such that $C \cap h(C) = \emptyset$, we can define $\hat{h} : M \to M$ by $\hat{h} \mid \bar{C} = h$, and $\hat{h}$ is the identity outside of C. Clearly $\bar{h}$ is proper and $\hat{h}(M) = M \setminus C$. Since $\hat{h}$ is not surjective, its degree is 0. Note that this degree is now with value in $\mathbb{Z}$, since M is orientable, see [3, Chapter III, §1 & 3, page 133]. On the other hand, if x is in the open set $h(C)$ then $\hat{h}^{-1}(x) = \{x_1, x_2\}$, with $x_1 \in C$ and $x_2 \in h(C)$. We can compute the degree of $\hat{h}$ using the point x. Since $\hat{h}$ is a local homeomorphism preserving the orientation near x_1 and x_2, we obtain that the degree of $\hat{h}$ is 2. This is a contradiction. □

It is possible to give another proof of the first part of Theorem 1.5 using Newman's theorems on transformation groups, namely using [2, Theorem 1, page 204]. Note however that the proofs of Newman's theorems requires degree theory. We sketch the proof. With the notations of the proof of Theorem 1.5, if $C \cap h(C) = \emptyset$, we can define a homeomorphism $\tilde{h} : M \to M$ of order 2, by

$$\tilde{h}(x) = \begin{cases} h(x), & \text{if } x \in C, \\ h^{-1}(x), & \text{if } x \in h(C), \\ x, & \text{if } x \notin C \cup h(C). \end{cases}$$

Since $\tilde{h}$ is not the identity, and $\tilde{h}$ is the identity outside $C \cup h(C)$, Theorem 1 in [2] implies that the closed set $M \setminus (C \cup h(C))$ has empty interior, and therefore C and $h(C)$ are the only connected component of $M \setminus F$.

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