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Optimal control/Probability theory

Existence of an optimal control for a system driven by a degenerate coupled forward–backward stochastic differential equations ☆

Existence d'un contrôle optimal pour des équations différentielles stochastiques progressives–rétrogrades couplées dégénérées

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A R T I C L E I N F O

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A B S T R A C T

We establish the existence of an optimal control for a system driven by a coupled forward–backward stochastic differential equation (FBDSE) whose diffusion coefficient may degenerate (i.e. are not necessary uniformly elliptic). The cost functional is defined as the initial value of the backward component of the solution. We construct a sequence of approximating controlled systems, for which we show the existence of a sequence of feedback optimal controls. By passing to the limit, we get the existence of a feedback optimal control. Filippov's convexity condition is used to ensure that the optimal control is strict. The present result extends those obtained in [2,4] to controlled systems of coupled SDE–BSDE.

R É S U M É

Nous établissons l'existence d'un contrôle optimal, pour un système modélisé par une équation différentielle stochastique progressive–rétrograde (EDSPR) couplée, dont le coefficient de diffusion peut dégénérer (i.e. est non nécessairement uniformément elliptique). Par une double régularisation, nous construisons une suite de contrôles optimaux markoviens. Nous passons ensuite à la limite pour établir l'existence d'un contrôle optimal markovien. L'hypothèse de convexité de Filippov est utilisée pour montrer que le contrôle optimal ainsi construit est strict. Le résultat étend en un sens ceux obtenus dans [2,4] aux systèmes d'EDS–EDSR couplés.

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Soit $T > 0$ et $t \in [0, T]$. Soit $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$ un espace de probabilisé filtré et W un $\mathbb{R}^k$-mouvement brownien défini sur cet espace. On considère le système d'équations différentielles stochastiques progressives–rétrogrades couplées et contrôlées, définie sur $[t, T]$ par :

$$\begin{cases} dX_s^{t,x,u} = b(X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u}, u_s)ds + \sigma(X_s^{t,x,u}, Y_s^{t,x,u}, u_s)dW_s, \\ dY_s^{t,x,u} = -f(X_s^{t,x,u}, Y_s^{t,x,u}, Z_s^{t,x,u}, u_s)ds + Z_s^{t,x,u}dW_s + dM_s^{t,x,u}, \\ \langle M^{t,x,u}, W\rangle_s = 0, \\ X_t^{t,x,u} = x, \quad Y_T^{t,x,u} = \Phi(X_T^{t,x,u}), \quad M_t^{t,x,u} = 0. \end{cases} \tag{1}$$

Le quadruplet $(X^{t,x,u}, Y^{t,x,u}, Z^{t,x,u}, M^{t,x,u})$ est une solution de l'équation (1), c'est à dire que $(X^{t,x,u}, Y^{t,x,u}, Z^{t,x,u}, M^{t,x,u})$ est un processus $(\mathcal{F}_t)$-adapté, de carré intégrable, vérifiant l'équation (1) et $M^{t,x,u}$ est une martingale orthogonale à W. La variable contrôle u est un processus $(\mathcal{F}_t)$-adapté qui prend ses valeurs dans un espace métrique compact $\mathbb{A}$ de $\mathbb{R}^m$. L'ensemble des contrôles admissibles $\mathbb{U}$ *est l'ensemble des processus $(\mathcal{F}_t)$-adaptés à valeurs dans $\mathbb{A}$*. Un contrôle $\hat{u}$ est dit optimal s'il vérifie :

$$Y_t^{t,x,\hat{u}} = \text{essinf}\left\{Y_t^{t,x,u}, \ u \in \mathbb{U}\right\} := V(t,x). \tag{2}$$

Si $\hat{u} \in \mathbb{U}$, il est alors appelé contrôle optimal strict.

L'objectif de ce travail est d'établir l'existence d'un contrôle optimal strict pour le problème (1)–(2). Le résultat principal est :

Théorème 0.1. *On suppose que les hypothèses* **(B1)–(B5)** *sont satisfaites et qu'il y a unicité des solutions de viscosité bornées de l'équation (4). Alors, il existe un contrôle optimal strict solution du problème (1)–(2).*

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$ be a filtered probability space. Let W be a k-dimensional Brownian motion with respect to the (not necessary Brownian) filtration $(\mathcal{F}_t)$. We consider the controlled system of coupled FBDSE (1). *A solution to equation (1) is a quadruplet $(X^{t,x,u}, Y^{t,x,u}, Z^{t,x,u}, M^{t,x,u})$ which is $(\mathcal{F}_t)$-adapted, square integrable, and $M^{t,x,u}$ is a martingale which is orthogonal to W*. The control variable u is an $(\mathcal{F}_t)$-adapted process with values in some compact metric space $\mathbb{A}$ of $\mathbb{R}^m$. It should be noted that the filtered probability space and the Brownian motion may change with the control u. The set $\mathbb{U}$ of admissible controls is a *set of $\mathcal{F}_t$-adapted processes with values in $\mathbb{A}$*. The objective is to minimize $Y_t^{t,x,u}$ over the class $\mathbb{U}$ of admissible controls. *A control $\widehat{u}$ is called optimal if it satisfies (2). If $\widehat{u}$ belongs to $\mathbb{U}$, we then say that $\widehat{u}$ is a strict control.* The aim of the present paper is to establish the existence of a strict optimal control for the system (1)–(2). The present result extends those obtained in [2,4] to the case where the state equation is a coupled FBSDE. In contrast to [2,4], the uniform Lipschitz condition on the coefficients is not sufficient to ensure the existence of a unique solution to equation (1) for an arbitrary duration. This fact is well explained in [1], where two illustrating examples are given. To ensure the existence and uniqueness of solutions to equation (1), we moreover assume that the coefficients satisfy the so-called G-monotony condition introduced in [11]. The second difficulty concerns the gradient estimate of the approximating value function. The G-monotony condition combined with the comparison theorem of BSDEs allow us to overcome this second difficulty.

2. Assumptions and the main result

Since the probability space can be change with the control, we then use the following notations. We put $\nu := (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), W)_{t\in[0,T]}$ and let

- $\mathcal{S}_\nu^2(t,T;\mathbb{R}^m)$ denote the set of $\mathbb{R}^m$-valued, $(\mathcal{F}_t)$-adapted, continuous processes $(X_s, s \in [t,T])$ that satisfy $\mathbb{E}[\sup_{t\le s\le T}|X_s|^2] < \infty$,
- $\mathcal{H}_\nu^2(t,T;\mathbb{R}^m)$ be the set of $\mathbb{R}^m$-valued, $(\mathcal{F}_t)$-predictable processes $(Z_s, s \in [t,T])$ that satisfy $\mathbb{E}[\int_t^T |Z_s|^2 ds] < \infty$,
- $\mathcal{M}_\nu^2(t,T;\mathbb{R}^m)$ denote the set of all $\mathbb{R}^m$-valued, square integrable $(\mathcal{F}_t)$-martingales $M = (M_s)_{s\in[t,T]}$ such that $M_t = 0$,
- $\mathcal{U}_\nu(t)$ denote the set of admissible strict controls,
- $\mathcal{R}_\nu(t)$ denote the set of admissible relaxed controls.

For a given $1 \times d$ matrix G and $\lambda := (x, y, z)$, we put

$$A(t,\lambda) := \begin{pmatrix} -G^T f \\ Gb \\ G\sigma \end{pmatrix}(t,\lambda),$$

where G^T is the transpose of G.

Assumption (B). Throughout the paper, we assume that there exists a $1 \times k$ full-rank matrix G such that the following assumptions are satisfied.

- **(B1)** (i) $A(t,\lambda)$ is uniformly Lipschitz in λ, and for any λ, $A(\cdot,\lambda) \in \mathcal{H}^2(0,T;\mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^k)$,
  (ii) Φ is uniformly Lipschitz.
  We denote by K the Lipschitz constant of A and Φ.
- **(B2)** (i) $\langle A(t,\lambda) - A(t,\widehat{\lambda}), \lambda - \widehat{\lambda}\rangle \le -\beta_1|G\overline{x}|^2 - \beta_2(|G^{\mathrm{T}}\overline{y}|^2 + |G^{\mathrm{T}}\overline{z}|^2)$,
  (ii) $\langle \Phi(x) - \Phi(\widehat{x}), G(x-\widehat{x})\rangle \ge \mu_1|G\overline{x}|^2$, $\overline{x} = x - \widehat{x}$, $\overline{y} = y - \widehat{y}$, $\overline{z} = z - \widehat{z}$,
  where β_1, β_2 and mu_1 are nonnegative constants with $\beta_1 + \beta_2 > 0$, $\beta_2 + \mu_1 > 0$.
- **(B3)** the functions b, σ, f and Φ are bounded.
- **(B4)** for every $(x,y,z) \in \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^k$ the functions $b(x,y,z,.)$, $\sigma(x,y,.)$ and $f(x,y,z,.)$ are continuous.

Under the previous assumptions, equation (1) has a unique solution in the space $\mathcal{S}^2_\nu(t,T;\mathbb{R}^k) \times \mathcal{S}^2_\nu(t,T;\mathbb{R}) \times \mathcal{H}^2_\nu(t,T;\mathbb{R}^k) \times \mathcal{M}^2_\nu(t,T;\mathbb{R})$.

The *Filippov convexity assumption* is given by:

$$
(\mathbf{B5}) \begin{cases} \text{For every } (x,y) \in \mathbb{R}^k \times \mathbb{R} \text{ the following set is convex:} \\ \{((\sigma\sigma^*)(x,y,u), z\sigma^*(x,y,u), b(x,y,z,u), f(x,y,z,u)) | (u,z) \in \mathbb{A} \times \bar{B}_C(0)\}, \end{cases}
$$

where $\bar{B}_C(0) \subset \mathbb{R}^k$ is the closed ball around 0 with radius C.

Theorem 2.1. *Assume that* **(B1)–(B5)** *are satisfied and the uniqueness holds for bounded viscosity solutions of equation (4). Then, there exists a strict optimal control which solves the problem (1)–(2) in some reference stochastic system* $\bar{\nu} := (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, (\bar{\mathcal{F}}_t), \bar{W})$.

3. Proof

3.1. Construction of an approximating control

Let $\mathbb{S}^k$ denotes the space of the real symmetric $k \times k$ matrices. Let H be defined on $[0,T] \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{S}^k \times \mathbb{A}$ by:

$$
\begin{aligned} H(t,x,y,p,A,v) := &\frac{1}{2}\mathrm{tr}\left((\sigma\sigma^*)(t,x,y,v)A\right) + b(t,x,y,p\,\sigma(t,x,y,v),v)p \\ &+ f(t,x,y,p\,\sigma(t,x,y,v),v). \end{aligned} \tag{3}
$$

According to Li and Wei [9], the field $V(t,x)$ possesses a continuous deterministic version which solves the following Hamilton–Jacobi–Bellman (HJB) equation in viscosity sense on the set $[0,T] \times \mathbb{R}^k$:

$$
\begin{cases} \dfrac{\partial}{\partial t}V(t,x) + \inf\limits_{v \in \mathbb{A}} H(t,x,V(t,x),\nabla_x V(t,x),\nabla_{xx}V(t,x),v) = 0, \\ V(T,x) = \Phi(x),\ x \in \mathbb{R}^k, \end{cases} \tag{4}
$$

where $\nabla_x V$ and $\nabla_{xx} V$ respectively denote the gradient and the Hessian matrix of V. For $\delta \in (0,1]$, we respectively denote by b_δ, σ_δ, f_δ and Φ_δ the classical mollifier by convolution of b, σ, f and Φ. If h is K-Lipschitz, then one can show that for every x, $y \in \mathbb{R}^k$ and $\delta, \delta' > 0$: *(i)* $|h_\delta(x) - h(x)| \le K\delta$, *(ii)* $|h_\delta(x) - h_{\delta'}(x)| \le K|\delta - \delta'|$, *(iii)* $|h_\delta(x) - h_\delta(y)| \le K|x-y|$.

For $(x,y,p,A,v) \in \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{S}^k \times \mathbb{A}$, we put

$$
\begin{aligned} H^\delta(x,y,p,A,v) := &\frac{1}{2}\left(\mathrm{tr}\left((\sigma_\delta\sigma^*_\delta)(x,y,v) + \delta^2 I_{\mathbb{R}^d}\right)A\right) \\ &+ b_\delta(x,y,p\sigma_\delta(x,y,v),v)\,p \\ &+ f_\delta(x,y,p\sigma_\delta(x,y,v),v). \end{aligned} \tag{5}
$$

For $(t,x) \in [0,T] \times \mathbb{R}^k$, let $V^\delta(t,x)$ be the unique bounded continuous viscosity solution to the HJB equation:

$$
\begin{cases} \dfrac{\partial}{\partial t}V^\delta(t,x) + \inf\limits_{v \in \mathbb{A}} H^\delta\left(x,(V^\delta,\nabla_x V^\delta,\nabla_{xx}V^\delta)(t,x),v\right) = 0, \\ V^\delta(T,x) = \Phi_\delta(x),\ x \in \mathbb{R}^k. \end{cases} \tag{6}
$$

Since H^δ is smooth and $((\sigma_\delta\sigma^*_\delta)(x,y,v) + \delta^2 I_{\mathbb{R}^k})$ is strictly elliptic, then, according to Krylov [7], the unique bounded continuous viscosity solution to (6) belongs to $C^{1,2}_b([0,T] \times \mathbb{R}^k)$. The regularity of V^δ and the compactness of the control state $\mathbb{A}$ show that there exists a measurable function $v^\delta : [0,T] \times \mathbb{R}^k \mapsto \mathbb{A}$ such that, for every $(t,x) \in [0,T] \times \mathbb{R}^k$,

$$
H^\delta\left(x,(V^\delta,\nabla_x V^\delta,\nabla_{xx}V^\delta)(t,x),v^\delta(t,x)\right) = \inf_{v \in \mathbb{A}} H^\delta\left(x,(V^\delta,\nabla_x V^\delta,\nabla_{xx}V^\delta)(t,x),v\right). \tag{7}
$$

Since all the coefficients and the terminal datum are bounded and uniformly Lipschitz, then standard arguments of BSDEs show that V^δ is bounded uniformly with respect to δ and there exists a constant $C > 0$ which depends on K, T and the bounds of b, σ, Φ and f such that for every $t \in [0, T]$; $x, x' \in \mathbb{R}^k$ and $\delta, \delta' \in (0, 1]$

$$|V^{\delta'}(t,x') - V^\delta(t,x)| \leq C(|\delta - \delta'|^{1/2} + |x - x'|). \tag{8}$$

We then deduce that:

- the gradient $\nabla_x V^\delta$ is uniformly bounded in (t, x, δ),
- V^δ converges to some continuous function V uniformly on every compact sets of $[0, T] \times \mathbb{R}^k$.

We fix an arbitrary initial datum $(t, x) \in [0, T] \times \mathbb{R}^k$. Let B be an $\mathbb{R}^k$-Brownian motion which is independent of W. Let X^δ be the solution to the forward stochastic differential equation defined on $[t, T]$ by:

$$\begin{cases} \mathrm{d}X_s^\delta = \sigma_\delta(X_s^\delta, V^\delta(s, X_s^\delta), v^\delta(s, X_s^\delta))\mathrm{d}W_s + \delta \mathrm{d}B_s \\ +b_\delta(X_s^\delta, V^\delta(s, X_s^\delta), \nabla_x V^\delta(s, X_s^\delta)\sigma_\delta(X_s^\delta, V^\delta(s, X_s^\delta), v^\delta(s, X_s^\delta)), v^\delta(s, X_s^\delta))\mathrm{d}s, \\ X_t^\delta = x, \end{cases}$$

and define the processes Y^δ, Z^δ and U^δ by putting:

$$Y_s^\delta := V^\delta(s, X_s^\delta),\ Z_s^\delta := \nabla_x V^\delta(s, X_s^\delta)\sigma_\delta(X_s^\delta, V^\delta(s, X_s^\delta), u_s^\delta),\ U_s^\delta := \delta \nabla_x V^\delta(s, X_s^\delta).$$

3.2. Auxiliary sequence and the passing to the limits

Let $(\delta_n)_{n\in\mathbb{N}^*} \subset (0, 1]$ be such that $\delta_n \downarrow 0$. Let $(X^{\delta_n}, Y^{\delta_n}, Z^{\delta_n}, u^{\delta_n})$ be a subsequence of $(X^\delta, Y^\delta, Z^\delta, u^\delta)$. The idea consists to define a sequence of processes (X^n, Y^n) which will be used as an intermediate sequence, which allows us to show the convergence of $(X^{\delta_n}, Y^{\delta_n})$. We put $w_s^n := \nabla_x V^{\delta_n}(s, X_s^{\delta_n})$ and define an auxiliary sequence of processes (X^n, Y^n) by:

$$\begin{cases} \mathrm{d}X_s^n = b\left(X_s^n, Y_s^n, w_s^n \sigma\left(X_s^n, Y_s^n\right), u_s^{\delta_n}\right)\mathrm{d}s + \sigma\left(X_s^n, Y_s^n\right)\mathrm{d}W_s^{\delta_n}, \\ \quad X_t^n = x, \end{cases} \tag{9}$$

and

$$\begin{cases} \mathrm{d}Y_s^n = -f\left(X_s^n, Y_s^n, w_s^n \sigma\left(X_s^n, Y_s^n\right), u_s^{\delta_n}\right)\mathrm{d}s + w_s^n\sigma\left(X_s^n, Y_s^n\right)\mathrm{d}W_s^{\delta_n}, \\ \quad Y_t^n = V^{\delta_n}(t, x). \end{cases} \tag{10}$$

Using standard arguments of BSDEs, one can show that there exists $C > 0$ independent of δ_n such that for every $n \in \mathbb{N}$

$$\begin{aligned} &\mathbb{E}[\sup_{s\in[t,T]} |X_s^{\delta_n} - X_s^n|^2] \leq C\delta_n^2, \\ &\mathbb{E}[\sup_{s\in[t,T]} |Y_s^{\delta_n} - Y_s^n|^2] \leq C\delta_n^2. \end{aligned} \tag{11}$$

For $(x, y, z, \theta, u) \in \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{A}$, we set

$$\Sigma(x, y, z, \theta, u) := \begin{pmatrix} \sigma(x, y, u) & 0 \\ z & \theta \end{pmatrix},\ \ \beta(x, y, z, u) := \begin{pmatrix} b(x, y, u) \\ -f(x, y, z, u) \end{pmatrix},$$

$$\chi_s^n := \begin{pmatrix} X_s^n \\ Y_s^n \end{pmatrix},\ \ r_s^n := (w_s^n, 0, u_s^{\delta_n}) \ \text{ and } \ \mathcal{W}^{\delta_n} := \begin{pmatrix} W^{\delta_n} \\ B^{\delta_n} \end{pmatrix},$$

and rewrite the system (9)–(10) as follows:

$$\begin{cases} \mathrm{d}\chi_s^n = \beta(\chi_s^n, r_s^n)\mathrm{d}s + \Sigma(\chi_s^n, r_s^n)\mathrm{d}\mathcal{W}_s^{\delta_n}, \\ \chi_t^n = \begin{pmatrix} x \\ V^{\delta_n}(t, x) \end{pmatrix}. \end{cases} \tag{12}$$

Since $\nabla_x V^\delta$ is uniformly bounded, we can interpret $(r_s^n, s \in [0, T])$ as a control which takes its values in the compact metric space $\mathbb{A}_1 := \bar{B}_C(0) \times [0, K] \times \mathbb{A}$. We identify the control process r^n with the random measure q^n defined for $(s, a) \in [0, T] \times \mathbb{A}_1$, $\omega \in \Omega$ by:

$$q^n(\omega, \mathrm{d}s, \mathrm{d}a) := \delta_{r_s^n(\omega)}(\mathrm{d}a)\mathrm{d}s. \tag{13}$$

Note that for every n, q^n belongs to the space $\mathbb{V}$ of all Borel measures q on $[0, T] \times \mathbb{A}_1$ whose projection $q(\cdot \times \mathbb{A}_1)$ is the Lebesgue measure on $[0, T]$. Since the set $\{(\Sigma(x, y, z, \theta, v), \beta(x, y, z, \theta, v)), (x, y, z, \theta, v) \in \mathbb{R}^k \times \mathbb{R} \times A\}$ is bounded (because the coefficients are bounded) and $\mathbb{V}$ is compact in the topology induced by the weak convergence of measures, then the sequence of laws of $(\chi^n, q^n)_{n\geq 1}$ is tight in the space $C([0,T];\mathbb{R}^k\times\mathbb{R})\times\mathbb{V}$. According to Prokhorov's theorem, there exists a subsequence (still denoted by (χ^n, q^n)) which converges in law to some process (χ, q). Assumption **(B5)** and the result of [6] allow us to show that there exists a stochastic reference system $\bar{\nu} := (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, (\bar{\mathcal{F}}_t), \bar{\mathcal{W}})$ and an $(\bar{\mathcal{F}}_t)$-adapted process $\bar{r}$ with values in $\mathbb{A}_1$ such that the process χ solves the following equation:

$$\begin{cases} \mathrm{d}\chi_s = \beta(\chi_s, \bar{r}_s)\mathrm{d}s + \Sigma(\chi_s, \bar{r}_s)\mathrm{d}\bar{\mathcal{W}}_s, \quad s\in[0,T], \\ \chi_t = \begin{pmatrix} x \\ V(t,x) \end{pmatrix}. \end{cases} \tag{14}$$

Replacing Σ and β by their definitions then setting $\chi := (\bar{X}, \bar{Y})$, $\bar{\mathcal{W}} := (\bar{W}, \bar{B})$ and $\bar{r} := (\bar{w}, \bar{\theta}, \bar{u})$, we get the following FBSDE:

$$\begin{cases} \mathrm{d}\bar{X}_s = b(\bar{X}_s, \bar{Y}_s, \bar{Z}_s, \bar{u}_s)\mathrm{d}s + \sigma(\bar{X}_s, \bar{Y}_s, \bar{u}_s)\mathrm{d}\bar{W}_s, \\ \mathrm{d}\bar{Y}_s = -f(\bar{X}_s, \bar{Y}_s, \bar{Z}_s, \bar{u}_s)\mathrm{d}s + \bar{Z}_s\mathrm{d}\bar{W}_s + \bar{\theta}_s\mathrm{d}\bar{B}_s, \quad s\in[t,T], \\ \bar{X}_t = x, \ \bar{Y}_T = \Phi(X_T). \end{cases}$$

If we take $\bar{M}_s := \int_t^s \bar{\theta}_r\mathrm{d}\bar{B}_r$, then $\langle \bar{M}, \bar{W}\rangle_s = \int_t^s \bar{\theta}_r\mathrm{d}\langle \bar{B}, \bar{W}\rangle_r = 0$. It follows that $(\bar{X}, \bar{Y}, \bar{Z}, \bar{M})$ satisfies our original FBSDE in $\bar{\nu}$, that is:

$$\begin{cases} \mathrm{d}\bar{X}_s = b(\bar{X}_s, \bar{Y}_s, \bar{Z}_s, \bar{u}_s)\mathrm{d}s + \sigma(\bar{X}_s, \bar{Y}_s, \bar{u}_s)\mathrm{d}\bar{W}_s, \\ \mathrm{d}\bar{Y}_s = -f(\bar{X}_s, \bar{Y}_s, \bar{Z}_s, \bar{u}_s)\mathrm{d}s + \bar{Z}_s\mathrm{d}\bar{W}_s + \mathrm{d}\bar{M}_s, \quad s\in[t,T], \\ \bar{X}_t = x, \ \bar{Y}_T = \Phi(X_T). \end{cases} \tag{15}$$

Using inequality (11), we conclude that the sequence $(X^{\delta_n}, Y^{\delta_n})$ converges in law to $(\bar{X}, \bar{Y})$ and we have $\bar{Y}_s = V(s, \bar{X}_s)$ for each $s\in[t,T]$, $\bar{\mathbb{P}}$-a.s. In particular, $Y_T = \Phi(X_T)$, $\bar{\mathbb{P}}$-a.s. Since V is bounded, then the uniqueness of bounded viscosity solutions for equation (4) allows us to deduce that:

$$V(t,x) = \mathrm{essinf}_{u\in\mathcal{U}_{\bar{\nu}}(t)} J(t,x,u), \quad \bar{\mathbb{P}}\text{-a.s},$$

and

$$J(t,x,\bar{u}) = \mathrm{essinf}_{u\in\mathcal{U}_{\bar{\nu}}(t)} J(t,x,u), \quad \bar{\mathbb{P}}\text{-a.s}.$$

Therefore, $\bar{u}$ is a strict optimal control.

Some remarks. It is well known from [1] that the uniform Lipschitz condition is not sufficient to ensure the existence of a solution to coupled FBSDEs. Nevertheless, there are results on the existence and uniqueness of solutions to coupled FBSDEs under the uniform Lipschitz condition and supplementary assumptions on the coefficients, see, e.g., [5,10,12,13].

When the coefficients are uniformly Lipschitz and σ is non-degenerate, the existence and the uniqueness of solutions were established in [5] for equation (1); in this case, the existence of an optimal control was recently established in [3] when the coefficients σ and b are independent of z, and σ is independent of the control u. The case where b depends on z and σ is independent of z and u, the existence of an optimal control can be performed as in [3]. In the case where σ depends upon (x, y, u), the problem of the existence of an optimal control seems difficult to obtain by the method we developed here. Indeed, when the control enters the diffusion coefficient σ, we lead to an FBSDE with a measurable diffusion matrix and, in this case, the uniqueness of solution (even in the law sense) may fails. We know from [8] that when the diffusion coefficient is merely measurable, then even the uniqueness in law fails in general for Itô's forward SDEs in dimension strictly greater than 2.

The existence of an optimal control under the conditions used in [13] can be obtained by using the method we developed here.

Recently, in order to study fully coupled FBSDEs, the authors of [10] consider the uniform Lipschitz condition, to which they add a supplementary hypothesis, which consists in assuming the existence of a decoupling function. This latter hypothesis seems rather implicit and of abstract nature. It can not be easily exploited for our problem.

When the coefficient σ depends upon z and u, the problem of the existence of an optimal control for controlled fully coupled FBSDEs remains open and it is a challenge. In this framework, the existence of solutions follows from [12], and the Bellman dynamic programming principle was established in [9]. In this case, the Bellman dynamic programming principle leads to an HJB equation coupled with a constraint given by an algebraic equation.

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