C. R. Acad. Sci. Paris, Ser. I 355 (2017) 607–617

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Number theory/Mathematical analysis

A new generalization of Apostol-type Laguerre–Genocchi polynomials

Une nouvelle généralisation des polynômes de Laguerre–Genocchi de type Apostol

Nabiullah Khan[a], Talha Usman[a], Junesang Choi[b]

[a] *Department of Applied Mathematics, Faculty of Engineering and Technology, Aligarh Muslim University, Aligarh, 202002, India*
[b] *Department of Mathematics, Dongguk University, Gyeongju 38066, Republic of Korea*

ARTICLE INFO

Article history:
Received 8 February 2017
Accepted after revision 19 April 2017
Available online 26 April 2017

Presented by the Editorial Board

ABSTRACT

Many extensions and variants of the so-called Apostol-type polynomials have recently been investigated. Motivated mainly by those works and their usefulness, we aim to introduce a new class of Apostol-type Laguerre–Genocchi polynomials associated with the modified Milne–Thomson's polynomials introduced by Derre and Simsek and investigate its properties, including, for example, various implicit formulas and symmetric identities in a systematic manner. The new family of polynomials introduced here, being very general, contains, as its special cases, many known polynomials. So the properties and identities presented here reduce to yield those results of the corresponding known polynomials.

RÉSUMÉ

Plusieurs extensions et variantes des polynômes dits de type Apostol ont été récemment étudiées. Motivés par ces travaux et leur utilité, notre but est d'introduire une nouvelle classe de polynômes de type Apostol généralisant les polynômes de Laguerre–Genochi associés aux polynômes de Milne–Thompson modifiés, introduits par Derre et Simsek, et d'en étudier de façon systématique les propriétés. Par exemple, nous donnons diverses formules implicites et des identités de symétrie. La nouvelle famille de polynômes introduite ici est très générale et contient comme cas particuliers beaucoup de polynômes connus. Les résultats présentés ici redonnent des propriétés et identités de ces polynômes connus.

E-mail addresses: nukhanmath@gmail.com (N. Khan), talhausman.maths@gmail.com (T. Usman), junesang@mail.dongguk.ac.kr (J. Choi).

http://dx.doi.org/10.1016/j.crma.2017.04.010
1631-073X/

1. Introduction and preliminaries

Two variable Laguerre polynomials $L_n(x, y)$ are defined by the following generating function (see [4])

$$\sum_{n=0}^{\infty} L_n(x, y)\frac{t^n}{n!} = e^{yt} C_0(xt), \tag{1.1}$$

where $C_0(x)$ is the 0-th order Tricomi function defined by (see [23])

$$C_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{(r!)^2}. \tag{1.2}$$

The $L_n(x, y)$ are represented by the series

$$L_n(x, y) = \sum_{s=0}^{n} \frac{n!(-1)^s y^{n-s} x^s}{(n-s)!(s!)^2}. \tag{1.3}$$

Recently, various generalizations of Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi polynomials have been extensively investigated (see, e.g., [6,9,10,12,13,15,19–21]). The generalized Apostol–Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$, the generalized Apostol–Euler polynomials $E_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$ and generalized Apostol–Genocchi polynomials $G_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$ are defined, respectively, by the following generating functions (see [24, Section 1.8])

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} \cdot e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda)\,\frac{t^n}{n!} \tag{1.4}$$

$$\big(|t| < 2\pi \quad \text{when} \quad \lambda = 1;\ |t| < |\log\lambda| \quad \text{when} \quad \lambda \neq 1;\ 1^{\alpha} := 1\big);$$

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} \cdot e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda)\,\frac{t^n}{n!} \quad \big(|t| < |\log(-\lambda)|;\ 1^{\alpha} := 1\big); \tag{1.5}$$

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} \cdot e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x;\lambda)\,\frac{t^n}{n!} \quad \big(|t| < |\log(-\lambda)|;\ 1^{\alpha} := 1\big). \tag{1.6}$$

Setting $x = 0$ in (1.4), (1.5), and (1.6), we have

$$B_n^{(\alpha)}(0;\lambda) := B_n^{(\alpha)}(\lambda),\ \ E_n^{(\alpha)}(0;\lambda) := E_n^{(\alpha)}(\lambda),\ \ G_n^{(\alpha)}(0;\lambda) = G_n^{(\alpha)}(\lambda), \tag{1.7}$$

which are called Apostol–Bernoulli numbers of order α, Apostol–Euler numbers of order α, and Apostol–Genocchi numbers of order α, respectively. Also

$$B_n^{(\alpha)}(x) := B_n^{(\alpha)}(x;1),\ \ E_n^{(\alpha)}(x) := E_n^{(\alpha)}(x;1),\ \ G_n^{(\alpha)}(x) = G_n^{(\alpha)}(x;1), \tag{1.8}$$

which are called Apostol–Bernoulli polynomials of order α, Apostol–Euler polynomials of order α, and Apostol–Genocchi polynomials of order α, respectively. Here and in the following, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}$ be the sets of complex numbers, real numbers, positive real numbers, positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

For systematic works about the Apostol-type polynomials, one may be referred, for example, to [6,9,10,15,19–21].

Derre and Simsek [5] modified the Milne–Thomson's polynomials $\Phi_n^{(\alpha)}(x)$ as $\Phi_n^{(\alpha)}(x, y)$ of degree n and order α by means of the following generating function:

$$\sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x, y)\frac{t^n}{n!} = f(t,\alpha)e^{xt+h(t,y)}, \tag{1.9}$$

where $f(t,\alpha)$ is a function of t and integer α. Observe that $\Phi_n^{(\alpha)}(x, 0) = \Phi_n^{(\alpha)}(x)$ (for details, see [22]). Setting $f(t,\alpha) = \left(\frac{2t}{\lambda e^t+1}\right)^{\alpha}$ in (1.9) gives

$$\sum_{n=0}^{\infty} G_n^{(\alpha)}(x, y;\lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} e^{xt+h(t,y)}, \tag{1.10}$$

where $G_n^{(\alpha)}(x, y;\lambda)$ denote the Apostol–Genocchi polynomials of higher order α based on Milne–Thomson's polynomials.

It immediately follows from (1.6) and (1.10) that

$$G_n^{(\alpha)}(0, 0;\lambda) = G_n^{(\alpha)}(\lambda).$$

Taking $h(t, y) = yt^2$ in (1.10) gives

$$\sum_{n=0}^{\infty} {}_H G_n^{(\alpha)}(x, y; \lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} e^{xt+yt^2}, \tag{1.11}$$

where ${}_H G_n^{(\alpha)}(x, y; \lambda)$ are called generalized Apostol–Hermite Genocchi polynomials (see [7]). The case $\alpha = 1$ in (1.11) reduces to the Apostol–Hermite Genocchi polynomials defined by Dattoli et al. [3] in the following form:

$$\sum_{n=0}^{\infty} {}_H G_n(x, y; \lambda)\frac{t^n}{n!} = \frac{2t}{\lambda e^t + 1} \cdot e^{xt+yt^2}. \tag{1.12}$$

Guo and Qi [8] and Luo et al. [17,18] gave the following generalizations of Bernoulli and Euler polynomials with a and b parameters, respectively:

$$\sum_{n=0}^{\infty} B_n(a, b)\frac{t^n}{n!} = \frac{t}{b^t - a^t} \quad \left(\left|t \log \frac{b}{a}\right| < 2\pi\right) \tag{1.13}$$

and

$$\sum_{n=0}^{\infty} E_n(a, b)\frac{t^n}{n!} = \frac{2}{b^t + a^t} \quad \left(\left|t \log \frac{b}{a}\right| < \pi\right). \tag{1.14}$$

The generalized Apostol–Genocchi polynomials with the parameters a and b are given by means of the following generating function (see [10]):

$$\sum_{n=0}^{\infty} G_n(x; a, b, e; \lambda)\frac{t^n}{n!} = \frac{2t}{\lambda b^t + a^t} \cdot e^{xt}. \tag{1.15}$$

For a real or complex α, the Apostol–Genocchi polynomials $G_n^{(\alpha)}(x; a, b, e; \lambda)$ of order α with parameters a and b are defined by means of the following generating function:

$$\sum_{n=0}^{\infty} G_n^{(\alpha)}(x; a, b, e; \lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda b^t + a^t}\right)^{\alpha} e^{xt}. \tag{1.16}$$

It is obvious that $G_n^{(1)}(x; a, b, e; \lambda) = G_n(x; a, b, e; \lambda)$.

The 2-variable Kampé–de Fériet generalization of the Hermite polynomials (see [1] and [3]) reads

$$H_n(x, y) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!}; \tag{1.17}$$

they are usually defined by the following generating function:

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x, y)\frac{t^n}{n!}. \tag{1.18}$$

The polynomials $H_n(x, y)$ in (1.18) reduce to the ordinary Hermite polynomials $H_n(x)$ when $y = -1$ and x is replaced by $2x$.

Motivated by their importance and potential for applications in certain problems in number theory, combinatorics, classical and numerical analysis and other fields of applied mathematics, several kinds of special numbers and polynomials have recently been studied by many authors (see the references).

Kurt and Kurt [14] first introduced the definition of Hermite–Apostol–Genocchi polynomials and derived some implicit formulas. Gaboury and Kurt [7] also gave the generating function of Hermite–Apostol–Genocchi polynomials with three parameters. Motivated by the above-cited works and mainly the work [7], in this paper, we introduce a new family of generalized Apostol-type Laguerre-Genocchi polynomials ${}_L G_n^{(\alpha)}(x, y, z; a, b, e; \lambda)$ in (2.1) and investigate their basic properties and certain relationships with Genocchi numbers G_n, Genocchi polynomials $G_n(x)$, the generalized Apostol–Genocchi numbers $G_n(a, b; \lambda)$, generalized Apostol–Genocchi polynomials $G_n(x; a, b, e; \lambda)$ [10], Hermite–Genocchi polynomials ${}_H G_n(x, y)$ [3], and generalized Apostol–Hermite–Genocchi polynomials ${}_H G_n^{(\alpha)}(x, y; \lambda)$. We also modify generating functions for the Milne–Thomson's polynomials [22] and derive some identities related to Laguerre polynomials, Hermite polynomials, and Genocchi polynomials. Some implicit summation formulae and general symmetry identities are derived by applying generating functions. These results are shown to extend some known summation formulas and identities for generalized Hermite–Bernoulli, Euler and Hermite–Genocchi polynomials that have been investigated by many authors (see, e.g., [3,7,10,15,16,26,27]).

2. Generalized Apostol-type Laguerre–Genocchi polynomials

Here, we introduce a new class of Apostol-type Laguerre–Genocchi polynomials and investigate its properties.

Definition 1. Let $a, b \in \mathbb{N}$ with $a \neq b$. A generalization of Apostol–Genocchi polynomials $G_n^{(\alpha)}(x, y, z; a, b, e; \lambda)$ $(n \in \mathbb{N}_0)$ is defined by the following generating function

$$\sum_{n=0}^{\infty} G_n^{(\alpha)}(x, y, z; a, b, e; \lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda b^t + a^t}\right)^{\alpha} e^{yt+h(t,z)}\, C_0(xt) \tag{2.1}$$

$$\left(|t| < \left|\frac{\log(-\lambda)}{\log \frac{b}{a}}\right|;\ a \in \mathbb{C}\setminus\{0\},\ b \in \mathbb{R}^+;\ 1^{\alpha} := 1\right),$$

where $C_0(xt)$ is given as in (1.2) and $h(t, z)$ is a function of t and z.

Setting $h(t, z) = zt^2$ in (2.1), we get Definition 2.

Definition 2. Let $a, b \in \mathbb{N}$ with $a \neq b$. A generalization of Apostol-type Laguerre–Genocchi polynomials ${}_LG_n^{(\alpha)}(x, y, z; a, b, e; \lambda)$ $(n \in \mathbb{N}_0)$ is defined by the following generating function

$$\sum_{n=0}^{\infty} {}_LG_n^{(\alpha)}(x, y, z; a, b, e; \lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda b^t + a^t}\right)^{\alpha} e^{yt+zt^2}\, C_0(xt) \tag{2.2}$$

$$\left(|t| < \left|\frac{\log(-\lambda)}{\log \frac{b}{a}}\right|;\ a \in \mathbb{C}\setminus\{0\},\ b \in \mathbb{R}^+;\ 1^{\alpha} := 1\right),$$

where $C_0(xt)$ is given as in (1.2).

Using (1.1) and (1.16) in (2.2), we find the following explicit formulas for the generalized Apostol–Laguerre–Genocchi polynomials ${}_LG_n^{(\alpha)}(x, y, z; a, b, e; \lambda)$: For $n \in \mathbb{N}_0$,

$${}_LG_n^{(\alpha)}(x, y, z; a, b, e; \lambda) = \sum_{m=0}^{n}\sum_{k=0}^{[\frac{m}{2}]} \frac{G_{n-m}^{(\alpha)}(0; a, b, e; \lambda)\, L_{m-2k}(x, y)z^k n!}{(m-2k)!k!(n-m)!}. \tag{2.3}$$

The case $x = 0$ in (2.2) reduces to the Apostol-type Hermite–Genocchi polynomials defined as follows (see [7]):

$$\sum_{n=0}^{\infty} {}_HG_n^{(\alpha)}(y, z; a, b, e; \lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda b^t + a^t}\right)^{\alpha} e^{yt+zt^2} \tag{2.4}$$

$$\left(|t| < \left|\frac{\log(-\lambda)}{\log \frac{b}{a}}\right|;\ a \in \mathbb{C}\setminus\{0\},\ b \in \mathbb{R}^+;\ 1^{\alpha} := 1\right).$$

Setting $x = y = z = 0$ and replacing e by 1 in (2.1), we obtain the extension of the generalized Apostol–Genocchi polynomials denoted by $G_n^{(\alpha)}(a, b; \lambda)$ $(n \in \mathbb{N}_0)$ (see [10]) and

$$G_n^{(\alpha+\beta)}(a, b; \lambda) = \sum_{k=0}^{n}\binom{n}{k} G_k^{(\alpha)}(a, b; \lambda)G_{n-k}^{(\beta)}(a, b; \lambda). \tag{2.5}$$

For easier use, we recall some formal manipulations of double series as in the following lemma (see, e.g., [2], [23, pp. 56–57], and [25, p. 52]).

Lemma 2.1. *The following identities hold true:*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_{k,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} A_{k,n-pk} \quad (p \in \mathbb{N}); \tag{2.6}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} A_{k,n} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A_{k,n+pk} \quad (p \in \mathbb{N}); \tag{2.7}$$

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(m+n)\frac{x^n}{n!}\frac{y^m}{m!}. \tag{2.8}$$

Remark 1. In Lemma 2.1, the $A_{k,n}$ and $f(N)$ $(k,\ n,\ N \in \mathbb{N}_0)$ are real or complex valued functions of the k, n and N, respectively, and x and y are real or complex numbers. Also, in order to guarantee certain rearrangements of the involved series, all the associated series should be absolutely convergent.

We give some identities for the generalized Apostol-type Laguerre–Genocchi polynomials ${}_LG_n^{(\alpha)}(x,y,z;a,b,e;\lambda)$ in (2.2), which are asserted by Theorem 2.2.

Theorem 2.2. *Let $a,\ b \in \mathbb{N}$ with $a \neq b$. Also, let $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Then*

$$\begin{aligned} &{}_LG_n^{(\alpha)}(x,y,z;1,e,e;\lambda) = {}_LG_n^{(\alpha)}(x,y,z;\lambda), \\ &{}_LG_n^{(\alpha)}(0,0,0;a,b,1;\lambda) = {}_LG_n^{(\alpha)}(a,b;\lambda), \\ &{}_LG_n^{(1)}(0,0,0;a,b,1;1) = G_n(a,b); \end{aligned} \tag{2.9}$$

$${}_LG_n^{(\alpha+\beta)}(x,y+z,v+u;a,b,e;\lambda) = \sum_{k=0}^{n}\binom{n}{k} G_{n-k}^{(\alpha)}(x,z,v;a,b,e;\lambda)\,{}_HG_k^{(\beta)}(y,u;a,b,e;\lambda); \tag{2.10}$$

$${}_LG_n^{(\alpha+\beta)}(x,y+v,z;a,b,e;\lambda) = {}_LG_{n-k}^{(\alpha)}(x,y,z;a,b,e;\lambda)G_k^{(\beta)}(v;a,b,e;\lambda). \tag{2.11}$$

Proof. The formulas in (2.9) are obvious.

By using (2.2), (2.4), and (2.6), we have:

$$\begin{aligned} &\sum_{n=0}^{\infty} {}_LG_n^{(\alpha+\beta)}(x,y+z,v+u;a,b,e;\lambda)\frac{t^n}{n!} \\ &= \left\{\left(\frac{2t}{\lambda b^t + a^t}\right)^{\alpha} \mathrm{e}^{zt+vt^2}\, C_0(xt)\right\}\left\{\left(\frac{2t}{\lambda b^t + a^t}\right)^{\beta} \mathrm{e}^{yt+ut^2}\right\} \\ &= \left(\sum_{n=0}^{\infty} {}_LG_n^{(\alpha)}(x,z,v;a,b,e;\lambda)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty} {}_HG_k^{(\beta)}(y,u;a,b,e;\lambda)\frac{t^k}{k!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} G_{n-k}^{(\alpha)}(x,z,v;a,b,e;\lambda)\,{}_HG_k^{(\beta)}(y,u;a,b,e;\lambda)\right)\frac{t^n}{(n-k)!\,k!}. \end{aligned}$$

From the first and last expressions, equating the coefficients of t^n, we obtain (2.10).

The proof of (2.11) would run parallel to that of (2.10), here, by using (2.2) and (1.16). We omit the details. □

3. Implicit summation formulae involving the generalized Apostol–Laguerre–Genocchi polynomials

Here, we give implicit summation formulae for the generalized Apostol–Laguerre–Genocchi polynomials (2.2) (cf., [11]).

Theorem 3.1. *Let $a,\ b \in \mathbb{N}$ with $a \neq b$. Also, let $x,\ y,\ v,\ z \in \mathbb{R}$ and $m,\ n \in \mathbb{N}_0$. Then*

$${}_LG_{m+n}^{(\alpha)}(x,v,z;a,b,e;\lambda) = \sum_{s=0}^{m}\sum_{k=0}^{n}\binom{m}{s}\binom{n}{k}(v-y)^{s+k}\,{}_LG_{m+n-s-k}^{(\alpha)}(x,y,z;a,b,e;\lambda). \tag{3.1}$$

Proof. Replacing t by $t+u$ in the generating function in (2.2), we have

$$\left(\frac{2(t+u)}{\lambda b^{t+u} + a^{t+u}}\right)^{\alpha} \mathrm{e}^{y(t+u)+z(t+u)^2}\, C_0(x(t+u)) = \sum_{n=0}^{\infty} {}_LG_n^{(\alpha)}(x,y,z;a,b,e;\lambda)\frac{(t+u)^n}{n!}. \tag{3.2}$$

By using (2.8) in the right-hand side of (3.2) and multiplying both sides of the resulting identity by $\mathrm{e}^{-y(t+u)}$, we obtain

$$\left(\frac{2(t+u)}{\lambda b^{t+u} + a^{t+u}}\right)^{\alpha} \mathrm{e}^{z(t+u)^2}\, C_0(x(t+u)) = \mathrm{e}^{-y(t+u)}\sum_{m,n=0}^{\infty} {}_LG_{m+n}^{(\alpha)}(x,y,z;a,b,e;\lambda)\,\frac{t^n}{n!}\,\frac{u^m}{m!}. \tag{3.3}$$

By observing that the left-hand side of (3.3) is independent of the variable y, we replace y by v to get

$$\left(\frac{2(t+u)}{\lambda b^{t+u} + a^{t+u}}\right)^{\alpha} \mathrm{e}^{z(t+u)^2}\, C_0(x(t+u)) = \mathrm{e}^{-v(t+u)}\sum_{m,n=0}^{\infty} {}_LG_{m+n}^{(\alpha)}(x,v,z;a,b,e;\lambda)\,\frac{t^n}{n!}\,\frac{u^m}{m!}. \tag{3.4}$$

Equating the right-hand sides of (3.3) and (3.4), we have

$$\sum_{m,n=0}^{\infty} {}_LG_{m+n}^{(\alpha)}(x,v,z;a,b,e;\lambda)\frac{t^n}{n!}\frac{u^m}{m!} = \mathrm{e}^{(v-y)(t+u)} \sum_{m,n=0}^{\infty} {}_LG_{m+n}^{(\alpha)}(x,y,z;a,b,e;\lambda)\frac{t^n}{n!}\frac{u^m}{m!}. \tag{3.5}$$

By using (2.8), we obtain

$$\mathrm{e}^{(v-y)(t+u)} = \sum_{N=0}^{\infty}(v-y)^N\frac{(t+u)^N}{N!} = \sum_{k,s=0}^{\infty}(v-y)^{s+k}\frac{t^k u^s}{k!\,s!}. \tag{3.6}$$

Inserting (3.6) into (3.5) and using (2.6) two times, we get

$$\begin{aligned}&\sum_{m,n=0}^{\infty} {}_LG_{m+n}^{(\alpha)}(x,v,z;a,b,e;\lambda)\frac{t^n}{n!}\frac{u^m}{m!}\\ &= \sum_{m,n=0}^{\infty}\sum_{s=0}^{m}\sum_{k=0}^{n}\frac{(v-y)^{k+s}}{k!s!}\,{}_LG_{m+n-k-s}^{(\alpha)}(x,y,z;a,b,e;\lambda)\frac{t^n}{(n-k)!}\frac{u^m}{(m-s)!},\end{aligned}$$

which, upon equating the coefficients of $t^n u^m$, yields the desired identity (3.1). □

We consider some interesting special cases of the result in Theorem 3.1, which are given in the following corollary.

Corollary 3.2. *Let $a, b \in \mathbb{N}$ with $a \neq b$. Also, let $x, y, v, z \in \mathbb{R}$. Then*

$${}_LG_n^{(\alpha)}(x,v,z;a,b,e;\lambda) = \sum_{k=0}^{n}\binom{n}{k}(v-y)^k\,{}_LG_{n-k}^{(\alpha)}(x,y,z;a,b,e;\lambda) \quad (n\in\mathbb{N}_0); \tag{3.7}$$

$${}_LG_{m+n}^{(\alpha)}(x,z;a,b,e;\lambda) = \sum_{s=0}^{m}\sum_{k=0}^{n}\binom{m}{s}\binom{n}{k}(-y)^{s+k}\,{}_LG_{m+n-s-k}^{(\alpha)}(x,y,z;a,b,e;\lambda), \tag{3.8}$$

where ${}_LG_{m+n}^{(\alpha)}(x,z;a,b,e;\lambda) := {}_LG_{m+n}^{(\alpha)}(x,0,z;a,b,e;\lambda)$ $(m, n\in\mathbb{N}_0)$*;*

$${}_LG_{m+n}^{(\alpha)}(v+y;a,b,e;\lambda) = \sum_{s=0}^{m}\sum_{k=0}^{n}\binom{m}{s}\binom{n}{k}v^{k+s}\,{}_LG_{m+n-k-s}^{(\alpha)}(y;a,b,e;\lambda), \tag{3.9}$$

where ${}_LG_{m+n}^{(\alpha)}(v+y;a,b,e;\lambda) := {}_LG_{m+n}^{(\alpha)}(0,v+y,0;a,b,e;\lambda)$, ${}_LG_{m+n-k-s}^{(\alpha)}(y;a,b,e;\lambda) := {}_LG_{m+n-k-s}^{(\alpha)}(0,y,0;a,b,e;\lambda)$ $(m, n\in\mathbb{N}_0)$.

Proof. Setting $m=0$ and $v=0$ in (3.1) yields (3.7) and (3.8), respectively.
Replacing v by $v+y$ and setting $x=z=0$ in (3.1) gives (3.9). □

Theorem 3.3. *Let $a, b \in \mathbb{N}$ with $a \neq b$. Also, let $x, y, z \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Then*

$${}_LG_n^{(\alpha)}(x,y+\alpha,z;a,b,e;\lambda) = \sum_{k=0}^{n}\sum_{m=0}^{[(n-k)/2]}\frac{(-1)^k x^k z^m G_{n-2m-k}^{(\alpha)}(y;\frac{a}{e},\frac{b}{e},e;\lambda)\,n!}{(n-2m-k)!\,m!\,(k!)^2}. \tag{3.10}$$

Proof. By using (2.2) and (1.16), we obtain

$$\begin{aligned}\mathcal{L}_1 &:= \sum_{n=0}^{\infty}{}_LG_n^{(\alpha)}(x,y+\alpha,z;a,b,e;\lambda)\frac{t^n}{n!}\\ &= \left[\left(\frac{2t}{\lambda(\frac{b}{e})^t + (\frac{a}{e})^t}\right)^{\alpha}\mathrm{e}^{yt}\right]\mathrm{e}^{zt^2}\,C_0(xt)\\ &= \left\{\sum_{n=0}^{\infty}G_n^{(\alpha)}\left(y;\frac{a}{e},\frac{b}{e},e;\lambda\right)\frac{t^n}{n!}\right\}\left\{\sum_{m=0}^{\infty}\frac{z^m t^{2m}}{m!}\right\}C_0(xt).\end{aligned}$$

Using (2.6) with $p=2$ and (1.2), we get

$$\mathcal{L}_1=\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{[n/2]} G_{n-m}^{(\alpha)}\left(y;\frac{a}{e},\frac{b}{e},e;\lambda\right)\frac{z^m}{(n-2m)!\,m!}\right\}t^n\cdot\sum_{k=0}^{\infty}\frac{(-1)^k x^k t^k}{(k!)^2},$$

which, upon using (2.6) with $p=1$, yields

$$\sum_{n=0}^{\infty}{}_LG_n^{(\alpha)}(x,y+\alpha,z;a,b,e;\lambda)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\sum_{m=0}^{[(n-k)/2]}\frac{(-1)^k x^k z^m G_{n-2m-k}^{(\alpha)}(y;\frac{a}{e},\frac{b}{e},e;\lambda)}{(n-2m-k)!\,m!\,(k!)^2}\right\}t^n.$$

Finally, equating the coefficients of t^n on both sides of the last identity yields the desired identity (3.10). □

Theorem 3.4. *Let $a,\ b\in\mathbb{N}$ with $a\neq b$. Also, let $x,\ y,\ z,\ u,\ w\in\mathbb{R}$ and $n\in\mathbb{N}_0$. Then*

$$ {}_LG_n^{(\alpha)}(x,y+u,z+w;a,b,e;\lambda)=\sum_{m=0}^{n}\binom{n}{m}{}_LG_{n-m}^{(\alpha)}(x,y,z;a,b,e;\lambda)\,H_m(u,w), \tag{3.11}$$

where $H_m(u,w)$ is given as in (1.18).

Proof. Using (2.2) and (1.18), similarly as above, we can prove (3.11). We omit the details. □

Theorem 3.5. *Let $a,\ b\in\mathbb{N}$ with $a\neq b$. Also, let $x,\ y,\ z,\ u,\ w\in\mathbb{R}$ and $n\in\mathbb{N}_0$. Then*

$$ {}_LG_n^{(\alpha)}(x,y,z;a,b,e;\lambda)=\sum_{m=0}^{n-2j}\sum_{j=0}^{[n/2]}\frac{G_m^{(\alpha)}(a,b,e;\lambda)L_{n-m-2j}(x,y)z^j n!}{m!\,j!(n-m-2j)!}, \tag{3.12}$$

where $G_m^{(\alpha)}(a,b,e;\lambda):=G_m^{(\alpha)}(0;a,b,e;\lambda)$.

Proof. We find from (2.2) that

$$\sum_{n=0}^{\infty}{}_LG_n^{(\alpha)}(x,y,z;a,b,e;\lambda)\frac{t^n}{n!}=e^{yt}C_0(xt)\cdot\left(\frac{2t}{\lambda b^t+a^t}\right)^{\alpha}\cdot e^{zt^2}.$$

By using (1.16) and (1.1), we have

$$\sum_{n=0}^{\infty}{}_LG_n^{(\alpha)}(x,y,z;a,b,e;\lambda)\frac{t^n}{n!}=\left\{\sum_{n=0}^{\infty}L_n(x,y)\frac{t^n}{n!}\right\}\cdot\left\{\sum_{m=0}^{\infty}G_m^{(\alpha)}(a,b,e;\lambda)\frac{t^m}{m!}\right\}\cdot\left(\sum_{j=0}^{\infty}z^j\frac{t^{2j}}{j!}\right).$$

Then, we apply (2.6) with $p=1$ to combine the first two summations, the result of which is combined into the third summation with the help of (2.6) with $p=2$. Finally, similarly as above, equating the coefficients of t^n in the starting summation and the last resulting summation yields the desired result (3.12). □

Theorem 3.6. *Let $a,\ b\in\mathbb{N}$ with $a\neq b$. Also, let $x,\ y,\ z,\ u,\ w\in\mathbb{R}$ and $n\in\mathbb{N}_0$. Then*

$$ {}_LG_n^{(\alpha)}(x,y+1,z;a,b,e;\lambda)=\sum_{j=0}^{n}\sum_{m=0}^{n-j}\frac{n!(-x)^j\,{}_HG_{n-m-j}^{(\alpha)}(y,z;a,b,e;\lambda)}{(n-m-j)!m!(j!)^2} \tag{3.13}$$

and

$$ {}_LG_n^{(\alpha)}(x,y+1,z;a,b,e;\lambda)=\sum_{m=0}^{n}\binom{n}{m}{}_LG_{n-m}^{(\alpha)}(x,y,z;a,b,e;\lambda). \tag{3.14}$$

Proof. We find from (2.2) that

$$\sum_{n=0}^{\infty}{}_LG_n^{(\alpha)}(x,y+1,z;a,b,e;\lambda)\frac{t^n}{n!}=\left(\frac{2t}{\lambda b^t+a^t}\right)^{\alpha}e^{yt+zt^2}\cdot e^t\cdot C_0(xt).$$

Using (2.4) and (1.2) to expand the three terms in series and applying (2.6) with $p = 1$ consecutively, similarly as in the proof of Theorem 3.5, we obtain the desired result (3.13). For (3.14), consider

$$\sum_{n=0}^{\infty} {}_LG_n^{(\alpha)}(x, y+1, z; a, b, e; \lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda b^t + a^t}\right)^{\alpha} \mathrm{e}^{yt+zt^2} C_0(xt)\cdot \mathrm{e}^t.$$

We omit the remaining details. □

Theorem 3.7. *Let $a, b \in \mathbb{N}$ with $a \neq b$. Also, let $x, y, z, u, w \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Then*

$$ {}_LG_n^{(\alpha)}(-x, -y, z; a, b, e; \lambda) = (-1)^{\alpha}(-1)^n \sum_{m=0}^{n} \binom{n}{m} \alpha^m \log^m(ab)\, {}_LG_{n-m}^{(\alpha)}(x, y, z; b, a, e; \lambda), \tag{3.15}$$

where $(-1)^{\alpha} := \exp(\alpha\pi i)$ $(i = \sqrt{-1})$;

$$ {}_LG_n^{(\alpha)}(x, y, z; a, b, e; \lambda) = {}_LG_n^{(\alpha)}(x, \alpha + y, z; ae, be, e; \lambda). \tag{3.16}$$

Proof. Replacing t by $-t$ in (2.2) and using (2.6) with $p = 1$, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} {}_LG_n^{(\alpha)}(x, y, z; a, b, e; \lambda)\frac{(-1)^n t^n}{n!} \\
&\quad = (-1)^{\alpha}(ab)^{\alpha t}\left(\frac{2t}{\lambda a^t + b^t}\right)^{\alpha} \mathrm{e}^{-yt+zt^2} C_0(-xt) \\
&\quad = (-1)^{\alpha}\left\{\sum_{n=0}^{\infty} {}_LG_n^{(\alpha)}(-x, -y, z; b, a, e; \lambda)\frac{t^n}{n!}\right\}\left\{\sum_{m=0}^{\infty}\frac{\alpha^m \log^m(ab)}{m!} t^m\right\} \\
&\quad = (-1)^{\alpha}\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n} {}_LG_{n-m}^{(\alpha)}(-x, -y, z; b, a, e; \lambda)\frac{\alpha^m \log^m(ab)}{(n-m)!\,m!}\right\} t^n.
\end{aligned}$$

Equating the coefficients of t^n and replacing x and y by $-x$ and $-y$, respectively, we get the identity (3.15).

Using (2.2), it is easy to prove the identity (3.16). □

4. General symmetry identities for the generalized Apostol-type Laguerre–Genocchi polynomials

Zhang and Yang [27] gave several symmetric identities on the generalized Apostol–Bernoulli polynomials by applying the generating functions (see also [26]). Here, we also present several symmetric identities on the generalized Apostol–Laguerre–Genocchi polynomials by applying the generating function (2.2).

Theorem 4.1. *Let $p, q, a, b \in \mathbb{N}$ with $a \neq b$. Also, let $x, y, z \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Then*

$$\begin{aligned}
&\sum_{m=0}^{n}\binom{n}{m} {}_LG_{n-m}^{(\alpha)}(x, ay, az; q^b, p^a, e; \lambda)\, {}_LG_m^{(\alpha)}(x, by, bz; q^a, p^b, e; \lambda) \\
&\quad = \sum_{m=0}^{n}\binom{n}{m} {}_LG_{n-m}^{(\alpha)}(x, by, az; q^b, p^a, e; \lambda)\, {}_LG_m^{(\alpha)}(x, ay, bz; q^a, p^b, e; \lambda) \\
&\quad = \sum_{m=0}^{n}\binom{n}{m} {}_LG_{n-m}^{(\alpha)}(x, by, bz; q^b, p^a, e; \lambda)\, {}_LG_m^{(\alpha)}(x, ay, az; q^a, p^b, e; \lambda),
\end{aligned} \tag{4.1}$$

which is symmetric in both *a and b* and *p and q*.

Proof. Consider the function

$$g(t) = \left\{\frac{4t^2}{\left(\lambda p^{at} + q^{bt}\right)\left(\lambda p^{bt} + q^{at}\right)}\right\}^{\alpha} \mathrm{e}^{(a+b)yt+(a+b)zt^2}\,\{C_0(xt)\}^2, \tag{4.2}$$

which is symmetric in *both* a and b *and* p and q. First, combining the factors in (4.2) and using (2.2) and (2.6) with $p = 1$, we obtain

$$\begin{aligned} g(t) &= \left\{ \left(\frac{2t}{\lambda p^{at} + q^{bt}} \right)^{\alpha} e^{ayt+azt^2} C_0(xt) \right\} \left\{ \left(\frac{2t}{\lambda p^{bt} + q^{at}} \right)^{\alpha} e^{byt+bzt^2} C_0(xt) \right\} \\ &= \sum_{n=0}^{\infty} {}_L G_n^{(\alpha)}(x, ay, az; q^b, p^a, e; \lambda) \frac{t^n}{n!} \cdot \sum_{m=0}^{\infty} {}_L G_m^{(\alpha)}(x, by, bz; q^a, p^b, e; \lambda) \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} {}_L G_{n-m}^{(\alpha)}(x, ay, az; q^b, p^a, e; \lambda) {}_L G_m^{(\alpha)}(x, by, bz; q^a, p^b, e; \lambda) \frac{1}{(n-m)!\,m!} \right\} t^n. \end{aligned} \tag{4.3}$$

Likewise,

$$\begin{aligned} g(t) &= \left\{ \left(\frac{2t}{\lambda p^{at} + q^{bt}} \right)^{\alpha} e^{byt+azt^2} C_0(xt) \right\} \left\{ \left(\frac{2t}{\lambda p^{bt} + q^{at}} \right)^{\alpha} e^{ayt+bzt^2} C_0(xt) \right\} \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} {}_L G_{n-m}^{(\alpha)}(x, by, az; q^b, p^a, e; \lambda) {}_L G_m^{(\alpha)}(x, ay, bz; q^a, p^b, e; \lambda) \frac{1}{(n-m)!\,m!} \right\} t^n \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} g(t) &= \left\{ \left(\frac{2t}{\lambda p^{at} + q^{bt}} \right)^{\alpha} e^{byt+bzt^2} C_0(xt) \right\} \left\{ \left(\frac{2t}{\lambda p^{bt} + q^{at}} \right)^{\alpha} e^{ayt+azt^2} C_0(xt) \right\} \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} {}_L G_{n-m}^{(\alpha)}(x, by, bz; q^b, p^a, e; \lambda) {}_L G_m^{(\alpha)}(x, ay, az; q^a, p^b, e; \lambda) \frac{1}{(n-m)!\,m!} \right\} t^n. \end{aligned} \tag{4.5}$$

Finally, equating the coefficients of t^n in the last expressions of (4.3), (4.4), and (4.5), we get the desired result. □

Theorem 4.2. *Let $p, q, a, b \in \mathbb{N}$ with $a \neq b$. Also, let $x, y, z \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Then*

$$\begin{aligned} &\sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^m \, {}_L G_{n-m}^{(\alpha)}(x/a, y, z; q, p, e; \lambda) \, {}_L G_m^{(\alpha)}(x/b, y, z; q, p, e; \lambda) \\ &= (ab)^{\alpha} \sum_{m=0}^{n} \binom{n}{m} {}_L G_{n-m}^{(\alpha)}(x, ay, a^2 z; q^a, p^a, e; \lambda) \, {}_L G_m^{(\alpha)}(x, by, b^2 z; q^b, p^b, e; \lambda), \end{aligned} \tag{4.6}$$

which is symmetric in a and b.

Proof. Consider the function

$$h(t) = \left\{ \frac{4abt^2}{(\lambda p^{at} + q^{at})(\lambda p^{bt} + q^{bt})} \right\}^{\alpha} e^{(a+b)yt + (a^2+b^2)zt^2} \{C_0(xt)\}^2, \tag{4.7}$$

which is symmetric in a and b. Then, a similar argument as in the proof of Theorem 4.1 will establish the result here. We omit the details. □

Theorem 4.3. *Let $p, q, a, b \in \mathbb{N}$ with $a \neq b$. Also, let $u, v, y_1, y_2, z_1, z_2 \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Then*

$$\begin{aligned} &\sum_{m=0}^{n} \binom{n}{m} {}_L G_{n-m}^{(\alpha)}(u, y_1, z_1; q^b, p^a, e; \lambda) \, {}_L G_m^{(\alpha)}(v, y_2, z_2; p^b, q^a, e; \lambda) \\ &= \sum_{m=0}^{n} \binom{n}{m} {}_L G_{n-m}^{(\alpha)}(v, y_1, z_1; q^b, p^a, e; \lambda) \, {}_L G_m^{(\alpha)}(u, y_2, z_2; p^b, q^a, e; \lambda) \\ &= \sum_{m=0}^{n} \binom{n}{m} {}_L G_{n-m}^{(\alpha)}(u, y_1, z_2; q^b, p^a, e; \lambda) \, {}_L G_m^{(\alpha)}(v, y_2, z_1; p^b, q^a, e; \lambda) \\ &= \sum_{m=0}^{n} \binom{n}{m} {}_L G_{n-m}^{(\alpha)}(v, y_1, z_2; q^b, p^a, e; \lambda) \, {}_L G_m^{(\alpha)}(u, y_2, z_1; p^b, q^a, e; \lambda), \end{aligned} \tag{4.8}$$

which is symmetric in the pairs (p, q), (a, b), (u, v), (y_1, y_2), and (z_1, z_2), respectively.

Proof. Consider the function

$$j(t) = \left\{ \frac{4t^2}{\left(\lambda p^{at} + q^{bt}\right)\left(\lambda q^{at} + p^{bt}\right)} \right\}^{\alpha} \mathrm{e}^{(y_1+y_2)t+(z_1+z_2t^2} C_0(ut)\, C_0(vt), \tag{4.9}$$

which is symmetric in the pairs (p,q), (a,b), (u,v), (y_1, y_2), and (z_1, z_2), respectively. Combining the factors in (4.9), and using (2.2) and (2.6) with $p = 1$, similarly as above, we can establish the results here. We omit the details. □

Setting $n - m = m'$ in each one of the results in Theorem 4.3 and dropping the prime on m, we obtain the corresponding identities as given in the following corollary.

Corollary 4.4. *Let $p, q, a, b \in \mathbb{N}$ with $a \neq b$. Also, let $u, v, y_1, y_2, z_1, z_2 \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Then*

$$\begin{aligned}
&\sum_{m=0}^{n} \binom{n}{m} {}_LG^{(\alpha)}_{n-m}(v, y_2, z_2; p^b, q^a, e; \lambda)\, {}_LG^{(\alpha)}_{m}(u, y_1, z_1; q^b, p^a, e; \lambda) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m} {}_LG^{(\alpha)}_{n-m}(u, y_2, z_2; p^b, q^a, e; \lambda)\, {}_LG^{(\alpha)}_{m}(v, y_1, z_1; q^b, p^a, e; \lambda) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m} {}_LG^{(\alpha)}_{n-m}(v, y_2, z_1; p^b, q^a, e; \lambda)\, {}_LG^{(\alpha)}_{m}(u, y_1, z_2; q^b, p^a, e; \lambda) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m} {}_LG^{(\alpha)}_{n-m}(u, y_2, z_1; p^b, q^a, e; \lambda)\, {}_LG^{(\alpha)}_{m}(v, y_1, z_2; q^b, p^a, e; \lambda),
\end{aligned} \tag{4.10}$$

which is symmetric in the pairs (p,q), (a,b), (u,v), (y_1, y_2), and (z_1, z_2), respectively.

5. Concluding remarks

The generalized Apostol-type Laguerre–Genocchi polynomials ${}_LG^{(\alpha)}_n(x, y, z; a, b, e; \lambda)$ $(n \in \mathbb{N}_0)$ (2.2), being very general, reduce to yield many known polynomials. For example, as already indicated in (2.4), ${}_LG^{(\alpha)}_n(0, y, z; a, b, e; \lambda) = {}_HG^{(\alpha)}_n(y, z; a, b, e; \lambda)$ are the Apostol-type Hermite–Genocchi polynomials; ${}_LG^{(\alpha)}_n(0, y, 0; a, b, e; \lambda) = G^{(\alpha)}_n(y; a, b, e; \lambda)$ are the Apostol–Genocchi polynomials of order α in (1.16); ${}_LG^{(\alpha)}_n(0, y, z; 1, e, e; \lambda) = {}_HG^{(\alpha)}_n(y, z; \lambda)$ are the generalized Apostol–Hermite–Genocchi polynomials in (1.11); ${}_LG^{(0)}_n(x, y, 0; a, b, e; \lambda) = L_n(x, y)$ are the two-variable Laguerre polynomials in (1.1).

So the results presented here reduce to those corresponding to the known polynomials. For example, setting $x = 0$ in Theorem 4.1 yields the corresponding symmetric identities associated with the Apostol-type Hermite–Genocchi polynomials in (2.4), which are asserted by the following corollary.

Corollary 5.1. *Let $p, q, a, b \in \mathbb{N}$ with $a \neq b$. Also, let $y, z \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Then*

$$\begin{aligned}
&\sum_{m=0}^{n} \binom{n}{m} {}_HG^{(\alpha)}_{n-m}(ay, az; q^b, p^a, e; \lambda)\, {}_HG^{(\alpha)}_{m}(by, bz; q^a, p^b, e; \lambda) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m} {}_HG^{(\alpha)}_{n-m}(by, az; q^b, p^a, e; \lambda)\, {}_HG^{(\alpha)}_{m}(ay, bz; q^a, p^b, e; \lambda) \\
&\quad = \sum_{m=0}^{n} \binom{n}{m} {}_HG^{(\alpha)}_{n-m}(by, bz; q^b, p^a, e; \lambda)\, {}_HG^{(\alpha)}_{m}(ay, az; q^a, p^b, e; \lambda),
\end{aligned} \tag{5.1}$$

which is symmetric in the pairs (a,b) and (p,q), respectively.

Acknowledgements

The authors would like to express their deep-felt thanks for the reviewer's detailed and useful comments.

References

[1] E.T. Bell, Exponential polynomials, Ann. Math. 35 (1934) 258–277.
[2] J. Choi, Notes on formal manipulations of double series, Commun. Korean Math. Soc. 18 (4) (2003) 781–789.

[3] G. Dattoli, S. Lorenzutta, C. Caserano, Finite sums and generalized forms of Bernoulli polynomials, Rend. Mat. 19 (1999) 385–391.
[4] G. Dattoli, A. Torre, Operational methods and two variable Laguerre polynomials, Atti Accad. Torino 132 (1998) 1–7.
[5] R. Dere, Y. Simsek, Bernoulli type polynomials on umbral algebra, Russ. J. Math. Phys. 23 (1) (2015) 1–6.
[6] R. Dere, Y. Simsek, H.M. Srivastava, A unified presentation of three families of generalized Apostol-type polynomials based upon the theory of the umbral calculus and the umbral algebra, J. Number Theory 133 (2013) 3245–3263.
[7] S. Gaboury, B. Kurt, Some relations involving Hermite-based Apostol–Genocchi polynomials, Appl. Math. Sci. 6 (82) (2012) 4091–4102.
[8] B.N. Guo, F. Qi, Generalization of Bernoulli polynomials, J. Math. Educ. Sci. Technol. 33 (3) (2002) 428–431.
[9] Y. He, S. Araci, H.M. Srivastava, M. Acikgoz, Some new identities for the Apostol–Bernoulli polynomials and the Apostol–Genocchi polynomials, Appl. Math. Comput. 262 (2015) 31–41.
[10] H. Jolany, H. Shari, R.E. Alikelaye, Some results for the Apostol–Genocchi polynomials of higher order, Bull. Malays. Math. Sci. Soc. 2 (2013) 465–479.
[11] S. Khan, M.A. Pathan, N.A.M.H. Makhboul, G. Yasmin, Implicit summation formula for Hermite and related polynomials, J. Math. Anal. Appl. 344 (2008) 408–416.
[12] T. Kim, On the analogs of Euler numbers and polynomials associated with p-adic q-integral on $\mathbb{Z}_p$ at $q=-1$, J. Math. Anal. Appl. 331 (2007) 779–792.
[13] M.-S. Kim, S. Hu, Sums of products of Apostol–Bernoulli numbers, Ramanujan J. 28 (2012) 113–123.
[14] V. Kurt, B. Kurt, On Hermite–Apostol–Genocchi polynomials, AIP Conf. Proc. 1389 (2011) 378–380, http://dx.doi.org/10.1063/1.3636741.
[15] Q.-M. Luo, q-extensions for the Apostol–Genocchi polynomials, Gen. Math. 17 (2) (2009) 113–125.
[16] Q.-M. Luo, Extensions for the Genocchi polynomials and their Fourier expansion and integral representations, Osaka J. Math. 48 (2011) 291–309, http://hdl.handle.net/11094/6673.
[17] Q.-M. Luo, B.N. Guo, F. Qi, L. Debnath, Generalization of Bernoulli numbers and polynomials, Int. J. Math. Math. Sci. 59 (2003) 3769–3776.
[18] Q.-M. Luo, F. Qi, L. Debnath, Generalization of Euler numbers and polynomials, Int. J. Math. Math. Sci. 61 (2003) 3893–3901.
[19] Q.-M. Luo, H.M. Srivastava, Some generalizations of the Apostol–Bernoulli and Apostol–Euler polynomials, J. Math. Anal. Appl. 308 (1) (2005) 290–302.
[20] Q.-M. Luo, H.M. Srivastava, Some relationships between the Apostol–Bernoulli and Apostol-Euler polynomials, Comput. Math. Appl. 51 (2006) 631–642.
[21] Q.-M. Luo, H.M. Srivastava, Some generalizations of the Apostol–Genocchi polynomials and the Stirling number of the second kind, Appl. Math. Comput. 217 (2011) 5702–5728.
[22] L.M. Thomsons, Two classes of generalized polynomials, Proc. Lond. Math. Soc. 35 (1) (1933) 514–522.
[23] E.D. Rainville, Special Functions, Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.
[24] H.M. Srivastava, J. Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, Elsevier Science Publishers, Amsterdam, London and New York, 2012.
[25] H.M. Srivastava, H.L. Manocha, A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.
[26] S. Yang, An identity of symmetry for the Bernoulli polynomials, Discrete Math. 308 (2008) 550–554.
[27] Z. Zhang, H. Yang, Several identities for the generalized Apostol–Bernoulli polynomials, Comput. Math. Appl. 56 (12) (2008) 2993–2999.