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Analytic geometry

Quot schemes and Ricci semipositivity

Schéma quot et semi-positivité de Ricci

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A R T I C L E I N F O

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A B S T R A C T

Let X be a compact connected Riemann surface of genus at least two, and let $\mathcal{Q}_X(r,d)$ be the quot scheme that parameterizes all the torsion coherent quotients of $\mathcal{O}_X^{\oplus r}$ of degree d. This $\mathcal{Q}_X(r,d)$ is also a moduli space of vortices on X. Its geometric properties have been extensively studied. Here we prove that the anticanonical line bundle of $\mathcal{Q}_X(r,d)$ is not nef. Equivalently, $\mathcal{Q}_X(r,d)$ does not admit any Kähler metric whose Ricci curvature is semipositive.

R É S U M É

Soit X une surface de Riemann compacte et connexe de genre au moins deux, et soit $\mathcal{Q}_X(r,d)$ le schéma quot qui paramétrise tous les quotients torsion cohérents de $\mathcal{O}_X^{\oplus r}$ de degré d. L'espace $\mathcal{Q}_X(r,d)$ est aussi un espace de modules de vortex sur X. Nous démontrons que le fibré anticanonique de X n'a pas la propriété nef. De façon équivalente, $\mathcal{Q}_X(r,d)$ n'admet aucune métrique kählérienne dont la courbure de Ricci soit semi-positive.

1. Introduction

Take a compact connected Riemann surface X. The genus of X, which will be denoted by g, is assumed to be at least two. We will not distinguish between the holomorphic vector bundles on X and the torsion-free coherent analytic sheaves on X. For a positive integer r, let $\mathcal{O}_X^{\oplus r}$ be the trivial holomorphic vector bundle on X of rank r. Fixing a positive integer d, let

$$\mathcal{Q} := \mathcal{Q}_X(r,d) \tag{1.1}$$

be the quot scheme that parametrizes all (torsion) coherent quotients of $\mathcal{O}_X^{\oplus r}$ of rank zero and degree d [17]. Equivalently, $\mathcal{Q}$ parametrizes all coherent subsheaves of $\mathcal{O}_X^{\oplus r}$ of rank r and degree $-d$, because these are precisely the kernels of coherent

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quotients of $\mathcal{O}_X^{\oplus r}$ of rank zero and degree d. This $\mathcal{Q}$ is a connected smooth complex projective variety of dimension rd. See [6,5,4] for the properties of $\mathcal{Q}$. It should be mentioned that $\mathcal{Q}$ is also a moduli space of vortices on X, and it has been extensively studied from this point of view of mathematical physics; see [3,9,12] and references therein.

Bökstedt and Romão proved some interesting differential geometric properties of $\mathcal{Q}$ (see [12]). In [10] and [11], we proved that $\mathcal{Q}$ does not admit Kähler metrics with semipositive or seminegative holomorphic bisectional curvature. In this note, we continue to study the question of the existence of metrics on $\mathcal{Q}$ whose curvature has a sign. Our aim here is to prove the following.

Theorem 1.1. *The quot scheme $\mathcal{Q}$ in (1.1) does not admit any Kähler metric such that the anticanonical line bundle $K_{\mathcal{Q}}^{-1}$ is Hermitian semipositive.*

Since semipositive holomorphic bisectional curvature implies semipositive Ricci curvature for a Kähler metric, Theorem 1.1 generalizes the main result of [11].

Recall that a holomorphic line bundle L on a compact complex manifold M is said to be *Hermitian semipositive* if L admits a smooth Hermitian structure such that the corresponding Hermitian connection has the property that its curvature form is semipositive. The anticanonical line bundle on M will be denoted by K_M^{-1}. Note that if M admits a Kähler metric such that the corresponding Ricci curvature is semipositive, then K_M^{-1} is Hermitian semipositive. Indeed, in that case, the Hermitian connection on K_M^{-1} for the Hermitian structure induced by such a Kähler metric has semipositive curvature. The converse statement, that the Hermitian semipositivity of K_M^{-1} implies the existence of Kähler metrics with semipositive Ricci curvature, is also true by Yau's solution to the Calabi's conjecture [1,2,19].

The proof of Theorem 1.1 is based on a recent work of Demailly, Campana, and Peternell on the classification of compact Kähler manifolds M with semipositive K_M^{-1} [15,14]. This classification implies that if K_M^{-1} is semipositive, then there is a nontrivial Abelian ideal in the Lie algebra of holomorphic vector fields on M, provided $b_1(M) > 0$. On the other hand, for $M = \mathcal{Q}$, this Lie algebra is isomorphic to $\mathfrak{sl}(r, \mathbb{C})$, which does not have any nontrivial Abelian ideal.

2. Proof of Theorem 1.1

2.1. Semipositive Ricci curvature

Let $J^d(X) = \mathrm{Pic}^d(X)$ be the connected component of the Picard group of X that parameterizes the isomorphism classes of holomorphic line bundles on X of degree d. Let $S^d(X)$ denote the space of all effective divisors on X of degree d, so $S^d(X) = X^d/P_d$ is the symmetric product, with P_d being the group of permutations of $\{1, \cdots, d\}$. Let

$$p : S^d(X) \longrightarrow \mathrm{Pic}^d(X) \tag{2.1}$$

be the natural morphism that sends a divisor on X to the holomorphic line bundle on X defined by it.

Take any coherent subsheaf $F \subset \mathcal{O}_X^{\oplus r}$ of rank r and degree $-d$. Let

$$s_F : \mathcal{O}_X^{\oplus r} = (\mathcal{O}_X^{\oplus r})^* \longrightarrow F^*$$

be the dual of the inclusion of F in $\mathcal{O}_X^{\oplus r}$. Its exterior product

$$\bigwedge^r s_F : \mathcal{O}_X = \bigwedge^r \mathcal{O}_X^{\oplus r} \longrightarrow \bigwedge^r F^*$$

is a holomorphic section of the holomorphic line bundle $\bigwedge^r F^*$ of degree d. Therefore, the divisor $\mathrm{div}(\bigwedge^r s_F)$ is an element of $S^d(X)$. Consequently, we have a morphism

$$\varphi : \mathcal{Q} \longrightarrow S^d(X), \quad F \longmapsto \mathrm{div}(\bigwedge^r s_F), \tag{2.2}$$

where $\mathcal{Q}$ is defined in (1.1). We note that when $r = 1$, then φ is an isomorphism.

Assume that $\mathcal{Q}$ admits a Kähler metric ω such that $K_{\mathcal{Q}}^{-1}$ is Hermitian semipositive. Then there is a connected finite étale Galois covering

$$f : \widetilde{\mathcal{Q}} \longrightarrow \mathcal{Q} \tag{2.3}$$

such that $(\widetilde{\mathcal{Q}}, f^*\omega)$ is holomorphically isometric to a product

$$\gamma : \widetilde{\mathcal{Q}} \longrightarrow A \times C \times H \times F, \tag{2.4}$$

where

- A is an Abelian variety,
- C is a simply connected Calabi–Yau manifold (holonomy is $\mathrm{SU}(c)$, where $c = \dim C$),

- H is a simply connected hyper-Kähler manifold (holonomy is $\mathrm{Sp}(h/2)$, where $h = \dim H$), and
- F is a rationally connected smooth projective variety such that K_F^{-1} is Hermitian semipositive.

(See [15, Theorem 3.1].) Henceforth, we will identify $\widetilde{\mathcal{Q}}$ with $A \times C \times H \times F$ using γ in (2.4). We note that F is simply connected because it is rationally connected [13, p. 545, Theorem 3.5], [18, p. 362, Proposition 2.3].

2.2. A lower bound of d

We know that $b_1(\mathcal{Q}) = 2g$, and the induced homomorphism

$$(p \circ \varphi)_* : H_1(\mathcal{Q}, \mathbb{Q}) \longrightarrow H_1(\mathrm{Pic}^d(X), \mathbb{Q}),$$

where p and φ are constructed in (2.1) and (2.2), respectively, is an isomorphism [5], [6, p. 649, Remark]. Since f in (2.3) is a finite étale covering, the induced homomorphism

$$f_* : H_1(\widetilde{\mathcal{Q}}, \mathbb{Q}) \longrightarrow H_1(\mathcal{Q}, \mathbb{Q})$$

is surjective. Therefore, the homomorphism

$$(p \circ \varphi \circ f)_* : H_1(\widetilde{\mathcal{Q}}, \mathbb{Q}) \longrightarrow H_1(\mathrm{Pic}^d(X), \mathbb{Q}) \tag{2.5}$$

is surjective.

There is no nonconstant holomorphic map from a compact simply connected Kähler manifold to an Abelian variety. In particular, there are no nonconstant holomorphic maps from C, H and F in (2.4) to $\mathrm{Pic}^d(X)$. Hence, the map $p \circ \varphi \circ f$ factors through a map

$$\beta : A \longrightarrow \mathrm{Pic}^d(X).$$

In other words, there is a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{Q}} = A \times C \times H \times F & \stackrel{p\circ\varphi\circ f}{\longrightarrow} & \mathrm{Pic}^d(X) \\ q\big\downarrow & & \|\mathrm{Id} \\ A & \stackrel{\beta}{\longrightarrow} & \mathrm{Pic}^d(X) \end{array} \tag{2.6}$$

where q is the projection of $A \times C \times H \times F$ to the first factor. Since $H_1(A \times C \times H \times F, \mathbb{Z}) = H_1(A, \mathbb{Z})$ (as C, H and F are simply connected), and $(p \circ \varphi \circ f)_*$ in (2.5) is surjective, it follows that the homomorphism

$$\beta_* : H_1(A, \mathbb{Q}) \longrightarrow H_1(\mathrm{Pic}^d(X), \mathbb{Q})$$

induced by β is surjective. This immediately implies that the map β is surjective. Since β is surjective, from the commutativity of (2.6) we know that the map p is surjective. This implies that

$$d = \dim S^d(X) \geq \dim \mathrm{Pic}^d(X) = g \geq 2. \tag{2.7}$$

2.3. Albanese for $\widetilde{\mathcal{Q}}$

The homomorphism of fundamental groups

$$\varphi_* : \pi_1(\mathcal{Q}) \longrightarrow \pi_1(S^d(X))$$

induced by φ in (2.2) is an isomorphism [8, Proposition 4.1]. Since $d \geq 2$ (see (2.7)), the homomorphism of fundamental groups

$$p_* : \pi_1(S^d(X)) \longrightarrow \pi_1(\mathrm{Pic}^d(X))$$

induced by p in (2.1) is an isomorphism. Indeed, $\pi_1(S^d(X))$ is the Abelianization

$$\pi_1(X)/[\pi_1(X), \pi_1(X)] = H_1(X, \mathbb{Z})$$

of $\pi_1(X)$ [16]. Combining these we conclude that the homomorphism of fundamental groups

$$(p \circ \varphi)_* : \pi_1(\mathcal{Q}) \longrightarrow \pi_1(\mathrm{Pic}^d(X)) \tag{2.8}$$

induced by $p \circ \varphi$ is an isomorphism.

Since the homomorphism in (2.8) is an isomorphism, the covering f in (2.3) is induced by a covering of $\mathrm{Pic}^d(X)$. In other words, there is a finite étale Galois covering

$$\mu : J \longrightarrow \mathrm{Pic}^d(X) \tag{2.9}$$

and a morphism $\lambda : \widetilde{\mathcal{Q}} \longrightarrow J$ such that the following diagram is commutative:

$$\begin{array}{ccc} \widetilde{\mathcal{Q}} & \xrightarrow{f} & \mathcal{Q} \\ \Big\downarrow \lambda & & \Big\downarrow p \circ \varphi \\ J & \xrightarrow{\mu} & \mathrm{Pic}^d(X) \end{array} \tag{2.10}$$

where f is the covering map in (2.3). The projection q in (2.6) is clearly the Albanese morphism for $\widetilde{\mathcal{Q}}$, because C, H and F are all simply connected. On the other hand, $p \circ \varphi$ is the Albanese morphism for $\mathcal{Q}$ [11, Corollary 2.2]. Therefore, its pullback, namely, λ, is the Albanese morphism for $\widetilde{\mathcal{Q}}$. Consequently, we have $A = J$ with λ coinciding with the projection q in (2.6). Henceforth, we will identify A and q with J and λ respectively.

2.4. Vector fields

The differential df of f identifies $T\widetilde{\mathcal{Q}}$ with $f^*T\mathcal{Q}$, because f is étale. Using the trace homomorphism $t : f_*\mathcal{O}_{\widetilde{\mathcal{Q}}} \longrightarrow \mathcal{O}_{\mathcal{Q}}$, we have

$$f_*T\widetilde{\mathcal{Q}} = f_*f^*T\mathcal{Q} \xrightarrow{p_f} (f_*\mathcal{O}_{\widetilde{\mathcal{Q}}}) \otimes T\mathcal{Q} \xrightarrow{t} \mathcal{O}_{\mathcal{Q}} \otimes T\mathcal{Q} = T\mathcal{Q},$$

where p_f is given by the projection formula. This produces a homomorphism

$$\Phi : H^0(\widetilde{\mathcal{Q}}, T\widetilde{\mathcal{Q}}) = H^0(\mathcal{Q}, f_*T\widetilde{\mathcal{Q}}) \longrightarrow H^0(\mathcal{Q}, T\mathcal{Q}) \tag{2.11}$$

(the equality $H^0(\widetilde{\mathcal{Q}}, T\widetilde{\mathcal{Q}}) = H^0(\mathcal{Q}, f_*T\widetilde{\mathcal{Q}})$ follows from the fact that f is a finite morphism). This homomorphism Φ is surjective. Indeed, as $f^*T\mathcal{Q} = T\widetilde{\mathcal{Q}}$, any section of $T\mathcal{Q}$ pulls back to a section of $T\widetilde{\mathcal{Q}}$.

Since $\widetilde{\mathcal{Q}} = A \times C \times H \times F$, we have

$$H^0(\widetilde{\mathcal{Q}}, T\widetilde{\mathcal{Q}}) = H^0(A, TA) \oplus H^0(C, TC) \oplus H^0(H, TH) \oplus H^0(F, TF). \tag{2.12}$$

Note that $H^0(\widetilde{\mathcal{Q}}, T\widetilde{\mathcal{Q}})$ is a Lie algebra under the operation of Lie bracket of vector fields, and the subspace

$$H^0(A, TA) \subset H^0(\widetilde{\mathcal{Q}}, T\widetilde{\mathcal{Q}})$$

(see (2.12)) is an ideal in this Lie algebra. Since $A = J$ is a covering of $\mathrm{Pic}^d(X)$, we have

$$\dim H^0(A, TA) = \dim \mathrm{Pic}^d(X) = g > 1. \tag{2.13}$$

Since $H^0(A, TA)$ is an ideal in $H^0(\widetilde{\mathcal{Q}}, T\widetilde{\mathcal{Q}})$, it follows immediately that

$$\Phi(H^0(A, TA)) \subset \Phi(H^0(\widetilde{\mathcal{Q}}, T\widetilde{\mathcal{Q}})) = H^0(\mathcal{Q}, T\mathcal{Q})$$

is an ideal, where Φ is constructed in (2.11). Note that $H^0(A, TA)$ is an Abelian Lie algebra, so the Lie algebra $\Phi(H^0(A, TA))$ is also Abelian.

Since $\mu : J = A \longrightarrow \mathrm{Pic}^d(X)$ in (2.9) is a covering map between Abelian varieties, the trace map $H^0(A, TA) \longrightarrow H^0(\mathrm{Pic}^d(X), T\mathrm{Pic}^d(X))$ is an isomorphism. In view of this, from the commutativity of the diagram in (2.10), it follows that the restriction

$$\Phi|_{H^0(A, TA)} : H^0(A, TA) \longrightarrow H^0(\mathcal{Q}, T\mathcal{Q})$$

is injective (see (2.12) and (2.11)). But $H^0(\mathcal{Q}, T\mathcal{Q}) = \mathfrak{sl}(r, \mathbb{C})$ [7, p. 1446, Theorem 1.1]. Hence the Lie algebra $H^0(\mathcal{Q}, T\mathcal{Q})$ does not contain any nonzero Abelian ideal. This is in contradiction with the earlier result that $\Phi(H^0(A, TA))$ is a nonzero Abelian ideal in $H^0(\mathcal{Q}, T\mathcal{Q})$ of dimension g (see (2.13)). This completes the proof of Theorem 1.1.

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