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# On non-Kähler compact complex manifolds with balanced and astheno-Kähler metrics

## *Sur les variétés compactes complexes non Kähler avec des métriques équilibrées et asthéno-Kähler*

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ABSTRACT

In this note, we construct, for every $n \geq 4$, a non-Kähler compact complex manifold $X$ of complex dimension $n$ admitting a balanced metric and an astheno-Kähler metric, which is in addition $k$-th Gauduchon for any $1 \leq k \leq n-1$.



RÉSUMÉ

Dans cette note, nous construisons, pour chaque $n \geq 4$, une variété compacte complexe non Kähler $X$ de dimension complexe $n$ admettant une métrique equilibrée et une métrique asthéno-Kähler ; de plus, cette métrique est $k$-ième Gauduchon pour $1 \leq k \leq n-1$.



## 1. Introduction

Let $X$ be a compact complex manifold of complex dimension $n$, and let $F$ be a Hermitian metric on $X$. It is well known that the metric $F$ is called *balanced* if the Lee form vanishes; equivalently, the form $F^{n-1}$ is closed. If $\partial\bar{\partial}F^{n-2}=0$, then the Hermitian metric $F$ is said to be *astheno-Kähler*. Balanced metrics are studied by Michelsohn in [15], and the class of astheno-Kähler metrics is considered by Jost and Yau in [13] to extend Siu's rigidity theorem to non-Kähler manifolds. This note is motivated by a question in the paper [16] by Székelyhidi, Tosatti and Weinkove about the existence of examples of non-Kähler compact complex manifolds admitting both balanced and astheno-Kähler metrics. Recently, two examples, in dimensions 4 and 11, have been constructed by Fino, Grantcharov and Vezzoni in [4]. Our goal is to present examples in any complex dimension $n \geq 4$. Moreover, we show that our astheno-Kähler metrics satisfy the stronger condition of being $k$-th Gauduchon for every $1 \leq k \leq n-1$.

When the Lee form is co-closed, equivalently $F^{n-1}$ is $\partial\bar{\partial}$-closed, the Hermitian metric $F$ is called *standard* or *Gauduchon*. By [10], there is a Gauduchon metric in the conformal class of every Hermitian metric on $X$. Fu, Wang and Wu introduce

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and study in [9] the following generalization of Gauduchon metrics. Let $k$ be an integer such that $1 \le k \le n-1$, a Hermitian metric $F$ on $X$ is called *k-th Gauduchon* if $\partial\bar{\partial}F^k \wedge F^{n-k-1} = 0$.

By definition, $(n-1)$-th Gauduchon metrics are the usual Gauduchon metrics. Astheno-Kähler metrics are particular examples of $(n-2)$-th Gauduchon metrics, and any *pluriclosed* (*SKT*) metric, i.e. a metric satisfying $\partial\bar{\partial}F = 0$, is in particular 1-st Gauduchon.

In [9] a unique constant $\gamma_k(F)$ is associated with any Hermitian metric $F$ on $X$. This constant is invariant by biholomorphisms and depends smoothly on $F$. Moreover, it is proved that $\gamma_k(F) = 0$ if and only if there exists a $k$-th Gauduchon metric in the conformal class of $F$.

On a compact complex surface any Hermitian metric is automatically astheno-Kähler, and the balanced condition is the same as the Kähler one. In complex dimension $n = 3$, the notion of astheno-Kähler metric coincides with that of SKT metric.

SKT or astheno-Kähler metrics on a compact complex manifold $X$ of complex dimension $n \ge 3$ cannot be balanced unless they are Kähler (see [1,14]). If the Lee form is exact, then the Hermitian structure is conformally balanced. By [5, 11], a conformally balanced SKT or astheno-Kähler metric whose Bismut connection has (restricted) holonomy contained in $SU(n)$ is necessarily Kähler. Similar results for 1-st Gauduchon metrics are proved in [6]. Ivanov and Papadopoulos [12] have extended these results to any generalized $k$-th Gauduchon metric, for $k \ne n-1$.

A recent conjecture in [7] asserts that if $X$ has an SKT metric and another metric which is balanced, then $X$ is Kähler. By a result of Chiose [3], a manifold in the Fujiki class $\mathcal{C}$ has no SKT metrics unless it is Kähler. In [8], the conjecture is studied on the class of *complex nilmanifolds* $X = (\Gamma\backslash G, J)$, i.e. on compact quotients of simply-connected nilpotent Lie groups $G$ by uniform discrete subgroups $\Gamma$ endowed with an invariant complex structure $J$. In this note, we construct, for every $n \ge 4$, a non-SKT complex nilmanifold $X$ of complex dimension $n$ admitting a balanced metric and an astheno-Kähler metric, which additionally satisfies the stronger condition of being $k$-th Gauduchon for every $1 \le k \le n-1$.

## 2. Generalized Gauduchon metrics on complex nilmanifolds

We first prove the following general result.

**Proposition 2.1.** *Let $X$ be a compact complex manifold of complex dimension $n \ge 3$, and $F$ any Hermitian metric on $X$. For any integer $k$ such that $1 \le k \le n-1$, we have*

$$\int_X \partial\bar{\partial}F^k \wedge F^{n-k-1} = \frac{k(n-k-1)}{n-2} \int_X \partial\bar{\partial}F \wedge F^{n-2}. \tag{1}$$

**Proof.** The equality (1) is trivial for $k = 1$ and for $k = n-1$. Let us then suppose that $2 \le k \le n-2$. By induction, one has $\partial F^k = k\,\partial F \wedge F^{k-1}$ and $\bar{\partial}F^k = k\,\bar{\partial}F \wedge F^{k-1}$. Therefore,

$$\partial\bar{\partial}F^k \wedge F^{n-k-1} = k\,\partial\bar{\partial}F \wedge F^{n-2} + k(k-1)\,\partial F \wedge \bar{\partial}F \wedge F^{n-3}. \tag{2}$$

On the other hand,

$$\begin{aligned}\partial\bar{\partial}F^k \wedge F^{n-k-1} &= \mathrm{d}\left(\bar{\partial}F^k \wedge F^{n-k-1}\right) + \bar{\partial}F^k \wedge \partial F^{n-k-1}\\ &= \mathrm{d}\left(\bar{\partial}F^k \wedge F^{n-k-1}\right) - k\,(n-k-1)\partial F \wedge \bar{\partial}F \wedge F^{n-3},\end{aligned}$$

so we get

$$\partial F \wedge \bar{\partial}F \wedge F^{n-3} = \frac{-1}{k(n-k-1)}\left[\partial\bar{\partial}F^k \wedge F^{n-k-1} - \mathrm{d}\left(\bar{\partial}F^k \wedge F^{n-k-1}\right)\right].$$

Now, if we substitute this expression in (2), we have

$$\partial\bar{\partial}F^k \wedge F^{n-k-1} = k\,\partial\bar{\partial}F \wedge F^{n-2} - \frac{k-1}{n-k-1}\,\partial\bar{\partial}F^k \wedge F^{n-k-1} + \frac{k-1}{n-k-1}\,\mathrm{d}\left(\bar{\partial}F^k \wedge F^{n-k-1}\right),$$

which leads to

$$(n-2)\,\partial\bar{\partial}F^k \wedge F^{n-k-1} = k(n-k-1)\,\partial\bar{\partial}F \wedge F^{n-2} + (k-1)\,\mathrm{d}\left(\bar{\partial}F^k \wedge F^{n-k-1}\right).$$

By Stokes' theorem, we arrive at (1). □

Next we apply the previous proposition to homogeneous compact complex manifolds $X$, of complex dimension $n$, endowed with an *invariant* Hermitian metric $F$. We recall that in [6, Lemma 4.7] the following duality result is proved: for each $k = 1, \ldots, \left[\frac{n}{2}\right] - 1$, the Hermitian metric $F$ is $k$-th Gauduchon if and only if it is $(n-k-1)$-th Gauduchon. As a consequence of Proposition 2.1, the relation among these metrics turns out to be stronger.

**Proposition 2.2.** *Let $F$ be an invariant Hermitian metric on a homogeneous compact complex manifold $X$ of complex dimension $n \geq 3$, and let $k$ be an integer such that $1 \leq k \leq n-2$. Then,*

(i) *$F$ is always Gauduchon, and*
(ii) *if $F$ is $k$-th Gauduchon for some $k$, then it is $k$-th Gauduchon for any other $k$.*

**Proof.** For any invariant Hermitian metric $F$ and any $1 \leq k \leq n-2$, the real $(n,n)$-form $\frac{i}{2}\partial\bar\partial F^k \wedge F^{n-k-1}$ is proportional to the volume form $F^n$, hence

$$\frac{i}{2}\partial\bar\partial F^k \wedge F^{n-k-1} = C_{F,k}\, F^n, \tag{3}$$

for some constant $C_{F,k} \in \mathbb{R}$ (notice that $C_{F,k}$ is a multiple of the constant $\gamma_k(F)$ in [9]).

If $k = n-1$ then $C_{F,n-1} = 0$, i.e. $F$ is Gauduchon, because otherwise the form $F^n$ would be exact. Now, let $k$ be such that $1 \leq k \leq n-2$. From (1) and (3) we get

$$C_{F,k}\int_X F^n = \frac{i}{2}\int_X \partial\bar\partial F^k \wedge F^{n-k-1} = \frac{k(n-k-1)}{n-2}\,\frac{i}{2}\int_X \partial\bar\partial F \wedge F^{n-2} = \frac{k(n-k-1)}{n-2}\,C_{F,1}\int_X F^n,$$

that is $\left(C_{F,k} - \frac{k(n-k-1)}{n-2}C_{F,1}\right)\int_X F^n = 0$. Therefore,

$$C_{F,k} = \frac{k(n-k-1)}{n-2}\,C_{F,1}\,, \tag{4}$$

for any $k$ such that $1 \leq k \leq n-2$. Hence, if $F$ is $k$-th Gauduchon for some $k$, then $C_{F,k} = 0$ and by (4) we get $C_{F,1} = 0$. Using again (4) we conclude that $C_{F,k} = 0$ for any other $k$, i.e. $F$ is $k$-th Gauduchon for any $1 \leq k \leq n-2$. □

**Corollary 2.3.** *Let $X$ be a homogeneous compact complex manifold of complex dimension $n \geq 3$ and let $F$ be an invariant Hermitian metric on $X$. If $F$ is SKT or astheno-Kähler, then $F$ is $k$-th Gauduchon for any $1 \leq k \leq n-1$.*

**Theorem 2.4.** *For each $n \geq 4$, there is a non-Kähler compact complex manifold $X$ of complex dimension $n$ admitting a balanced metric $\tilde F$ and an astheno-Kähler metric $F$, which is additionally $k$-th Gauduchon for any $1 \leq k \leq n-1$.*

**Proof.** We will construct such an $X$ using the class of complex nilmanifolds. Let $(a_1,\ldots,a_{n-1}) \in (\mathbb{R}\backslash\{0\})^{n-1}$, and let $\{\omega^j\}_{j=1}^n$ be a basis of forms of type $(1,0)$ satisfying

$$d\omega^1 = \cdots = d\omega^{n-1} = 0, \quad d\omega^n = \sum_{j=1}^{n-1} a_j\,\omega^{j\bar j}. \tag{5}$$

(See Remark 2.5 below for more details.) We impose the “canonical” metric $\tilde F = \frac{i}{2}(\omega^{1\bar 1} + \cdots + \omega^{n\bar n})$ to be balanced, i.e. $d\tilde F^{n-1} = 0$. This condition is equivalent to

$$a_1 + \cdots + a_{n-1} = 0. \tag{6}$$

Let us now consider a generic “diagonal” metric

$$F = \frac{i}{2}(b_1\,\omega^{1\bar 1} + \cdots + b_{n-1}\,\omega^{n-1\overline{n-1}}) + \frac{i}{2}\,\omega^{n\bar n}, \tag{7}$$

where $b_1,\ldots,b_{n-1} \in \mathbb{R}^+$.

Let $r \leq n-1$. We denote $A_r = a_1\,\omega^{1\bar 1} + \cdots + a_r\,\omega^{r\bar r}$ and $B_r = b_1\,\omega^{1\bar 1} + \cdots + b_r\,\omega^{r\bar r}$. Hence, in (5) and (7) we can write $d\omega^n = A_{n-1}$ and $F = \frac{i}{2}B_{n-1} + \frac{i}{2}\omega^{n\bar n}$.

Let us calculate $\partial\bar\partial F^{n-2}$. Using that the form $B_{n-1}$ is closed, we get

$$\begin{aligned}(-2i)^{n-2}\partial\bar\partial F^{n-2} &= \partial\bar\partial\left(B_{n-1} + \omega^{n\bar n}\right)^{n-2} = \partial\bar\partial(B_{n-1})^{n-2} + (n-2)\,\partial\bar\partial\left((B_{n-1})^{n-3}\wedge\omega^{n\bar n}\right)\\ &= (n-2)\,(B_{n-1})^{n-3}\wedge\partial\bar\partial(\omega^{n\bar n}) = -(n-2)\,(A_{n-1})^2\wedge(B_{n-1})^{n-3},\end{aligned}$$

where in the last equality we have used that $\partial\bar\partial(\omega^{n\bar n}) = \bar\partial\omega^n \wedge \partial\omega^{\bar n} = -A_{n-1}\wedge A_{n-1}$.

Therefore, $F$ is astheno-Kähler if and only if $(A_{n-1})^2 \wedge (B_{n-1})^{n-3} = 0$.

We now use the balanced condition (6), i.e. $a_{n-1} = -a_1 - \cdots - a_{n-2}$. Writing $A_{n-1} = A_{n-2} + a_{n-1}\,\omega^{n-1\overline{n-1}}$ and $B_{n-1} = B_{n-2} + b_{n-1}\,\omega^{n-1\overline{n-1}}$, and noting that $(A_{n-2})^2 \wedge (B_{n-2})^{n-3} = 0$, one has that the astheno-Kähler condition is equivalent to

$$\begin{aligned} 0 &= (A_{n-1})^2 \wedge (B_{n-1})^{n-3} = (A_{n-2} + a_{n-1}\,\omega^{n-1\overline{n-1}})^2 \wedge (B_{n-2} + b_{n-1}\,\omega^{n-1\overline{n-1}})^{n-3} \\ &= \left[(A_{n-2})^2 + 2\,a_{n-1}\,A_{n-2} \wedge \omega^{n-1\overline{n-1}}\right] \wedge \left[(B_{n-2})^{n-3} + (n-3)\,b_{n-1}\,(B_{n-2})^{n-4} \wedge \omega^{n-1\overline{n-1}}\right] \\ &= [(n-3)\,b_{n-1}\,A_{n-2} - 2\,(a_1 + \cdots + a_{n-2})\,B_{n-2}] \wedge A_{n-2} \wedge (B_{n-2})^{n-4} \wedge \omega^{n-1\overline{n-1}}. \end{aligned}$$

Let us observe that in order to simplify this equation, one can take $a_1, \ldots, a_{n-2} > 0$ and $b_j = a_j$ for $1 \le j \le n-2$. Indeed, in this case, we have that $B_{n-2} = A_{n-2}$, so it is enough to choose $b_{n-1} = \frac{2}{n-3}(a_1 + \cdots + a_{n-2})$ to get an astheno-Kähler metric $F$ given by (7).

Finally, by Corollary 2.3, the metric $F$ is in addition $k$-th Gauduchon for any $1 \le k \le n-1$. We notice that it can be directly proved that these complex nilmanifolds do not admit any SKT metric. Let us also note that the canonical bundle is holomorphically trivial, since the $(n,0)$-form $\Omega = \omega^{1\cdots n}$ is closed. □

**Remark 2.5.** The (real) nilmanifolds in (5) correspond to the Lie algebras $\mathfrak{g} = \mathfrak{h}_{2n+1} \times \mathbb{R}$, where $\mathfrak{h}_{2n+1}$ is the $(2n+1)$-dimensional Heisenberg algebra. Andrada, Barberis and Dotti proved in [2, Proposition 2.2] that every invariant complex structure $J$ on these nilmanifolds is *Abelian*, i.e. $[Jx, Jy] = [x, y]$ for any $x, y \in \mathfrak{g}$. Moreover, there are exactly $\left[\frac{n}{2}\right] + 1$ complex structures up to isomorphism. Let $J_0$ be the complex structure defined by taking all the coefficients $a_j$ positive numbers, i.e. $a_1, \ldots, a_{n-1} > 0$ in (5). One can prove the following result: *for any $J$ not isomorphic to $J_0$, the complex nilmanifold admits a balanced metric and an astheno-Kähler metric which is k-th Gauduchon for any k.*

**Remark 2.6.** The complex structure in the 4-dimensional example given in [4] as well as those given in (5) are all Abelian. Here we present a more general family of 4-dimensional complex nilmanifolds where the complex structure is not of that special type. Let us consider the complex structure equations

$$d\omega^1 = d\omega^2 = d\omega^3 = 0, \quad d\omega^4 = A\,\omega^{12} + B\,\omega^{13} + C\,\omega^{23} + \omega^{1\bar{1}} + \omega^{2\bar{2}} - 2\,\omega^{3\bar{3}}, \tag{8}$$

where we require the coefficients $A$, $B$, $C$ to belong to $\mathbb{Q}(i)$ in order to ensure the existence of a lattice, so that equations (8) define a complex nilmanifold. Consider a metric $F_{\alpha,\beta,\gamma}$ of the form

$$F_{\alpha,\beta,\gamma} = \frac{i}{2}(\alpha\,\omega^{1\bar{1}} + \beta\,\omega^{2\bar{2}} + \gamma\,\omega^{3\bar{3}} + \omega^{4\bar{4}}),$$

with $\alpha, \beta, \gamma \in \mathbb{R}^+$. On the one hand, it is easy to see that $\alpha = \beta = \gamma = 1$ provides a balanced metric. On the other hand, the astheno-Kähler condition is satisfied if and only if $\gamma = \frac{\alpha\,(|C|^2+4)+\beta\,(|B|^2+4)}{2-|A|^2} > 0$, so it suffices to take any complex structure in (8) with $|A| < \sqrt{2}$. This provides a family of 4-dimensional complex nilmanifolds $X_{A,B,C}$ with balanced and astheno-Kähler metrics that are $k$-th Gauduchon for any $k$. Notice that if $(A, B, C) \neq (0, 0, 0)$, then the Lie algebra underlying $X_{A,B,C}$ is not isomorphic to $\mathfrak{h}_7 \times \mathbb{R}$.


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## References


[1] B. Alexandrov, S. Ivanov, Vanishing theorems on Hermitian manifolds, Differ. Geom. Appl. 14 (2001) 251–265.

[2] A. Andrada, M.L. Barberis, I. Dotti, Classification of Abelian complex structures on 6-dimensional Lie algebras, J. Lond. Math. Soc. (2) 83 (1) (2011) 232–255;
A. Andrada, M.L. Barberis, I. Dotti, J. Lond. Math. Soc. (2) 87 (1) (2013) 319–320 (Corrigendum).

[3] I. Chiose, Obstructions to the existence of Kähler structures on compact complex manifolds, Proc. Amer. Math. Soc. 142 (2014) 3561–3568.

[4] A. Fino, G. Grantcharov, L. Vezzoni, Astheno-Kähler and balanced structures on fibrations, arXiv:1608.06743 [math.DG].

[5] A. Fino, A. Tomassini, On astheno-Kähler metrics, J. Lond. Math. Soc. 83 (2011) 290–308.

[6] A. Fino, L. Ugarte, On generalized Gauduchon metrics, Proc. Edinb. Math. Soc. 56 (2013) 733–753.

[7] A. Fino, L. Vezzoni, Special Hermitian metrics on compact solvmanifolds, J. Geom. Phys. 91 (2015) 40–53.

[8] A. Fino, L. Vezzoni, On the existence of balanced and SKT metrics on nilmanifolds, Proc. Amer. Math. Soc. 144 (2016) 2455–2459.

[9] J. Fu, Z. Wang, D. Wu, Semilinear equations, the $\gamma_k$ function, and generalized Gauduchon metrics, J. Eur. Math. Soc. 15 (2013) 659–680.

[10] P. Gauduchon, La 1-forme de torsion d'une variété hermitienne compacte, Math. Ann. 267 (1984) 495–518.

[11] S. Ivanov, G. Papadopoulos, Vanishing theorems and string backgrounds, Class. Quantum Gravity 18 (2001) 1089–1110.

[12] S. Ivanov, G. Papadopoulos, Vanishing theorems on $(l|k)$-strong Kähler manifolds with torsion, Adv. Math. 237 (2013) 147–164.

[13] J. Jost, S.-T. Yau, A non-linear elliptic system for maps from Hermitian to Riemannian manifolds and rigidity theorems in Hermitian geometry, Acta Math. 170 (1993) 221–254;
J. Jost, S.-T. Yau, Acta Math. 173 (1994) 307 (Corrigendum).

[14] K. Matsuo, T. Takahashi, On compact astheno-Kähler manifolds, Colloq. Math. 89 (2001) 213–221.

[15] M.L. Michelsohn, On the existence of special metrics in complex geometry, Acta Math. 149 (1982) 261–295.

[16] G. Székelyhidi, V. Tosatti, B. Weinkove, Gauduchon metrics with prescribed volume form, arXiv:1503.04491v1 [math.DG].