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Mean ergodic theorem in symmetric spaces

Théorème ergodique moyen dans les espaces symétriques

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ABSTRACT

We investigate the validity of the Mean Ergodic Theorem in symmetric Banach function spaces E. The assertion of that theorem always holds when E is separable, whereas the situation is more delicate when E is non-separable. To describe positive results in the latter setting, we use the connections with the theory of singular traces.

RÉSUMÉ

Nous étudions la validité du théorème de la moyenne ergodique dans les espaces de fonctions symétriques E. Nous montrons que ce théorème est toujours vérifié lorsque E est séparable ; cependant, la situation est plus délicate dans le cas non séparable. Les résultats positifs obtenus dans ce cadre utilisent des connexions avec la théorie des traces singulières.

Version française abrégée

On munit l'intervalle $(0, 1)$ de la mesure de Lebesgue. Soit $T : (0, 1) \to (0, 1)$ un automorphisme, c'est-à-dire une bijection préservant la mesure. On pose :

$$S_n^T f = \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k, \quad n \geqslant 1.$$

Un théorème de moyenne ergodique classique (voir, e.g., [3, Theorème VIII.5.9]) affirme que, pour tout $1 \leqslant p < \infty$ et pour tout $f \in L_p$, les moyennes de Cesaro $\{S_n^T f\}_{n\geqslant 1}$ convergent dans l'espace L_p lorsque $n \to \infty$. Dans cet article, nous étudions la convergence des moyennes de Cesaro $\{S_n^T f\}_{n\geqslant 1}$ dans le cas où f appartient à un espace symétrique $(E, \|\cdot\|_E)$ sur $(0, 1)$, distinct de $L_\infty(0, 1)$. Les résultats principaux sont énoncés ci-dessous (la terminologie utilisée est expliquée dans la section suivante).

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Théorème 1. *Soit $E \neq L_\infty$ un espace symétrique sur $(0,1)$, supposé non séparable. Soit E_0 la fermeture de $L_\infty(0,1)$ dans E.*

(a) *Supposons T non périodique. Si $f \in E$ est tel que pour tout $h \in E$ équimesurable avec f, la suite $\left\{S_n^T h\right\}_{n\geqslant 1}$ converge dans E, alors $f \in E_0$.*
(b) *Supposons T périodique. Alors la suite $\{S_n^T f\}_{n\geqslant 1}$ converge dans E pour tout $f \in E$.*

Pour $m \geqslant 1$ entier, on note σ_m l'opérateur de dilatation sur $S(0,1)$ défini par (4). On note f^* le réarrangement continu à droite de $|f|$ (voir, e.g., [1]).

Théorème 2. *Si $f \in E$ est tel que*

$$\frac{1}{n}\|\sigma_n(f^*)\|_E \to 0, \quad n \to \infty, \tag{1}$$

alors il existe $h \in E$ équimesurable avec f tel que $\{S_n^T h\}_{n\geqslant 1}$ converge dans E.

La condition (1) joue un rôle important dans la théorie des traces singulières. Elle est équivalente à l'absence de forme linéaire singulière, positive et symétrique sur E (voir [6] et/ou Chapitre 4 dans [5]).

Corollaire 3. *Si E est un espace symétrique séparable sur $(0,1)$, alors, pour tout $f \in E$, la suite $\{S_n^T f\}_{n\geqslant 1}$ converge dans E.*

Théorème 4. *Soit E un espace symétrique sur $(0,1)$ et soit $T : (0,1) \to (0,1)$ un automorphisme sans composante ergodique de mesure strictement positive. Pour tout $f \in E$, il existe $h \in E$ équimesurable avec f tel que la suite $\{S_n^T h\}_{n\geqslant 1}$ converge dans E.*

1. Introduction

Let (Ω, Σ, ν) be an atomless Lebesgue probability space and let $S(\Omega, \Sigma, \nu)$ be the space of all real-valued Σ-measurable functions on Ω. For $f \in S(\Omega, \Sigma, \nu)$, let d_f be its distribution function given by the formula

$$d_f(t) := \nu\left(\{s \in \Omega : f(s) > t\}\right), \quad t \in \mathbb{R}.$$

Two measurable functions f_1 and f_2 are called *equimeasurable* if and only if $d_{f_1} = d_{f_2}$. The function f^* on $(0,1)$ given by the formula

$$f^*(t) = \inf\left\{\tau > 0 : d_{|f|}(\tau) \leqslant t\right\}, \tag{2}$$

is called a *rearrangement* of $|f|$.
Let E be a linear subspace in $S(\Omega, \Sigma, \nu)$. A Banach space $(E, \|\cdot\|_E)$ is said to be *symmetric* if

(a) $f_1 \in S(\Omega, \Sigma, \nu)$, $f_2 \in E$ and $f_1^* = f_2^*$ imply that $f_1 \in E$ and $\|f_1\|_E = \|f_2\|_E$;
(b) $f_1 \in S(\Omega, \Sigma, \nu)$, $f_2 \in E$ and $|f_1| \leqslant |f_2|$ imply that $f_1 \in E$ and $\|f_1\|_E \leqslant \|f_2\|_E$.

Recall that a symmetric space E is an intermediate space for the Banach couple (L_1, L_∞), that is the following continuous embeddings hold

$$L_\infty = L_\infty(\Omega, \Sigma, \nu) \subseteq E \subseteq L_1 = L_1(\Omega, \Sigma, \nu).$$

Let $T : \Omega \to \Omega$ be an automorphism, that is a measure preserving bijection. We write:

$$S_n^T f = \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k, \quad n \geqslant 1.$$

A classical mean ergodic theorem (see, e.g., [3, Theorem VIII.5.9]) asserts that, for every $f \in L_p$, the Cesaro averages $\{S_n^T f\}_{n\geqslant 1}$ converge in every L_p-space, $1 \leqslant p < \infty$, when $n \to \infty$. In this paper, we investigate the convergence of Cesaro averages $\{S_n^T f\}_{n\geqslant 1}$ in the setting when f belongs to a symmetric space $(E, \|\cdot\|_E)$ on Ω distinct from $L_\infty(\Omega)$. Some of the results presented in this article develop those obtained in 1989 by the second-named author in collaboration with A. Fedorov [7] and published in [8].
Our first main result is stated below.

Theorem 5. *Let $E \neq L_\infty$ be a non-separable symmetric space on Ω; let E_0 be the closure of $L_\infty(\Omega)$ in E and let $T : \Omega \to \Omega$ be an automorphism.*

(a) *Let T be non-periodic. If $f \in E$ is such that, for every $h \in E$ equimeasurable with f, the sequence $\{S_n^T h\}_{n\geqslant 1}$ converges in E, then $f \in E_0$.*
(b) *If T is periodic, then the sequence $\{S_n^T f\}_{n\geqslant 1}$ converges in E for every $f \in E$.*

In order to prove Theorem 5(a), we employ an integral representation of T. First we outline this construction. We refer the reader to pp. 20–22 in [2] for a detailed account of the construction presented below.

Let $\Omega_0 \in \Sigma$ be such that $\mathrm{Orb}(\Omega_0) = \Omega$. Define the function r_{Ω_0} on Ω_0 by setting

$$r_{\Omega_0}(s) := \inf\{k \in \mathbb{N} : T^k s \in \Omega_0\}, \quad s \in \Omega_0.$$

Define a new measure space

$$\Xi = \Big\{(s,k) : s \in \Omega_0, \ 1 \leqslant k \leqslant r_{\Omega_0}(s)\Big\}.$$

Measure on Ξ is defined as follows: if $A \subset \Omega_0$ is such that $r_{\Omega_0}|_A \geqslant k$, then

$$\lambda(A \times \{k\}) = \nu(A). \tag{3}$$

This definition coincides with the one on p. 21 in [2]. Indeed, we have $\mathrm{Orb}(\Omega_0) = \Omega$. Hence, the denominator in the definition of measure on p. 21 in [2] equals 1 due to Lemma 1 on p. 20 in [2].

Applying Lemma 1 on p. 20 in [2] once again, we obtain

$$\lambda(\Xi) = \int_\Omega r_{\Omega_0}(s)\, \mathrm{d}\nu(s) = 1.$$

We conclude that (Ξ, λ) is a probability space.

Define an automorphism $T_{\mathrm{spec}} : \Xi \to \Xi$ by setting

$$T_{\mathrm{spec}}(s,k) := \begin{cases} (s, k+1) & k < r_{\Omega_0}(s) \\ (T^{r_{\Omega_0}(s)}(s), 1) & k = r_{\Omega_0}(s) \end{cases}$$

In order to prove Theorem 5(a), we fix a function $f \in E(\Xi)$ and explicitly construct a measurable function h on Ω which is equimeasurable with f. Assuming that the sequence $\{S_n^{T_{\mathrm{spec}}} h\}_{n\geqslant 1}$ converges, we derive that $h \in E_0$ and, therefore, $f \in E_0$.

By σ_m, $m \in \mathbb{N}$, we denote a dilation operator on $S(0,1)$ (whenever we consider a measure space $(0,1)$, we always assume that it is equipped with Lebesgue measure) acting f on $(0,1)$ by the formula

$$(\sigma_m f)(t) = f(\frac{t}{m}) \quad t \in (0,1). \tag{4}$$

The key fact about the mapping σ_m is that (see (4.16) in [4])

$$\|\sigma_m f\|_E \leqslant m\|f\|_E. \tag{5}$$

Theorem 6. *Let E be a symmetric space on Ω and let $T : \Omega \to \Omega$ be an automorphism. If $f \in E$ is such that*

$$\frac{1}{n}\|\sigma_n(f^*)\|_E \to 0, \quad n \to \infty, \tag{6}$$

then there exists $h \in E$ which is equimeasurable with f and such that $\{S_n^T h\}_{n\geqslant 1}$ converges in E.

Condition (6) plays an important role in the theory of singular traces. It is equivalent to the lack of positive singular symmetric functionals on E (see [6] and/or Chapter 4 in [5]). Using this connection, we are able to show that every symmetric Orlicz space satisfies (6) and fully characterizes Marcinkiewicz spaces (see [1,4]) satisfying that condition.

Corollary 7. *Let E be a separable symmetric space on Ω and let $T : \Omega \to \Omega$ be an automorphism. If $f \in E$, then $\{S_n^T f\}_{n\geqslant 1}$ converges in E.*

Theorem 8. *Let E be a symmetric space on Ω and let $T : \Omega \to \Omega$ be an automorphism without ergodic components of positive measure. For every $f \in E$, there exists $h \in E$ which is equimeasurable with f and such that $\{S_n^T h\}_{n\geqslant 1}$ converges in E.*

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