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Mathematical analysis

On polynomial interpolation of bivariate harmonic polynomials

Sur l'interpolation polynomiale des polynômes harmoniques à deux variables

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ABSTRACT

We use Kergin and Hakopian interpolants to give some bases for the dual space of bivariate harmonic polynomials.

RÉSUMÉ

Nous utilisons les interpolations de Kergin et d'Hakopian pour construire des bases du dual de l'espace des polynômes harmoniques à deux variables.

1. Introduction

Let $\mathbb{P}$ be the vector space of all polynomials (with real coefficients) in $\mathbb{R}^2$ and $\mathbb{P}_n$ the subspace consisting of all polynomials of degree at most n. Let $\mathbb{H}_n$ be the set of all harmonic polynomials of (total) degree at most n in $\mathbb{R}^2$. It is well known that

$$\dim \mathbb{P}_n = \frac{(n+1)(n+2)}{2} \quad \text{and} \quad \dim \mathbb{H}_n = 2n+1.$$

A useful basis for $\mathbb{H}_n$ is given by $\left\{1, x, y, \dots, \Re(x+iy)^n, \Im(x+iy)^n\right\}$. We are concerned with the problem:

Find bases for the dual space $\mathbb{H}_n'$?

Many explicit bases are known. For example, if $\theta_1, \dots, \theta_{2n+1} \in [0, 2\pi)$ are $2n+1$ distinct angles, then $\{\delta_{(\cos\theta_k, \sin\theta_k)} : k = 1, \dots, 2n+1\}$ is a basis for $\mathbb{H}_n'$, where $\delta_{\mathbf{a}}$ is the point evaluation functional given by $\delta_{\mathbf{a}}(f) = f(\mathbf{a})$, because $\mathbb{H}_n$, restricted to the unit circle, becomes the space of all trigonometric polynomials of degree at most n. In recent works [4,5], Georgieva

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and Hofreither studied the interpolation of harmonic functions based on Radon projections along chords of the unit disk. They gave certain sets of $2n+1$ Radon projections that form dual bases for $\mathbb{H}_n$. Note that, in the above-mentioned results, the points and chords have special configurations. The aim of this note is to show that some bases for $\mathbb{H}'_n$ can be taken from the interpolation conditions for Kergin and Hakopian interpolants. In our results, the points are only assumed to be in general position, that is, no line contains three of them. For convenience, we recall some facts about Kergin and Hakopian interpolants.

Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_d\}$ be d points in general position in $\mathbb{R}^2$ with $d \geq 3$. This means that no three points of A are collinear. Let f be a continuous function defined in a neighborhood of the convex hull of A. Hakopian showed in [7] that there exists a unique polynomial $p \in \mathbb{P}_{d-2}$ such that

$$\int\limits_{[\mathbf{a}_j,\mathbf{a}_k]} p = \int\limits_{[\mathbf{a}_j,\mathbf{a}_k]} f, \quad \forall 1 \leq j < k \leq d,$$

where $\int\limits_{[\mathbf{a},\mathbf{b}]}$ is the simplex functional with respect to $\mathbf{a}$ and $\mathbf{b}$,

$$\int\limits_{[\mathbf{a},\mathbf{b}]} g = \int_0^1 g\big(\mathbf{a} + t(\mathbf{b} - \mathbf{a})\big)\,\mathrm{d}t.$$

Remark that the simplex functional is symmetric, i.e., $\int\limits_{[\mathbf{a},\mathbf{b}]} g = \int\limits_{[\mathbf{b},\mathbf{a}]} g$. The polynomial p will be called the Hakopian interpolation polynomial of f at A and denoted by $\mathscr{H}[A; f]$. Under the same assumption on A, Bos and Calvi showed in the proof of [2, Theorem 2.3] that the set of $\frac{(d+1)d}{2}$ functionals $\delta_{\mathbf{a}_k}$ for $k = 1, \ldots, d$ and $\int\limits_{[\mathbf{a}_j,\mathbf{a}_k]} D_{(\mathbf{a}_k - \mathbf{a}_j)^\perp}$ for $1 \leq j < k \leq d$ is a dual basis for $\mathbb{P}_{d-1}$, where $D_{\mathbf{a}^\perp}$ is the usual directional derivative along $\mathbf{a}^\perp = (-a_2, a_1)$ if $\mathbf{a} = (a_1, a_2)$, i.e., $D_{\mathbf{a}^\perp} g = -a_2 \frac{\partial g}{\partial x} + a_1 \frac{\partial g}{\partial y}$. Hence, if f is a continuously differentiable function in a neighborhood of the convex hull of A, there exists a unique $q \in \mathbb{P}_{n-1}$ such that

$$q(\mathbf{a}_k) = f(\mathbf{a}_k), \quad \forall k = 1, \ldots, d \quad \text{and} \quad \int\limits_{[\mathbf{a}_j,\mathbf{a}_k]} D_{(\mathbf{a}_k - \mathbf{a}_j)^\perp} q = \int\limits_{[\mathbf{a}_j,\mathbf{a}_k]} D_{(\mathbf{a}_k - \mathbf{a}_j)^\perp} f, \quad \forall 1 \leq j < k \leq d.$$

The polynomial q is called the Kergin interpolation polynomial of f at A and denoted by $\mathscr{K}[A; f]$. Compact formulas for $\mathscr{H}[A; f]$ and $\mathscr{K}[A; f]$ can be found in [9,2,8]. Kergin and Hakopian interpolants can be defined at any set of distinct points (without the assumption that the points are in general position). But it requires the existence of derivatives of higher order for the interpolated functions. Moreover, such types of interpolants are special cases of mean-value interpolation introduced in [6]. It is important to note that Kergin and Hakopian interpolants preserve all homogeneous partial differential relations. Here, we state the simplest case of this property.

Theorem 1.1. *Kergin and Hakopian interpolation polynomials of harmonic functions are harmonic polynomials.*

It seems that the above result appeared for the first time in [1, Lemma 12.4]. A stronger version may be found in [3, Corollary 1].

2. Main results

As usual, to $\mathbf{x} = (x, y) \in \mathbb{R}^2$, we associate the complex number $x + yi$ which we still denote by $\mathbf{x}$. It is understood that $\mathbf{x}$ and $\mathbf{y}$ are complex numbers when we write the product $\mathbf{xy}$. We start with a result that plays an important role in our arguments.

Lemma 2.1. *Let $A = \{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ be a set of four points in general position in $\mathbb{R}^2$.*

(1) *If $p \in \mathbb{H}_2$ and five out of six numbers $\int\limits_{[\mathbf{a}_j,\mathbf{a}_k]} p$, $0 \leq j < k \leq 3$, are equal to 0, then $p \equiv 0$.*

(2) *If five out of six functionals $\int\limits_{[\mathbf{a}_j,\mathbf{a}_k]}$, $0 \leq j < k \leq 3$, vanish at a harmonic polynomial Q, then the remaining functional also vanishes at Q.*

Proof. 1) Without loss of generality, we may assume that $\int\limits_{[\mathbf{a}_0,\mathbf{a}_1]} p = \int\limits_{[\mathbf{a}_0,\mathbf{a}_2]} p = \int\limits_{[\mathbf{a}_0,\mathbf{a}_3]} p = \int\limits_{[\mathbf{a}_1,\mathbf{a}_3]} p = \int\limits_{[\mathbf{a}_2,\mathbf{a}_3]} p = 0$. Since harmonicity is invariant under the change of variables of the complex form $\mathbf{x} \mapsto \eta\mathbf{x} + \xi$ with $\eta, \xi \in \mathbb{C}$, $\eta \neq 0$, we can assume that $\mathbf{a}_0 = (0,0)$ and $\mathbf{a}_3 = (1,0)$. We can find a quadratic polynomial $q(\mathbf{x}) = 3\alpha\mathbf{x}^2 + 2\beta\mathbf{x} + \gamma$ with $\alpha, \beta \in \mathbb{C}$ and $\gamma \in \mathbb{R}$ such that $p(x, y) = \Re q(\mathbf{x})$. For $0 \le j < k \le 3$, we have

$$\int_{[\mathbf{a}_j,\mathbf{a}_k]} p = \int_0^1 p\big(\mathbf{a}_j + t(\mathbf{a}_k - \mathbf{a}_j)\big)\,\mathrm{d}t = \int_0^1 \Re q\big(\mathbf{a}_j + t(\mathbf{a}_k - \mathbf{a}_j)\big)\,\mathrm{d}t = \Re\Big(\int_0^1 q\big(\mathbf{a}_j + t(\mathbf{a}_k - \mathbf{a}_j)\big)\,\mathrm{d}t\Big)$$
$$= \Re\Big(\alpha(\mathbf{a}_j^2 + \mathbf{a}_j\mathbf{a}_k + \mathbf{a}_k^2) + \beta(\mathbf{a}_j + \mathbf{a}_k) + \gamma\Big).$$

From the above computations and the hypothesis, we obtain

$$\begin{cases} \Re\Big(\alpha\mathbf{a}_1^2 + \beta\mathbf{a}_1 + \gamma\Big) & = 0 \\ \Re\Big(\alpha\mathbf{a}_2^2 + \beta\mathbf{a}_2 + \gamma\Big) & = 0 \\ \Re\Big(\alpha + \beta + \gamma\Big) & = 0 \\ \Re\Big(\alpha(\mathbf{a}_1^2 + \mathbf{a}_1 + 1) + \beta(\mathbf{a}_1 + 1) + \gamma\Big) & = 0 \\ \Re\Big(\alpha(\mathbf{a}_2^2 + \mathbf{a}_2 + 1) + \beta(\mathbf{a}_2 + 1) + \gamma\Big) & = 0 \end{cases} \tag{1}$$

Combining the first, third and fourth equations in (1) we get

$$\Re\big(\alpha\mathbf{a}_1 - \gamma\big) = 0. \tag{2}$$

Similarly, the first, third and fifth equations in (1) follow that

$$\Re\big(\alpha\mathbf{a}_2 - \gamma\big) = 0. \tag{3}$$

We write $\alpha = \alpha_1 + \mathrm{i}\alpha_2$, $\beta = \beta_1 + \mathrm{i}\beta_2$, $\mathbf{a}_1 = u_1 + \mathrm{i}u_2$ and $\mathbf{a}_2 = v_1 + \mathrm{i}v_2$. Since $\mathbf{a}_1$ and $\mathbf{a}_2$ does not lie on the line passing through $\mathbf{a}_0$ and $\mathbf{a}_3$, we have $u_2 v_2 \neq 0$. Substituting the algebraic forms of $\alpha, \beta, \mathbf{a}_1, \mathbf{a}_2$ into the first three equations in (1), and (2), (3), we obtain

$$\begin{cases} (u_1^2 - u_2^2)\alpha_1 - 2u_1u_2\alpha_2 + u_1\beta_1 - u_2\beta_2 + \gamma & = 0 \\ (v_1^2 - v_2^2)\alpha_1 - 2v_1v_2\alpha_2 + v_1\beta_1 - v_2\beta_2 + \gamma & = 0 \\ \alpha_1 + \beta_1 + \gamma & = 0 \\ u_1\alpha_1 - u_2\alpha_2 - \gamma & = 0 \\ v_1\alpha_1 - v_2\alpha_2 - \gamma & = 0 \end{cases} \tag{4}$$

We can regard (4) as a system of five linear equations where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma$ are unknowns. Easy computations show that the determinant of the coefficient matrix is equal to $-u_2v_2\big((u_1 - v_1)^2 + (u_2 - v_2)^2\big) \neq 0$. It follows that (4) has a unique solution $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \gamma = 0$. This forces $q = 0$, and hence $p = 0$.

2) We define $q = \mathscr{H}[\{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}; Q]$. Then, by Theorem 1.1, q is a harmonic polynomial of degree at most 2. We have

$$\int_{[\mathbf{a}_j,\mathbf{a}_k]} q = \int_{[\mathbf{a}_j,\mathbf{a}_k]} Q, \quad 0 \le j < k \le 3. \tag{5}$$

The hypothesis yields that five out of six numbers $\int_{[\mathbf{a}_j,\mathbf{a}_k]} q$, $0 \le j < k \le 3$, are equal to 0. The first part of this lemma implies $q \equiv 0$. Hence, from (5), we conclude that the remaining functional vanishes at Q. The proof is complete. □

Definition 2.1. We say a function f is five-vanishing at the set $\{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ if five out of six numbers $\int_{[\mathbf{a}_j,\mathbf{a}_k]} f$, $0 \le j < k \le 3$, equal 0.

Theorem 2.2. *If $\{\mathbf{a}_1, \dots, \mathbf{a}_d\}$ is a set of d points in general position in $\mathbb{R}^2$ with $d \ge 3$, then both sets of functionals*

$$S_1 = \Big\{ \int_{[\mathbf{a}_1,\mathbf{a}_2]}, \int_{[\mathbf{a}_2,\mathbf{a}_3]}, \dots, \int_{[\mathbf{a}_d,\mathbf{a}_1]}, \int_{[\mathbf{a}_1,\mathbf{a}_3]}, \int_{[\mathbf{a}_1,\mathbf{a}_4]}, \dots, \int_{[\mathbf{a}_1,\mathbf{a}_{d-1}]} \Big\},$$
$$S_2 = \Big\{ \int_{[\mathbf{a}_1,\mathbf{a}_2]}, \int_{[\mathbf{a}_2,\mathbf{a}_3]}, \dots, \int_{[\mathbf{a}_d,\mathbf{a}_1]}, \int_{[\mathbf{a}_1,\mathbf{a}_3]}, \int_{[\mathbf{a}_2,\mathbf{a}_4]}, \int_{[\mathbf{a}_3,\mathbf{a}_5]}, \dots, \int_{[\mathbf{a}_{d-3},\mathbf{a}_{d-1}]} \Big\}$$

are bases for the dual space $\mathbb{H}'_{d-2}$.

Proof. For convenience, we set $\mathbf{a}_{d+1} = \mathbf{a}_1$. We first prove the assertion for S_1. Since $\#S_1 = \dim \mathbb{H}_{d-2} = 2d-3$, it suffices to show that if $Q \in \mathbb{H}_{d-2}$ satisfies the following relations

$$\int\limits_{[\mathbf{a}_k,\mathbf{a}_{k+1}]} Q = 0, \quad \forall k = 1,\dots,d \quad \text{and} \quad \int\limits_{[\mathbf{a}_1,\mathbf{a}_k]} Q = 0, \quad \forall k = 3,\dots,d-1, \tag{6}$$

then $Q \equiv 0$. The case $d = 3$ is trivial. Hence, we can assume that $d \geq 4$. It is easily seen from (6) that

$$\int\limits_{[\mathbf{a}_1,\mathbf{a}_2]} Q = \int\limits_{[\mathbf{a}_1,\mathbf{a}_3]} Q = \int\limits_{[\mathbf{a}_1,\mathbf{a}_4]} Q = \int\limits_{[\mathbf{a}_2,\mathbf{a}_3]} Q = \int\limits_{[\mathbf{a}_3,\mathbf{a}_4]} Q = 0.$$

In other words, Q is five-vanishing at $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$. Lemma 2.1 leads to $\int_{[\mathbf{a}_2,\mathbf{a}_4]} Q = 0$. We now apply this argument again, with $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ replaced by $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4, \mathbf{a}_5\}$, to obtain $\int_{[\mathbf{a}_2,\mathbf{a}_5]} Q = 0$. In the same manner we can see that $\int_{[\mathbf{a}_2,\mathbf{a}_k]} Q = 0$ for $k = 4,\dots,n$. We now apply the above arguments repeatedly to obtain $\int_{[\mathbf{a}_j,\mathbf{a}_k]} Q = 0$, $1 \leq j < k \leq d$. The uniqueness of Hakopian interpolation gives $Q = 0$.

Similarly, to prove the assertion for S_2, we need only to verify that a polynomial $P \in \mathbb{H}_{d-2}$ satisfying relations below must be identically zero,

$$\int\limits_{[\mathbf{a}_k,\mathbf{a}_{k+1}]} P = 0, \quad \forall k = 1,\dots,d \quad \text{and} \quad \int\limits_{[\mathbf{a}_k,\mathbf{a}_{k+2}]} P = 0, \quad \forall k = 1,\dots,d-3. \tag{7}$$

The cases where $k = 3, 4, 5$ are trivial. We assume that $d \geq 6$. We will prove by induction that $\int_{[\mathbf{a}_2,\mathbf{a}_k]} P = 0$ for $k = 5,\dots,d-1$. Indeed, by hypothesis, it is easily seen that P is five-vanishing at $\{\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$. Lemma 2.1 yields $\int_{[\mathbf{a}_2,\mathbf{a}_5]} P = 0$. Assume that the assertion holds up to $k < d-1$; we will prove it for $k+1$. By the induction hypothesis, P is also five-vanishing at $\{\mathbf{a}_2, \mathbf{a}_{k-1}, \mathbf{a}_k, \mathbf{a}_{k+1}\}$, we conclude from Lemma 2.1 that $\int_{[\mathbf{a}_2,\mathbf{a}_{k+1}]} P = 0$. Now, since S_1 is a dual basis for $\mathbb{H}'_{d-2}$, we need only to show that $\int_{[\mathbf{a}_1,\mathbf{a}_k]} Q = 0$ for $k = 4,\dots,d-1$. We have $\int_{[\mathbf{a}_1,\mathbf{a}_4]} Q = 0$, because P is five-vanishing at $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$. Assume that $\int_{[\mathbf{a}_1,\mathbf{a}_k]} P = 0$ for k with $4 \leq k < d-1$. Then P is five-vanishing at $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_k, \mathbf{a}_{k+1}\}$. Hence $\int_{[\mathbf{a}_1,\mathbf{a}_{k+1}]} P = 0$ due to Lemma 2.1. The proof is complete. □

The method of proof of Theorem 2.2 enables us to find other bases for $\mathbb{H}'_{d-2}$. The following example shows that there is a set consisting of $2d-3$ functionals, which is not a basis for $\mathbb{H}'_{d-2}$.

Example 1. We consider four points $\mathbf{a}_1 = (0,0)$, $\mathbf{a}_2 = (1,0)$, $\mathbf{a}_3 = (0,1)$, $\mathbf{a}_4 = (1,1)$ and $p \in \mathbb{H}_3$ given by

$$p(x,y) = -3x + 3y + 6(x^2 - y^2) - 2(x^3 - 3xy^2) - 2(3x^2y - y^3).$$

It is easily seen that $\int_{[\mathbf{a}_j,\mathbf{a}_k]} p = 0$ for $1 \leq j < k \leq 4$. We have $\int_{[\mathbf{a}_1,(u,-u)]} p = \int_0^1 \big(-6ut + 8u^3t^3\big)\,\mathrm{d}t = -3u + 2u^3$, because $p(u,-u) = -6u + 8u^3$. The cubic equation $-3u + 2u^3 = 0$ has a solution $\sqrt{\frac{3}{2}}$. Set $\mathbf{a}_5 = (\sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}})$. Then the set of five points $\{\mathbf{a}_1,\dots,\mathbf{a}_5\}$ is in general position in $\mathbb{R}^2$. Moreover, seven functionals $\int_{[\mathbf{a}_j,\mathbf{a}_k]}$ for $1 \leq j < k \leq 4$ and $\int_{[\mathbf{a}_1,\mathbf{a}_5]}$ vanish at p.

Corollary 2.3. *Let $A = \{\mathbf{a}_1,\dots,\mathbf{a}_d\}$ be d points in general position in $\mathbb{R}^2$. Let f be a harmonic function in a neighborhood of the convex hull of A. Then the harmonic polynomial $p \in \mathbb{H}_{d-2}$ is the Hakopian interpolation polynomial of f at A if and only if*

$$\mu(p) = \mu(f), \quad \forall \mu \in S_1 \quad \text{or} \quad \nu(p) = \nu(f), \quad \forall \nu \in S_2.$$

Proof. We will prove the assertion for S_1. Let p be a harmonic polynomial of degree at most $d-2$ such that $\mu(p) = \mu(f)$ for all $\mu \in S_1$. Set $Q = p - \mathscr{H}[A; f]$. Then $Q \in \mathbb{H}_{d-2}$ and $\mu(p) = 0$ for all $\mu \in S_1$. Theorem 2.2 implies $Q = 0$. Hence, $p = \mathscr{H}[A; f]$. The proof for S_2 is similar and we omit it. □

Next, we give another type of basis for $\mathbb{H}'_{d-1}$. Let $\mathbf{a}$ and $\mathbf{b}$ be two distinct points in $\mathbb{R}^2$. The line segment with end points $\mathbf{a}$ and $\mathbf{b}$ is denoted by $[\mathbf{a},\mathbf{b}]$. Let $\mu_{[\mathbf{a},\mathbf{b}]}$ denote the functional

$$\mu_{[\mathbf{a},\mathbf{b}]}(f) = \int_{[\mathbf{a},\mathbf{b}]} D_{(\mathbf{b}-\mathbf{a})^{\perp}} f.$$

Note that $\mu_{[\mathbf{a},\mathbf{b}]} = -\mu_{[\mathbf{b},\mathbf{a}]}$.

Definition 2.2. Let $A = \{\mathbf{a}_1, \dots, \mathbf{a}_d\}$ be a set of d points in general position in $\mathbb{R}^2$. We say that a subset I of $\{[\mathbf{a}_j, \mathbf{a}_k] : 1 \le j < k \le d\}$ is regular if it satisfies the following conditions:

(1) $\#I = d-1$;
(2) The union of segments in I forms a pathwise connected set;
(3) Every point in A is an end point of a segment in I.

There are many examples of regular sets, e.g.,

$$\{[\mathbf{a}_1,\mathbf{a}_2],[\mathbf{a}_1,\mathbf{a}_3],\dots,[\mathbf{a}_1,\mathbf{a}_n]\}, \quad \{[\mathbf{a}_1,\mathbf{a}_2],[\mathbf{a}_2,\mathbf{a}_3],\dots,[\mathbf{a}_{n-1},\mathbf{a}_n]\}.$$

Theorem 2.4. *Let $A = \{\mathbf{a}_1, \dots, \mathbf{a}_d\}$ be a set of d points in general position in $\mathbb{R}^2$. Let I be a regular subset of $\{[\mathbf{a}_j, \mathbf{a}_k] : 1 \le j < k \le d\}$. Then the set of functionals*

$$\left\{\delta_{\mathbf{a}_k} : \quad k = 1, \dots, d\right\} \cup \left\{\mu_{[\mathbf{a}_j,\mathbf{a}_k]} : [\mathbf{a}_j,\mathbf{a}_k] \in I\right\}$$

forms a dual basis for $\mathbb{H}'_{d-1}$.

Proof. Since the number of functionals is equal to $2d-1 = \dim \mathbb{H}_{d-1}$, it is sufficient to prove that if $Q \in \mathbb{H}_{d-2}$ satisfies the following relations

$$\delta_{\mathbf{a}_k}(Q) = 0 \quad \forall k = 1, \dots, d, \quad \text{and} \quad \mu_{[\mathbf{a}_j,\mathbf{a}_k]}(Q) = 0, \quad \forall [\mathbf{a}_j,\mathbf{a}_k] \in I,$$

then $Q \equiv 0$. Let us fix $1 \le l < m \le d$. If $[\mathbf{a}_l, \mathbf{a}_m] \in I$, then we have $\mu_{[\mathbf{a}_l,\mathbf{a}_m]}(Q) = 0$ by the above. Otherwise, since I is regular, we can find n points $\mathbf{a}_{s_1}, \dots, \mathbf{a}_{s_n}$ such that

$$[\mathbf{a}_{s_k}, \mathbf{a}_{s_{k+1}}] \in I \quad \text{or} \quad [\mathbf{a}_{s_{k+1}}, \mathbf{a}_{s_k}] \in I, \quad k = 0, \dots, n,$$

where $\mathbf{a}_{s_0} = \mathbf{a}_l$ and $\mathbf{a}_{s_{n+1}} = \mathbf{a}_m$. Let $\widetilde{Q}$ be a harmonic conjugate to Q, i.e., $Q + i\widetilde{Q}$ is a polynomial in $\mathbb{C}$. Using [2, Lemma 2.4], we see that

$$\begin{aligned}
\mu_{[\mathbf{a}_l,\mathbf{a}_m]}(Q) &= \widetilde{Q}(\mathbf{a}_m) - \widetilde{Q}(\mathbf{a}_l) = \left(\widetilde{Q}(\mathbf{a}_m) - \widetilde{Q}(\mathbf{a}_{s_n})\right) + \sum_{i=1}^{n-1}\left(\widetilde{Q}(\mathbf{a}_{s_{i+1}}) - \widetilde{Q}(\mathbf{a}_{s_i})\right) + \left(\widetilde{Q}(\mathbf{a}_{s_1}) - \widetilde{Q}(\mathbf{a}_l)\right) \\
&= \mu_{[\mathbf{a}_{s_n},\mathbf{a}_m]}(Q) + \sum_{i=1}^{n-1} \mu_{[\mathbf{a}_{s_i},\mathbf{a}_{s_{i+1}}]}(Q) + \mu_{[\mathbf{a}_l,\mathbf{a}_{s_1}]}(Q) = 0.
\end{aligned}$$

Consequently,

$$Q(\mathbf{a}_k) = 0, \quad \forall k = 1, \dots, d, \quad \text{and} \quad \mu_{[\mathbf{a}_j,\mathbf{a}_k]}(Q) = 0, \quad \forall 1 \le j < k \le d.$$

The uniqueness of Kergin interpolation implies that $Q = 0$, and the proof is complete. □

Example 2. We consider four points in general position $\mathbf{a}_1 = (0,0)$, $\mathbf{a}_2 = (1,0)$, $\mathbf{a}_3 = (0,1)$, $\mathbf{a}_4 = (\frac{1}{2}, -\frac{1}{2})$. The polynomial $p(x,y) = x - y - 3(x^2 - y^2) + 2(x^3 - 3xy^2) + 2(3x^2y - y^3)$ is in $\mathbb{H}_2$. Easy computations shows that

$$\delta_{\mathbf{a}_i}(p) = 0, \quad \forall i = 1, \dots, 4 \quad \text{and} \quad \mu_{[\mathbf{a}_j,\mathbf{a}_k]}(p) = 0, \quad 1 \le i < j \le 3. \tag{8}$$

Remark that $\dim \mathbb{H}_3 = 7$ and the set I of three functionals in (8) is not regular. Hence the assumption of regularity cannot be removed.

The proof of the following result is similar to that given in Corollary 2.3.

Corollary 2.5. *Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_d\}$ be a set of d points in general position in $\mathbb{R}^2$. Let I be a regular subset of $\{[\mathbf{a}_j, \mathbf{a}_k] : 1 \le j < k \le d\}$. Let f be a harmonic function in a neighborhood of A. Then the harmonic polynomial $p \in \mathbb{H}_{d-1}$ is the Kergin interpolation polynomial of f at A if and only if it satisfies the following relations*

$$\delta_{\mathbf{a}_k}(p) = \delta_{\mathbf{a}_k}(f), \quad \forall k = 1, \ldots, d, \quad \textit{and} \quad \mu_{[\mathbf{a}_j,\mathbf{a}_k]}(p) = \mu_{[\mathbf{a}_j,\mathbf{a}_k]}(f), \quad \forall [\mathbf{a}_j, \mathbf{a}_k] \in I.$$

We end up the paper with the following questions.

Open questions.

(1) Characterize all subsets S of $\{\int_{[\mathbf{a}_j,\mathbf{a}_k]} : 1 \le j < k \le d\}$ such that $\#S = 2d - 3$ and S forms a basis for $\mathbb{H}'_{d-2}$.
(2) Find analogous bases for the dual spaces of harmonic polynomials in $\mathbb{R}^n$?
(3) Find dual bases of the bases for $\mathbb{H}'_n$ given in Theorems 2.2 and 2.4?

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References

[1] B. Bojanov, H. Hakopian, A. Sahakian, Spline Functions and Multivariate Interpolations, Springer-Verlag, 1993.
[2] L. Bos, J.-P. Calvi, Kergin interpolant at the roots of unity approximate C^2 functions, J. Anal. Math. 72 (1997) 203–221.
[3] J.-P. Calvi, L. Filipsson, The polynomial projectors that preserve homogeneous differential relations: a new characterization of Kergin interpolation, East J. Approx. 10 (2004) 441–454.
[4] I. Georgieva, C. Hofreither, Interpolation of harmonic functions bases on Radon projections, Numer. Math. 127 (2014) 423–445.
[5] I. Georgieva, C. Hofreither, New results on regularity and errors of harmonic interpolation using Radon projections, J. Comput. Appl. Math. 29 (2016) 73–81.
[6] T.N.T. Goodman, Interpolation in minimum seminorm and multivariate B-spline, J. Approx. Theory 37 (1983) 212–223.
[7] H.A. Hakopian, Multivariate divided differences and multivariate interpolation of Lagrange and Hermite type, J. Approx. Theory 34 (1982) 286–305.
[8] C.A. Micchelli, A constructive approach to Kergin interpolation in R^k: multivariate B-spline and Lagrange interpolation, Rocky Mt. J. Math. 10 (1980) 485–497.
[9] V.M. Phung, On the convergence of Kergin and Hakopian interpolants at Leja sequences for the disk, Acta Math. Hung. 136 (2012) 165–188.