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The global monodromy property for $K3$ surfaces allowing a triple-point-free model

La propriété de monodromie globale pour les surfaces $K3$ ayant un modèle sans point triple

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ABSTRACT

Inspired by the motivic monodromy conjecture, Halle and Nicaise defined the global monodromy property for Calabi–Yau varieties over a discretely valued field. In this note, we discuss this property for $K3$ surfaces allowing a strict normal crossings model where no three components in the special fiber have a common intersection. The main result is that the global monodromy property holds for a $K3$ surface with a so-called flowerpot degeneration. It also holds for $K3$ surfaces with a chain degeneration under certain conditions.

RÉSUMÉ

Inspirés par la conjecture de monodromie motivique, Halle et Nicaise ont défini la propriété de monodromie globale pour les variétés de Calabi–Yau définies sur un corps de valuation discrète. Dans cette note, nous étudions cette propriété pour les surfaces $K3$ ayant un modèle sans point triple. Le résultat principal est que la propriété de monodromie globale est satisfaite pour les surfaces $K3$ ayant une dégénérescence en pot de fleurs. Elle est également satisfaite pour les surfaces $K3$ ayant une dégénérescence en chaîne sous une condition supplémentaire.

1. Introduction

For every non-constant polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$, Igusa's p-adic monodromy conjecture expresses some properties of the p-adic zeta functions that predict in a quantitative way how the singularities of the complex hypersurface $f = 0$ affect

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the number of solutions to the congruence $f \equiv 0$ modulo powers of primes p^m for m big. Based on Kontsevich's theory of motivic integration, Denef and Loeser formulated in the nineties in [3, Conjecture 3.4.1] the *motivic monodromy conjecture*: a generalization of the p-adic monodromy conjecture in terms of the motivic zeta functions. We refer to [8] for more details about the motivic monodromy conjecture and its relation with the p-adic monodromy conjecture.

In [5], Halle and Nicaise define a motivic zeta function $Z_{X,\omega}(T)$ associated with a Calabi–Yau variety X over a discretely valued field and a volume form ω on X and they address, in this context, an analogue of the motivic monodromy conjecture: do the poles of the power series $Z_{X,\omega}(T)$ correspond to monodromy eigenvalues on the cohomology of X? Calabi–Yau varieties with this property are said to satisfy the *global monodromy property* (GMP). Halle and Nicaise proved that abelian varieties satisfy the GMP and that the motivic zeta function of an abelian variety has exactly one pole.

In this note, we investigate the GMP for $K3$ surfaces with a triple-point-free degeneration, i.e. $K3$ surfaces allowing a strict normal crossings model such that three irreducible components of the special fiber never meet simultaneously. In [2], Crauder and Morrison classify such degenerate fibers into two main classes: so-called flowerpot degenerations and chain degenerations. This classification is very precise, which allows us to use a combination of geometrical and combinatorial arguments to prove the GMP for $K3$ surfaces allowing a flowerpot degeneration or allowing a chain degeneration under an extra condition. This result constitutes the first example of Calabi–Yau varieties, beyond abelian varieties, that satisfy the global monodromy property. We also show that there exist Calabi–Yau varieties with a motivic zeta function with more than one pole, in contrast with the abelian case.

This note presents the results, without proof, obtained by the author in her PhD thesis [7].

For the rest of this note, fix the notation $K = \mathbb{C}((t))$, $k = \mathbb{C}$ and $R = \mathbb{C}[[t]]$. A variety is a reduced, separated scheme of finite type over a field.

2. The global monodromy property for Calabi–Yau varieties

2.1. Calabi–Yau varieties

Definition 2.1. A *Calabi–Yau variety* is a smooth, proper, geometrically connected variety with trivial canonical sheaf.

A well-known class of examples of Calabi–Yau varieties are the abelian varieties. In this note, we will focus on $K3$ surfaces.

Definition 2.2. A $K3$ *surface* X is a 2-dimensional Calabi–Yau variety with $H^1(X, \mathcal{O}_X) = 0$.

2.2. The Grothendieck ring of varieties

The *Grothendieck group* $K_0(Var_k)$ of k-varieties is the abelian group generated by the isomorphism classes of separated k-schemes of finite type, modulo the scissor relations

$$[X] = [Y] + [X \setminus Y],$$

for Y a closed subscheme of X and where we use the notation $[X]$ to denote the isomorphism class of the scheme X. This group is endowed with a ring structure by considering the formula

$$[X] \cdot [X'] = [X \times_k X']$$

for every pair (X, X') of separated k-schemes of finite type.

We introduce the symbol $\mathbb{L}$, which stands for the class of $\mathbb{A}^1_k$. The localisation of $K_0(Var_k)$ with respect to $\mathbb{L}$ is denoted by $\mathcal{M}_k$. For more details on the Grothendieck ring, we refer the reader to [10].

2.3. Motivic zeta function for Calabi–Yau varieties

Let X be a Calabi–Yau variety over K and let ω be a volume form on X, i.e. a nowhere-vanishing differential form of maximal degree. By analogy with [9, §7], Halle and Nicaise associate with X and ω in [5, Definition 6.1.4] a motivic zeta function as follows.

Definition 2.3. The *motivic zeta function* $Z_{X,\omega}(T)$ of X is defined as

$$Z_{X,\omega}(T) = \sum_{d \in \mathbb{Z}_{>0}} \left(\int_{X(d)} |\omega(d)| \right) T^d \in \mathcal{M}_k[[T]],$$

where $X(d) = X \times_K \mathbb{C}((\sqrt[d]{t}))$ and where $\omega(d)$ is the pullback of ω to $X(d)$.

The integral $\int_{X(d)} |\omega(d)|$ is a motivic integral measuring the space of $\mathbb{C}((\sqrt[d]{t}))$-rational points on X.

The motivic zeta function of X can be computed from an *snc*-model $\mathcal{X}$ of X. An *snc*-model is a regular proper algebraic space over R such that $\mathcal{X} \times_R K \simeq X$ and such that the special fiber $\mathcal{X}_k = \sum_{i\in I} N_i E_i$ is a divisor with strict normal crossings. Such a model always exists by Nagata's compactification theorem and the embedded resolution of singularities. We define μ_i as the order of $\mathrm{div}(\omega)$ along E_i when ω is viewed as a rational section of the log-relative canonical bundle $\omega_{\mathcal{X}/R}(\mathcal{X}_{k,\mathrm{red}} - \mathcal{X}_k)$, where $\mathcal{X}_{k,\mathrm{red}}$ is the divisor $\sum_{i\in I} E_i$.

For every non-empty subset $J \subseteq I$, we define

$$E_J = \cap_{j\in J} E_j \qquad \text{and} \qquad E_J^\circ = E_J \setminus \left(\cup_{i\in I\setminus J} E_i\right).$$

In [1, Corollary 4.3.2], Bultot and Nicaise prove the following Denef–Loeser-type formula for $\mathcal{X}$ a scheme:

$$Z_{X,\omega}(T) = \sum_{\emptyset\neq J\subseteq I} (\mathbb{L}-1)^{|J|-1} [\widetilde{E_J^\circ}] \prod_{j\in J} \frac{\mathbb{L}^{-\mu_j} T^{N_j}}{1-\mathbb{L}^{-\mu_j} T^{N_j}}, \tag{1}$$

where $\widetilde{E_J^\circ}$ is a certain finite étale cover of E_J°. See also [9, Corollary 7.7] for a similar result for formal schemes. This formula is easily generalized to the case where $\mathcal{X}$ is an algebraic space; see [6].

This formula immediately implies that $Z_{X,\omega}$ is a rational function and that all poles of $Z_{X,\omega}(\mathbb{L}^{-s})$ are of the form $-\mu_i/N_i$ for some $i \in I$. Since a normal crossing model is not unique, one cannot expect all 'candidate poles' $-\mu_i/N_i$ to be actual poles of $Z_{X,\omega}$. But even candidate poles that appear in *every* model will not always be actual poles. This phenomenon is intimately related with the global monodromy property for Calabi–Yau varieties.

2.4. *The global monodromy property for Calabi–Yau varieties*

Let X be a proper smooth variety over K. For every $m \geq 0$, the *monodromy transformation* M_m for X is the action of the canonical topological generator σ of $\mathrm{Gal}(K^{\mathrm{alg}}/K)$ on the ℓ-adic cohomology $H^m(X \times_K K^{\mathrm{alg}}, \mathbb{Q}_\ell)$. A *monodromy eigenvalue for* X is an eigenvalue of M_m for some $m \geq 0$.

Informally, the global monodromy property (GMP) for Calabi–Yau varieties expresses that the poles of the motivic zeta function correspond to monodromy eigenvalues. By subsection 2.3, one can interpret the GMP for a Calabi–Yau variety X as a precise relation between its cohomology and the geometry of its *snc*-models.

Definition 2.4 *([5, Definition 6.4.1]).* Let X be a Calabi–Yau variety over K with volume form ω, and let σ be the canonical topological generator of the monodromy group $\mathrm{Gal}(K^{\mathrm{alg}}/K)$. We say that X satisfies the *global monodromy property* if there exists a finite subset S of $\mathbb{Z} \times \mathbb{Z}_{>0}$ such that

$$Z_{X,\omega}(T) \in \mathcal{M}_k \left[T, \frac{1}{1-\mathbb{L}^{\mu} T^N} \right]_{(\mu,N)\in S},$$

and such that for each $(\mu, N) \in S$, we have that $\exp(2\pi i \mu/N)$ is an eigenvalue of σ on $H^m(X \times_K K^{\mathrm{alg}}, \mathbb{Q}_\ell)$, for every embedding of $\mathbb{Q}_\ell$ into $\mathbb{C}$.

Remark 1. Whether a Calabi–Yau variety X satisfies the global monodromy property does not depend on the choice of volume form ω. Indeed, it follows immediately from the definition that

$$Z_{X,u\cdot\omega}(T) = Z_{X,\omega}(\mathbb{L}^{-\mathrm{ord}_t(u)}\, T),$$

for every unit $u \in K^\times$. Hence, changing ω amounts to shifting the poles by an integer value.

Halle and Nicaise proved that abelian varieties satisfy the GMP.

Theorem 2.5 *([4, Theorem 8.5]). Let X be an abelian variety over K and let ω be a volume form on X. The motivic zeta function $Z_{X,\omega}(T)$ has a unique pole and the global monodromy property holds for X.*

Remark 2. For a suitable choice of ω, the unique pole of $Z_{X,\omega}(T)$ coincides with Chai's base change conductor.

To investigate the GMP in dimension 2, the only remaining case is that of $K3$-surfaces, i.e. 2-dimensional Calabi–Yau varieties X with $H^1(X, \mathcal{O}_X) = 0$. Indeed, if X is a 2-dimensional Calabi–Yau variety with $H^1(X, \mathcal{O}_X) \neq 0$, then X is an abelian surface and hence the GMP holds for X by Theorem 2.5.

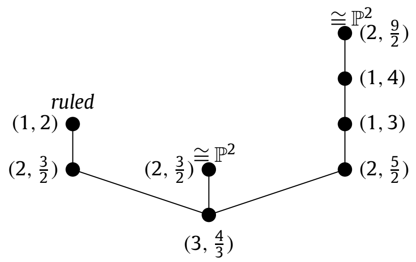

Fig. 1. An example of a flowerpot degeneration with three flowers.

Fig. 1. Un exemple d'une dégénérescence en pot de fleurs avec trois fleurs.

3. The Crauder–Morrison classification

Let X be a proper smooth surface over K with $\omega_{X/K}^{\otimes m} \simeq \mathcal{O}_X$ for some $m \geq 1$. Let $\mathcal{X}$ be an *snc*-model of X and write the special fiber as $\mathcal{X}_k = \sum_{i \in I} N_i E_i$. Suppose that $\mathcal{X}$ is triple point free; this means that $E_i \cap E_j \cap E_k = \emptyset$ when i, j, k are pairwise distinct. Assume furthermore that $\mathcal{X}$ is relatively minimal for this property, i.e. it is not possible to contract components in the special fiber such that the result is still a triple-point-free *snc*-model.

Let ω be a nowhere vanishing section of $\omega_{X/K}^{\otimes m}$. We can view ω as a rational section of $\omega_{\mathcal{X}/R}(\mathcal{X}_{k,\mathrm{red}} - \mathcal{X}_k)^{\otimes m}$, where $\mathcal{X}_{k,\mathrm{red}}$ is the divisor $\sum_{i \in I} E_i$. Define μ_i such that $m\mu_i$ is the multiplicity of $\operatorname{div}(\omega)$ along E_i. We define the *weight* ρ_i of the component E_i to be

$$\rho_i = \frac{\mu_i}{N_i} + 1.$$

Let Γ be the dual graph of the special fiber $\mathcal{X}_k$. Denote by $\Gamma_{\min}$ the graph spanned by the vertices corresponding to components E_i with minimal weight ρ_i.

Theorem 3.1 *(Crauder–Morrison Classification [2]). Let $\mathcal{X}$ be a relatively minimal triple-point-free model of X. Then $\mathcal{X}$ has the following properties:*

- *(i) $\Gamma_{\min}$ is a connected subgraph of Γ and it is either a single vertex, a cycle or a chain. We call $\mathcal{X}$ a flowerpot degeneration, a cycle degeneration or a chain degeneration respectively.*
- *(ii) Each connected component of $\Gamma \setminus \Gamma_{\min}$ is a chain (called a* flower*) $F_0 - F_1 - \cdots - F_l$, with only F_l meeting $\Gamma_{\min}$, and it meets a unique vertex of $\Gamma_{\min}$. The weights strictly increase: $\rho_0 > \rho_1 > \cdots > \rho_l$. The surfaces $F_1, \ldots, F_l$ are minimal ruled and F_0 is either minimal ruled or is isomorphic to $\mathbb{P}^2$. Flowers have been classified into 21 types.*
- *(iii) Suppose $\Gamma_{\min}$ is a single vertex P (called a* flowerpot*). Then the surface P is isomorphic to $\mathbb{P}^2$, or a ruled surface, or it has Kodaira dimension 0.*
- *(iv) Suppose $\Gamma_{\min}$ is a cycle $V_1 - V_2 - \cdots - V_k$. Then there are no flowers and all components have the same multiplicity, i.e. there exists an integer $N \geq 1$ such that the special fiber can be written as $\mathcal{X}_k = N\left(\sum_{i=1}^{k} V_i\right)$. Furthermore, all V_i are elliptic minimal ruled surfaces.*
- *(v) Suppose $\Gamma_{\min}$ is a chain $V_0 - V_1 - \cdots - V_k - V_{k+1}$. Then the surfaces $V_1, \ldots, V_k$ are elliptic ruled and if $i = 0$ or $k+1$, the surface V_i is either isomorphic to $\mathbb{P}^2$, or it is a rational or elliptic ruled surface.*

Example 1. The dual graph in Fig. 1 shows a flowerpot degeneration with three flowers. Two of the flowers have a top isomorphic to $\mathbb{P}^2$ and the third flower has a ruled top. The labels next to the vertices denote (N_i, ρ_i).

In [7], a proof of the following will appear:

Proposition 3.2. *If X is a $K3$ surface, then $\Gamma_{\min}$ is either a flowerpot or a chain.*

4. The monodromy property for $K3$ surfaces with a triple-point-free model

Let X be a $K3$ surface over K. Suppose that $\mathcal{X}$ is a relatively minimal triple-point-free model of X with special fiber $\mathcal{X}_k = \sum_{i \in I} N_i E_i$. Let ω be a volume form on X and let μ_i be the multiplicity of ω along E_i when viewed as a rational section of the log-relative canonical bundle $\omega_{\mathcal{X}/R}(\mathcal{X}_{k,\mathrm{red}} - \mathcal{X}_k)$. It is possible to determine the poles of the motivic zeta function of (X, ω) in terms of $\mathcal{X}$.

Theorem 4.1. *The poles of $Z_{X,\omega}(T)$ are exactly $-\mu_i/N_i$ for $i \in I$ such that ρ_i is minimal or such that $E_i \simeq \mathbb{P}^2$ and it is the top of a flower meeting the next component in a conic.*

The motivic zeta function of such $K3$ surfaces can have more than one pole, in contrast with abelian varieties. An easy example of a $K3$ surface with a motivic zeta function with more than one pole is the following:

Example 2. Let X be the $K3$ surface defined by the homogeneous equation

$$x^2w^2 + y^2w^2 + z^2w^2 + x^4 + y^4 + z^4 + tw^4 = 0 \tag{2}$$

in $\mathrm{Proj}\, K[x, y, z, w]$. Let $\mathcal{Y}$ be the closed subscheme of $\mathrm{Proj}\, R[x, y, z, w]$ defined by the homogeneous equation (2). We construct an *snc*-model $\mathcal{X}$ of X by blowing up $\mathcal{Y}$ at $P = (0 : 0 : 0 : 1)$. The special fiber of $\mathcal{X}$ is of the form $\mathcal{X}_k = D + 2E$, where D is the strict transform of $\mathcal{Y}_k$ and $E \simeq \mathbb{P}^2_k$ is the exceptional divisor. The strict transform D is smooth and intersects E transversally along a smooth conic C. For a suitable choice of volume form ω, one has $\mu_D = 0$ and $\mu_E = 1$. This means that $\mathcal{X}$ is a flowerpot degeneration, with D the flowerpot and E a flower. Using formula (1), the motivic zeta function can be computed as

$$Z_{X,\omega}(T) = [\widetilde{D^\circ}]\frac{T}{1-T} + [\widetilde{E^0}]\frac{\mathbb{L}^{-1}T^2}{1-\mathbb{L}^{-1}T^2} + [C]\frac{\mathbb{L}^{-1}T^3}{(1-T)(1-\mathbb{L}^{-1}T^2)}.$$

One checks that 0 and $-1/2$ are poles of $Z_{X,\omega}(T)$, which is in agreement with Theorem 4.1.

Our main result is that $K3$ surfaces with a flowerpot degeneration satisfy the GMP.

Theorem 4.2. *Let X be a $K3$ surface over K with relatively minimal triple-point-free model $\mathcal{X}$ over R with special fiber $\mathcal{X}_k$ a flowerpot degeneration. Then X satisfies the global monodromy property.*

We also have some partial results for chain degenerations.

Theorem 4.3. *Let X be a $K3$ surface over K with relatively minimal triple-point-free model $\mathcal{X}$ over R with special fiber $\mathcal{X}_k$ a chain degeneration. Denote by $V_0 — V_1 — \cdots — V_k — V_{k+1}$ the chain. The $K3$ surface X satisfies the global monodromy property if one of the following conditions hold:*

(i) the components $V_0, V_1, \ldots, V_{k+1}$ all have the same multiplicity N, or
(ii) neither V_0 nor V_{k+1} is a rational ruled surface.

It is not known which special fibers can occur as a chain degeneration of a $K3$ surface, which makes it hard to verify the GMP for $K3$ surfaces with a chain degeneration in general. Therefore, it is an open question to determine whether all $K3$-surfaces with a chain degeneration satisfy the GMP.

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