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A sharp weighted anisotropic Poincaré inequality for convex domains

Une inégalité de Poincaré anisotrope pondérée pour les domaines convexes

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ABSTRACT

We prove an optimal lower bound for the best constant in a class of weighted anisotropic Poincaré inequalities.

RÉSUMÉ

Nous prouvons une limite inférieure optimale pour la meilleure constante dans une classe d'inégalités de Poincaré anisotropes pondérées.

1. Introduction

In this paper, we prove a sharp lower bound for the optimal constant $\mu_{p,\mathcal{H},\omega}(\Omega)$ in the Poincaré-type inequality

$$\inf_{t\in\mathbb{R}} \|u-t\|_{L^p_\omega(\Omega)} \le \frac{1}{[\mu_{p,\mathcal{H},\omega}(\Omega)]^{\frac{1}{p}}} \|\mathcal{H}(\nabla u)\|_{L^p_\omega(\Omega)},$$

with $1<p<+\infty$; Ω is a bounded convex domain of $\mathbb{R}^n$, $\mathcal{H}\in\mathscr{H}(\mathbb{R}^n)$, where $\mathscr{H}(\mathbb{R}^n)$ is the set of lower semicontinuous functions, positive in $\mathbb{R}^n\setminus\{0\}$ and positively 1-homogeneous, and ω is a log-concave function.

If $\mathcal{H}$ is the Euclidean norm of $\mathbb{R}^n$ and $\omega=1$, then $\mu_p(\Omega)=\mu_{p,\mathcal{E},\omega}(\Omega)$ is the first nontrivial eigenvalue of the Neumann p-Laplacian:

$$\begin{cases} -\Delta_p u = \mu_p |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

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Then, for a convex set Ω, it holds that

$$\mu_p(\Omega) \geq \left(\frac{\pi_p}{D_{\mathcal{E}}(\Omega)}\right)^p,$$

where

$$\pi_p = 2\int_0^{+\infty} \frac{1}{1+\frac{1}{p-1}s^p}\,\mathrm{d}s = 2\pi\frac{(p-1)^{\frac{1}{p}}}{p\sin\frac{\pi}{p}}, \qquad D_{\mathcal{E}}(\Omega) \text{ being the Euclidean diameter of } \Omega.$$

This estimate, proved in the case $p = 2$ in [13] (see also [3]), has been generalized to the case $p \neq 2$ in [1,10,12,15] and for $p \to \infty$ in [9,14]. Moreover, the constant $\left(\frac{\pi_p}{D_{\mathcal{E}}(\Omega)}\right)^p$ is the optimal constant of the one-dimensional Poincaré–Wirtinger inequality, with $\omega = 1$, on a segment of length $D_{\mathcal{E}}(\Omega)$. When $p = 2$ and $\omega = 1$, in [4] an extension of the estimate in the class of suitable non-convex domains has been proved.

The aim of the paper is to prove an analogous sharp lower bound for $\mu_{p,\mathcal{H},\omega}(\Omega)$, in a general anisotropic case. More precisely, our main result is:

Theorem 1. *Let $\mathcal{H} \in \mathscr{H}(\mathbb{R}^n)$, $\mathcal{H}^o$ be its polar function. Let us consider a bounded convex domain $\Omega \subset \mathbb{R}^n$, $1 < p < \infty$, and take a positive* log*-concave function ω defined in Ω. Then, given that*

$$\mu_{p,\mathcal{H},\omega}(\Omega) = \inf_{\substack{u \in W^{1,\infty}(\Omega) \\ \int_\Omega |u|^{p-2}u\omega\,\mathrm{d}x = 0}} \frac{\displaystyle\int_\Omega \mathcal{H}(\nabla u)^p \omega\,\mathrm{d}x}{\displaystyle\int_\Omega |u|^p \omega\,\mathrm{d}x},$$

it holds that

$$\mu_{p,\mathcal{H},\omega}(\Omega) \geq \left(\frac{\pi_p}{D_{\mathcal{H}}(\Omega)}\right)^p, \tag{1}$$

where $D_{\mathcal{H}}(\Omega) = \sup_{x,y\in\Omega} \mathcal{H}^o(y-x)$.

This result has been proved in the case $p = 2$ and $\omega = 1$, when $\mathcal{H}$ is a strongly convex, smooth norm of $\mathbb{R}^n$ in [17], with a completely different method than the one presented here.

In Section 2 below, we give the precise definition of $\mathcal{H}^o$ and give some details on the set $\mathscr{H}(\mathbb{R}^n)$. In Section 3, we give the proof of the main result.

2. Notation and preliminaries

A function

$$\xi \in \mathbb{R}^n \mapsto \mathcal{H}(\xi) \in [0, +\infty[$$

belongs to the set $\mathscr{H}(\mathbb{R}^n)$ if it verifies the following assumptions:

(1) $\mathcal{H}$ is positively 1-homogeneous, that is

$$\text{if } \xi \in \mathbb{R}^n \text{ and } t \geq 0, \text{ then } \mathcal{H}(t\xi) = t\mathcal{H}(\xi);$$

(2) if $\xi \in \mathbb{R}^n \setminus \{0\}$, then $\mathcal{H}(\xi) > 0$;
(3) $\mathcal{H}$ is lower semi-continuous.

If $\mathcal{H} \in \mathscr{H}(\mathbb{R}^n)$, properties (1), (2), (3) give that there exists a positive constant a such that

$$a|\xi| \leq \mathcal{H}(\xi), \quad \xi \in \mathbb{R}^n.$$

The polar function $\mathcal{H}^o : \mathbb{R}^n \to [0, +\infty[$ of $\mathcal{H} \in \mathscr{H}(\mathbb{R}^n)$ is defined as

$$\mathcal{H}^o(\eta) = \sup_{\xi\neq 0} \frac{\langle \xi, \eta\rangle}{\mathcal{H}(\xi)}.$$

The function $\mathcal{H}^o$ belongs to $\mathscr{H}(\mathbb{R}^n)$. Moreover, it is convex on $\mathbb{R}^n$, and then continuous. If $\mathcal{H}$ is convex, it holds that

$$\mathcal{H}(\xi) = (\mathcal{H}^o)^o(\xi) = \sup_{\eta \neq 0} \frac{\langle \xi, \eta \rangle}{\mathcal{H}^o(\eta)}.$$

If $\mathcal{H}$ is convex and $\mathcal{H}(\xi) = \mathcal{H}(-\xi)$ for all $\xi \in \mathbb{R}^n$, then $\mathcal{H}$ is a norm on $\mathbb{R}^n$, and the same holds for $\mathcal{H}^o$.

We recall that if $\mathcal{H}$ is a smooth norm of $\mathbb{R}^n$ such that $\nabla^2(\mathcal{H}^2)$ is positive definite on $\mathbb{R}^n \setminus \{0\}$, then $\mathcal{H}$ is called a Finsler norm on $\mathbb{R}^n$.

If $\mathcal{H} \in \mathscr{H}(\mathbb{R}^n)$, by definition, we have

$$\langle \xi, \eta \rangle \leq \mathcal{H}(\xi)\mathcal{H}^o(\eta), \qquad \forall \xi, \eta \in \mathbb{R}^n. \tag{2}$$

Remark 2. Let $\mathcal{H} \in \mathscr{H}(\mathbb{R}^n)$, and consider the convex envelope of $\mathcal{H}$, that is the largest convex function $\overline{\mathcal{H}}$ such that $\overline{\mathcal{H}} \leq \mathcal{H}$. It holds that $\overline{\mathcal{H}}$ and $\mathcal{H}$ have the same polar function:

$$(\overline{\mathcal{H}})^o = \mathcal{H}^o \quad \text{in } \mathbb{R}^n.$$

Indeed, being $\overline{\mathcal{H}} \leq \mathcal{H}$, by definition it holds that $(\overline{\mathcal{H}})^o \geq \mathcal{H}^o$. To show the reverse inequality, it is enough to prove that $(\mathcal{H}^o)^o \leq \mathcal{H}$. Then, being $\overline{\mathcal{H}}$ the convex envelope of $\mathcal{H}$, it must be $(\mathcal{H}^o)^o \leq \overline{\mathcal{H}}$, that implies $(\overline{\mathcal{H}})^o \leq \mathcal{H}^o$. Denoting by $G(x) = (\mathcal{H}^o)^o(x)$, for any x there exists $\overline{v}_x$ such that

$$G(x) = \frac{\langle x, \overline{v}_x \rangle}{\mathcal{H}^o(\overline{v}_x)}, \quad \text{and} \quad \langle x, \overline{v}_x \rangle \leq \mathcal{H}^o(\overline{v}_x)\mathcal{H}(x), \quad \text{that implies} \quad G(x) \leq \mathcal{H}(x).$$

Let $\mathcal{H} \in \mathscr{H}(\mathbb{R}^n)$, and consider a bounded convex domain Ω of $\mathbb{R}^n$. Throughout the paper $D_{\mathcal{H}}(\Omega) \in]0, +\infty[$ will be

$$D_{\mathcal{H}}(\Omega) = \sup_{x, y \in \Omega} \mathcal{H}^o(y - x).$$

We explicitly observe that since $\mathcal{H}^o$ is not necessarily even, in general $\mathcal{H}^o(y - x) \neq \mathcal{H}^o(x - y)$. When $\mathcal{H}$ is a norm, then $D_{\mathcal{H}}(\Omega)$ is the so-called anisotropic diameter of Ω with respect to $\mathcal{H}^o$. In particular, if $\mathcal{H} = \mathcal{E}$ is the Euclidean norm in $\mathbb{R}^n$, then $\mathcal{E}^o = \mathcal{E}$ and $D_{\mathcal{E}}(\Omega)$ is the standard Euclidean diameter of Ω. We refer the reader, for example, to [5,11] for remarkable examples of convex not even functions in $\mathscr{H}(\mathbb{R}^n)$. On the other hand, in [16] some results on isoperimetric and optimal Hardy–Sobolev inequalities for a general function $\mathcal{H} \in \mathscr{H}(\mathbb{R}^n)$ have been proved, by using a generalization of the so-called convex symmetrization introduced in [2] (see also [6–8]).

Remark 3. In general, $\mathcal{H}$ and $\mathcal{H}^o$ are not rotational invariant. Anyway, if $A \in SO(n)$, defining

$$\mathcal{H}_A(x) = \mathcal{H}(Ax), \tag{3}$$

and being $A^{\mathrm{T}} = A^{-1}$, then $\mathcal{H}_A \in \mathscr{H}(\mathbb{R}^n)$ and

$$(\mathcal{H}_A)^o(\xi) = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\langle x, \xi \rangle}{\mathcal{H}_A(x)} = \sup_{y \in \mathbb{R}^n \setminus \{0\}} \frac{\langle A^{\mathrm{T}} y, \xi \rangle}{\mathcal{H}(y)} = \sup_{y \in \mathbb{R}^n \setminus \{0\}} \frac{\langle y, A\xi \rangle}{\mathcal{H}(y)} = (\mathcal{H}^o)_A(\xi).$$

Moreover,

$$D_{\mathcal{H}_A}(A^{\mathrm{T}}\Omega) = \sup_{x, y \in A^{\mathrm{T}}\Omega} (\mathcal{H}^o)_A(y - x) = \sup_{\bar{x}, \bar{y} \in \Omega} \mathcal{H}^o(\bar{y} - \bar{x}) = D_{\mathcal{H}}(\Omega). \tag{4}$$

3. Proof of the Payne–Weinberger inequality

In this section, we state and prove Theorem 1. To this aim, the following Wirtinger-type inequality, contained in [12] is needed.

Proposition 4. *Let f be a positive log-concave function defined on $[0, L]$ and $p > 1$, then*

$$\inf \left\{ \frac{\int_0^L |u'|^p f \, dx}{\int_0^L |u|^p f \, dx}, \; u \in W^{1,p}(0, L), \; \int_0^L |u|^{p-2} u f \, dx = 0 \right\} \geq \frac{\pi_p^p}{L^p}.$$

The proof of the main result is based on a slicing method introduced in [13] in the Laplacian case. The key ingredient is the following Lemma. For a proof, we refer the reader, for example, to [13,3,12].

Lemma 5. *Let Ω be a convex set in $\mathbb{R}^n$ having (Euclidean) diameter $D_{\mathcal{E}}(\Omega)$, let ω be a positive log-concave function on Ω, and let u be any function such that $\int_\Omega |u|^{p-2}u\omega\,\mathrm{d}x = 0$. Then, for all positive ε, there exists a decomposition of the set Ω in mutually disjoint convex sets Ω_i $(i = 1, \dots, k)$ such that*

$$\bigcup_{i=1}^{k} \overline{\Omega_i} = \overline{\Omega}$$

$$\int_{\Omega_i} |u|^{p-2} u\,\omega\,\mathrm{d}x = 0$$

and for each i there exists a rectangular system of coordinates such that

$$\Omega_i \subset \{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 \le x_1 \le d_i, |x_l| \le \varepsilon, \ l = 2, \dots, n\},$$

where $d_i \le D_{\mathcal{E}}(\Omega)$, $i = 1, \dots, k$.

Proof of Theorem 1. By density, it is sufficient to consider a smooth function u with uniformly continuous first derivatives and $\int_\Omega |u|^{p-2}u\,\omega\,\mathrm{d}x = 0$.

Hence, we can decompose the set Ω in k convex domains Ω_i as in Lemma 5. In order to prove (1), we will show that, for any $i \in \{1, \dots, k\}$, it holds that

$$\int_{\Omega_i} H^p(\nabla u)\omega\,\mathrm{d}x \ge \frac{\pi_p^p}{D_{\mathcal{H}}(\Omega)^p} \int_{\Omega_i} |u|^p \omega\,\mathrm{d}x. \tag{5}$$

By Lemma 5, for each fixed $i \in \{1, \dots, k\}$, there exists a rotation $A_i \in SO(n)$ such that

$$A_i\Omega_i \subset \{(x_1, \dots, x_n) \in \mathbb{R}^n : 0 \le x_1 \le d_i, \ |x_l| \le \varepsilon, \ l = 2, \dots, n\}.$$

By changing the variable $y = A_i x$, recalling the notation (3) and using (4), it holds that

$$\int_{\Omega_i} \mathcal{H}^p(\nabla u(x))\,\omega(x)\,\mathrm{d}x = \int_{A_i\Omega_i} \mathcal{H}_{A_i^{\mathrm{T}}}(\nabla u(A_i^{\mathrm{T}} y))^p\,\omega(A_i^{\mathrm{T}} y)\,\mathrm{d}y; \qquad D_{\mathcal{H}}(\Omega) = D_{\mathcal{H}_{A_i^{\mathrm{T}}}}(A_i\Omega).$$

We deduce that it is not restrictive to suppose that for any $i \in \{1, \dots, n\}$ A_i is the identity matrix, and the decomposition holds with respect to the x_1-axis.

Now we may argue as in [12]. For any $t \in [0, d_i]$ let us denote by $v(t) = u(t, 0, \dots, 0)$, and $f_i(t) = g_i(t)\omega(t, 0, \dots, 0)$, where $g_i(t)$ will be the $(n-1)$ volume of the intersection of Ω_i with the hyperplane $x_1 = t$. By the Brunn–Minkowski inequality, g_i, and then f_i, is a log-concave function in $[0, d_i]$. Since u, u_{x_1} and ω are uniformly continuous in Ω, there exists a modulus of continuity $\eta(\cdot)$ with $\eta(\varepsilon) \searrow 0$ for $\varepsilon \to 0$, independent of the decomposition of Ω and such that

$$\left| \int_{\Omega_i} |u_{x_1}|^p \omega\,\mathrm{d}x - \int_0^{d_i} |v'|^p f_i\,\mathrm{d}t \right| \le \eta(\varepsilon)|\Omega_i|, \qquad \left| \int_{\Omega_i} |u|^p \omega\,\mathrm{d}x - \int_0^{d_i} |v|^p f_i\,\mathrm{d}t \right| \le \eta(\varepsilon)|\Omega_i|,$$

and

$$\left| \int_0^{d_i} |v|^{p-2} v f_i\,\mathrm{d}t \right| \le \eta(\varepsilon)|\Omega_i|.$$

Now, by property (2), we deduce that for any vector $\eta \in \mathbb{R}^n$

$$|\langle \nabla u, \eta \rangle| \le \mathcal{H}(\nabla u) \max\{\mathcal{H}^o(\eta), \mathcal{H}^o(-\eta)\}.$$

Then choosing $\eta = e_1$ and denoting by $M = \max\{\mathcal{H}^o(e_1), \mathcal{H}^o(-e_1)\}$, Proposition 4 gives

$$\int_{\Omega_i} \mathcal{H}^p(\nabla u)\omega\,\mathrm{d}x \ge \frac{1}{M^p} \int_{\Omega_i} |u_{x_1}|^p \omega\,\mathrm{d}x \ge \frac{1}{M^p} \int_0^{d_i} |v'|^p f_i\,\mathrm{d}t - \frac{\eta(\varepsilon)|\Omega_i|}{M^p}$$

$$\ge \frac{\pi_p}{d_i^p M^p} \int_0^{d_i} |v|^p f_i\,\mathrm{d}t + C\eta(\varepsilon)|\Omega_i| \ge \frac{\pi_p^p}{d_i^p M^p} \int_{\Omega_i} |u|^p \omega\,\mathrm{d}x + C\eta(\varepsilon)|\Omega_i|,$$

where C is a constant which does not depend on ε. Being $d_i \le D_{\mathcal{E}}(\Omega)$, and then $d_i M \le D_{\mathcal{H}}(\Omega)$, by letting ε to zero, we get (5). Hence, by summing over i, we get the thesis.

Remark 6. In order to prove an estimate for $\mu_{p,\mathcal{H},\omega}$, we could use directly property (2) with $\nu = \frac{\nabla u}{|\nabla u|}$, and the Payne–Weinberger inequality in the Euclidean case, obtaining that

$$\int_\Omega \mathcal{H}^p(\nabla u)\omega\,\mathrm{d}x \ge \int_\Omega \frac{|\nabla u|^p}{\mathcal{H}^o(\nu)^p}\omega\,\mathrm{d}x \ge \frac{\pi_p^p}{D_{\mathcal{E}}(\Omega)^p \mathcal{H}^o(\nu_m)^p}\int_\Omega |u|^p\omega\,\mathrm{d}x,$$

where $\mathcal{H}^o(\nu_m) = \max_{|\nu|=1} \mathcal{H}^o(\nu)$. However, we have a worse estimate than (1) because $D_{\mathcal{E}}(\Omega)\cdot\mathcal{H}^o(\nu_m)$ is, in general, strictly larger than $D_{\mathcal{H}}(\Omega)$, as shown in the following example.

Example 1. Let $\mathcal{H}(x,y) = \sqrt{a^2x^2 + b^2y^2}$, with $a < b$. Then $\mathcal{H}$ is a even, smooth norm with $\mathcal{H}^o(x,y) = \sqrt{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}}$ and the Wulff shapes $\{\mathcal{H}^o(x,y) < R\}$, $R > 0$, are ellipses. Clearly, we have:

$$D_{\mathcal{E}}(\Omega) = 2b \quad \text{and} \quad D_{\mathcal{H}}(\Omega) = 2.$$

Let us compute $\mathcal{H}^o(\nu_m)$. We have:

$$\max_{|\nu|=1} \mathcal{H}^o(\nu) = \max_{\vartheta\in[0,2\pi]} \sqrt{\frac{(\cos\vartheta)^2}{a^2} + \frac{(\sin\vartheta)^2}{b^2}} = \mathcal{H}^o(0,\pm 1) = \frac{1}{a}.$$

Then $D_{\mathcal{E}}(\Omega)\cdot\mathcal{H}^o(\nu_m) = 2\dfrac{b}{a} > 2$.

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