C. R. Acad. Sci. Paris, Ser. I 354 (2016) 1241–1247

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Mathematical problems in mechanics

A mathematical model of Koiter's type for a nonlinearly elastic "almost spherical" shell

Un modèle mathématique du type de Koiter pour une coque non linéairement élastique « presque sphérique »

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ARTICLE INFO

Article history:
Received 14 October 2016
Accepted 14 October 2016
Available online 31 October 2016

Presented by Philippe G. Ciarlet

ABSTRACT

We propose a new nonlinear shell model of Koiter's type, i.e. one that combines membrane and flexural strains, which can be used in the case where the middle surface of the undeformed shell is "almost spherical", in the sense that its Gaussian curvature K_ε and mean curvature H_ε satisfy $K_\varepsilon = H_\varepsilon^2 + \mathcal{O}(\varepsilon^2)$, where 2ε denotes the thickness of the shell.

RÉSUMÉ

Nous proposons un nouveau modèle non linéaire de coques du type de Koiter, c'est-à-dire combinant les déformations membranaires et en flexion, qui peut être utilisé lorsque la surface moyenne de la coque non déformée est « presque sphérique », au sens que sa courbure gaussiene K_ε et sa courbure moyenne H_ε satisfont $K_\varepsilon = H_\varepsilon^2 + \mathcal{O}(\varepsilon^2)$, où 2ε désigne l'épaisseur de la coque.

1. Introduction

In the well-known *nonlinear Koiter shell model* [7], the deformation of the middle surface of a thin elastic shell subjected to applied forces should be the solution to a specific minimization problem (Section 3). This model is often used in numerical simulations of shells in spite of the fact that it is yet to be justified mathematically by an existence theorem. This situation has been partially remedied in 2015 by Bunoiu and the authors [1] in the particular case where the middle surface is a *portion of a sphere* that can be parametrized by a single chart and its admissible deformations are subjected to Dirichlet boundary conditions. We address here the situation where the middle surface of a shell is a *surface without boundary* (so it cannot be parametrized by a single chart), which is "sufficiently close to a sphere".

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http://dx.doi.org/10.1016/j.crma.2016.10.011

More specifically, a shell with thickness 2ε is "*almost spherical*" if its middle surface is a closed (compact without boundary) surface $S \subset \mathbb{R}^3$ of class $\mathcal{C}^2$ whose Gaussian and mean curvatures, respectively denoted K_ε and H_ε, satisfy

$$K_\varepsilon = \mathcal{O}(1) > 0, \quad H_\varepsilon = \mathcal{O}(1), \text{ and } K_\varepsilon = H_\varepsilon^2 + \mathcal{O}(\varepsilon^2) \text{ uniformly on } S.$$

Note that such a definition is similar *in its spirit* to that of a "shallow shell" given in [5]. There, a shell is deemed "shallow" if, when considered as imbedded in a *family* of shells parametrized by $\varepsilon > 0$, its "geometry" is likewise (in a specific sense, completely different from that used here) of a given order with respect to ε as ε approaches zero.

In all that follows, we study the deformation of an almost spherical shell with middle surface $S \subset \mathbb{R}^3$ and thickness $2\varepsilon > 0$, made of a homogeneous and isotropic nonlinearly elastic material whose Lamé constants λ and μ satisfy $3\lambda + 2\mu > 0$ and $\mu > 0$. Note that, for notational brevity, the index "ε" is not attached to "S".

More detailed proofs will be provided in a forthcoming paper [4], where the results of this note will be also generalized to more general surfaces, with or without boundary.

2. Kinematics

We describe here the notions used to define the nonlinear shell models of Section 3. For more details about the notions from differential geometry of surfaces used here, we refer the reader to, e.g., [3] or [6].

In all that follows, except when otherwise explicitly specified, Greek indices range in the set $\{1, 2\}$, while Latin indices range in the set $\{1, 2, 3\}$. The summation convention with respect to repeated indices is used in conjunction with the above rule.

The three-dimensional Euclidean space containing the surface S is identified with $\mathbb{R}^3$ by choosing an origin and a Euclidean basis. The Euclidean norm, the inner product, and the vector product, of vectors in $\mathbb{R}^3$ are respectively denoted $|\cdot|$, $\cdot$, and $\wedge$.

A generic point in $\mathbb{R}^3$ is denoted by $x = (x_i)$. A generic point in $\mathbb{R}^2$ is denoted by $y = (y_\alpha)$, and partial derivatives with respect to y_α are denoted by ∂_α. Vector and tensor fields are denoted by boldface letters.

Given any local chart $\boldsymbol{\theta} \in \mathcal{C}^2(\omega; \mathbb{R}^3)$ of S, where $\omega \subset \mathbb{R}^2$ is a connected open set, the vectors fields $\boldsymbol{a}_\alpha : S \to \mathbb{R}^3$ defined by

$$\boldsymbol{a}_\alpha \circ \boldsymbol{\theta} := \partial_\alpha \boldsymbol{\theta}$$

span the tangent planes to S at the points contained in the image $\boldsymbol{\theta}(\omega) \subset S$ of ω under $\boldsymbol{\theta}$. Then the unit area element along the portion $\boldsymbol{\theta}(\omega)$ of S is

$$\mathrm{d}S(x) := |\partial_1 \boldsymbol{\theta}(y) \wedge \partial_2 \boldsymbol{\theta}(y)|\, dy \text{ for all } x = \boldsymbol{\theta}(y),\ y \in \omega.$$

Since the surface S is compact, there exists a finite number of local charts $\boldsymbol{\theta}_k \in \mathcal{C}^2(\omega_k; \mathbb{R}^3)$, $1 \le k \le N$, where $\omega_k \subset \mathbb{R}^2$ are connected open sets, such that

$$S \subset \bigcup_{k=1}^N S_k, \text{ where } S_k := \boldsymbol{\theta}_k(\omega_k).$$

Let $\alpha_k \in \mathcal{C}^2(S)$, $k \in \{1, 2, ..., N\}$, denote a partition of unity subordinated to the above covering of S.

The Lebesgue integral of a $\mathrm{d}S$-measurable nonnegative function $f : S \to \mathbb{R}$ is defined in $\mathbb{R} \cup \{+\infty\}$ by

$$\int_S f \,\mathrm{d}S := \sum_{k=1}^N \int_{\omega_k} (f \circ \boldsymbol{\theta}_k)\,(\alpha_k \circ \boldsymbol{\theta}_k)\,|\partial_1 \boldsymbol{\theta}_k \wedge \partial_2 \boldsymbol{\theta}_k|\,\mathrm{d}y.$$

Then, for each $1 \le p < \infty$, the $L^p(S)$-norm of a $\mathrm{d}S$-measurable function $f : S \to \mathbb{R}$ is defined by

$$\|f\|_{L^p(S)} := \Big(\int_S |f|^p \,\mathrm{d}S \Big)^{1/p}.$$

A $\mathrm{d}S$-measurable mapping $\boldsymbol{\varphi} : S \to \mathbb{R}^3$ belongs to the Lebesgue space $L^p(S; \mathbb{R}^3)$, $1 \le p < \infty$, if

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{L}^p(S)} := \| |\boldsymbol{\varphi}| \|_{L^p(S)} < \infty.$$

The tangent space $T_x\mathbb{R}^3$ to $\mathbb{R}^3$ at any given point $x \in \mathbb{R}^3$ is identified with $\mathbb{R}^3$. The tangent space T_xS to the surface S at any given point $x \in S$ is identified with the plane parallel to T_xS and passing through the origin of $\mathbb{R}^3$.

A mapping $\boldsymbol{\varphi} \in L^p(S; \mathbb{R}^3)$ belongs to the Sobolev space $W^{1,p}(S; \mathbb{R}^3)$, $1 \le p < \infty$, if its tangent maps $\mathrm{d}\boldsymbol{\varphi}(x) : T_xS \to T_{\boldsymbol{\varphi}(x)}\mathbb{R}^3$ exist for $\mathrm{d}S$-almost all $x \in S$ and satisfy

$$\|\mathrm{d}\boldsymbol{\varphi}\|_{\boldsymbol{L}^p(S)} := \Big(\int_S |\mathrm{d}\boldsymbol{\varphi}|^p \,\mathrm{d}S \Big)^{1/p} < \infty,$$

where $|\mathrm{d}\boldsymbol{\varphi}(x)|$ denotes at almost all $x \in S$ the operator norm of the linear mapping $\mathrm{d}\boldsymbol{\varphi}(x) : T_xS \to T_{\boldsymbol{\varphi}(x)}\mathbb{R}^3$.

A continuous vector field $\boldsymbol{a}_3 : S \to \mathbb{R}^3$ with unit length and normal to the tangent planes to S is given once and for all. Then all the local bases $\{\boldsymbol{a}_\alpha(x)\}$ in the tangent planes T_xS, $x \in S$, considered below are positively oriented, in the sense that the basis $\{\boldsymbol{a}_i(x)\}$ of $\mathbb{R}^3$ has the same orientation as the chosen Cartesian basis of $\mathbb{R}^3$. The dual basis of the basis $\{\boldsymbol{a}_i(x)\}$ of $\mathbb{R}^3$ is denoted $\{\boldsymbol{a}^i(x)\}$. Note that the two vectors $\{\boldsymbol{a}^\alpha(x)\}$ belong to T_xS and form the dual basis of the basis $\{\boldsymbol{a}_\alpha(x)\}$ in T_xS.

The *first*, *second*, and *third*, *fundamental forms* of the surface S are defined at each point $x \in S$ as the bilinear forms $\boldsymbol{I}(x), \boldsymbol{II}(x), \boldsymbol{III}(x) : T_xS \times T_xS \to \mathbb{R}$ defined for each $(\boldsymbol{\zeta}, \boldsymbol{\eta}) \in T_xS \times T_xS$ by

$$\boldsymbol{I}(x)(\boldsymbol{\zeta}, \boldsymbol{\eta}) := \boldsymbol{\zeta} \cdot \boldsymbol{\eta},$$
$$\boldsymbol{II}(x)(\boldsymbol{\zeta}, \boldsymbol{\eta}) := -\mathrm{d}\boldsymbol{a}_3(x)\boldsymbol{\zeta} \cdot \boldsymbol{\eta},$$
$$\boldsymbol{III}(x)(\boldsymbol{\zeta}, \boldsymbol{\eta}) := \mathrm{d}\boldsymbol{a}_3(x)\boldsymbol{\zeta} \cdot \mathrm{d}\boldsymbol{a}_3(x)\boldsymbol{\eta}.$$

The inverse of the first fundamental form of the surface S is the twice-contravariant symmetric tensor field $\boldsymbol{I}^{-1}$ whose components $\boldsymbol{I}^{-1}(\boldsymbol{a}^\alpha, \boldsymbol{a}^\beta)$ in a local basis are defined by

$$\big(\boldsymbol{I}^{-1}(\boldsymbol{a}^\alpha, \boldsymbol{a}^\beta)\big) := \big(\boldsymbol{I}(\boldsymbol{a}_\alpha, \boldsymbol{a}_\beta)\big)^{-1}.$$

Let $\boldsymbol{A}$ be any second-order mixed tensor field on the surface S. Then the trace and the determinant of $\boldsymbol{A}$ are denoted and defined by

$$\mathrm{tr}\,\boldsymbol{A} := \sum_\alpha \boldsymbol{A}(\boldsymbol{a}^\alpha, \boldsymbol{a}_\alpha) \text{ and } \det \boldsymbol{A} := \det\big(\boldsymbol{A}(\boldsymbol{a}^\alpha, \boldsymbol{a}_\beta)\big),$$

and the norm of $\boldsymbol{A}$ is denoted and defined by

$$|\boldsymbol{A}| := \Big\{ \sum_{\alpha,\beta,\sigma,\tau} \boldsymbol{I}(\boldsymbol{a}_\alpha, \boldsymbol{a}_\sigma)\boldsymbol{I}^{-1}(\boldsymbol{a}^\beta, \boldsymbol{a}^\tau)\boldsymbol{A}(\boldsymbol{a}^\alpha, \boldsymbol{a}_\beta)\boldsymbol{A}(\boldsymbol{a}^\sigma, \boldsymbol{a}_\tau) \Big\}^{1/2},$$

for any given dual bases $\{\boldsymbol{a}_\alpha\}$ and $\{\boldsymbol{a}^\alpha\}$ in the tangent spaces to the surface S.

The product of a twice-contravariant tensor field $\boldsymbol{A}$ by a twice-covariant tensor field $\boldsymbol{B}$ on the surface S is the second-order mixed tensor field $\boldsymbol{AB}$ defined by

$$(\boldsymbol{AB})(\boldsymbol{a}^\alpha, \boldsymbol{a}_\beta) := \sum_\sigma \boldsymbol{A}(\boldsymbol{a}^\alpha, \boldsymbol{a}^\sigma)\boldsymbol{B}(\boldsymbol{a}_\sigma, \boldsymbol{a}_\beta).$$

The product of two second-order mixed tensor fields $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ on the surface S is the second-order mixed tensor field $\boldsymbol{FG}$ defined by

$$(\boldsymbol{F}_1\boldsymbol{F}_2)(\boldsymbol{a}^\alpha, \boldsymbol{a}_\beta) := \sum_\sigma \boldsymbol{F}_1(\boldsymbol{a}^\alpha, \boldsymbol{a}_\sigma)\boldsymbol{F}_2(\boldsymbol{a}^\sigma, \boldsymbol{a}_\beta).$$

The square of a second-order mixed tensor field $\boldsymbol{F}$ is denoted and defined by

$$\boldsymbol{F}^2 := \boldsymbol{FF}.$$

Note that all of the above definitions do not depend on the choice of the dual bases $\{\boldsymbol{a}\}$ and $\{\boldsymbol{a}\}$.

The *mean*, resp. *Gaussian*, *curvatures* of the surface S are denoted and defined by

$$H := \frac{1}{2}\mathrm{tr}\,\boldsymbol{S} \text{ and } K := \det \boldsymbol{S},$$

where $\boldsymbol{S} := \boldsymbol{I}^{-1}\boldsymbol{II}$ denotes the *shape operator* of S. If K is > 0 at all points of S, the third fundamental form of S is invertible, and its inverse is the twice-contravariant symmetric tensor field $\boldsymbol{III}^{-1}$ whose components $\boldsymbol{III}^{-1}(\boldsymbol{a}^\alpha, \boldsymbol{a}^\beta)$ in a local basis are defined by

$$\big(\boldsymbol{III}^{-1}(\boldsymbol{a}^\alpha, \boldsymbol{a}^\beta)\big) := \big(\boldsymbol{III}(\boldsymbol{a}_\alpha, \boldsymbol{a}_\beta)\big)^{-1}.$$

A mapping $\boldsymbol{\varphi} : S \to \mathbb{R}^3$ is an *immersion* if it is of class $W^{1,1}$ and if the tangent map $\mathrm{d}\boldsymbol{\varphi}(x) : T_xS \to T_{\boldsymbol{\varphi}(x)}\mathbb{R}^3$ has rank two for almost all $x \in S$. It follows that any immersion $\boldsymbol{\varphi}$ possesses a unique, up to a negligible subset of S, "*orientation-preserving unit normal vector field*", hereafter denoted $\boldsymbol{a}_3(\boldsymbol{\varphi}) : S \to \mathbb{R}^3$, defined at almost all $x \in S$ as the unique vector $\boldsymbol{a}_3(\boldsymbol{\varphi})(x) \in \mathbb{R}^3$ such that

$$|\boldsymbol{a}_3(\boldsymbol{\varphi})(x)| = 1, \quad \boldsymbol{a}_3(\boldsymbol{\varphi})(x) \cdot \mathrm{d}\boldsymbol{\varphi}(x)\boldsymbol{\zeta} = 0 \text{ for all } \boldsymbol{\zeta} \in T_xS,$$

and such that, for any basis $\{\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2\}$ in T_xS, the bases $\{\mathrm{d}\boldsymbol{\varphi}(x)\boldsymbol{\zeta}_1, \mathrm{d}\boldsymbol{\varphi}(x)\boldsymbol{\zeta}_2, \boldsymbol{a}_3(\boldsymbol{\varphi})(x)\}$ and $\{\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \boldsymbol{a}_3(x)\}$ have the same orientation in $\mathbb{R}^3$.

In all that follows, a *deformation of S is defined as an immersion* $\boldsymbol{\varphi} \in W^{1,1}(S;\mathbb{R}^3)$ *such that* $\boldsymbol{a}_3(\boldsymbol{\varphi}) \in W^{1,1}(S;\mathbb{R}^3)$. Given any deformation $\boldsymbol{\varphi}: S \to \mathbb{R}^3$, the *pullbacks* on S of the first, second, and third, fundamental forms of the immersed surface $\boldsymbol{\varphi}(S) \subset \mathbb{R}^3$ are defined at almost all point $x \in S$ as the bilinear forms $\boldsymbol{I}(\boldsymbol{\varphi})(x), \boldsymbol{II}(\boldsymbol{\varphi})(x), \boldsymbol{III}(\boldsymbol{\varphi})(x) : T_xS \times T_xS \to \mathbb{R}$ defined for each $(\boldsymbol{\zeta}, \boldsymbol{\eta}) \in T_xS \times T_xS$ by

$$\boldsymbol{I}(\boldsymbol{\varphi})(x)(\boldsymbol{\zeta},\boldsymbol{\eta}) := \mathrm{d}\boldsymbol{\varphi}(x)\boldsymbol{\zeta} \cdot \mathrm{d}\boldsymbol{\varphi}(x)\boldsymbol{\eta},$$
$$\boldsymbol{II}(\boldsymbol{\varphi})(x)(\boldsymbol{\zeta},\boldsymbol{\eta}) := -\mathrm{d}(\boldsymbol{a}_3(\boldsymbol{\varphi}))(x)\boldsymbol{\zeta} \cdot \mathrm{d}\boldsymbol{\varphi}(x)\boldsymbol{\eta},$$
$$\boldsymbol{III}(\boldsymbol{\varphi})(x)(\boldsymbol{\zeta},\boldsymbol{\eta}) := \mathrm{d}(\boldsymbol{a}_3(\boldsymbol{\varphi}))(x)\boldsymbol{\zeta} \cdot d(\boldsymbol{a}_3(\boldsymbol{\varphi}))(x)\boldsymbol{\eta}.$$

Finally, the *change of first fundamental form*, the *change of second fundamental form*, and the *change of third fundamental form, tensor fields* associated with a deformation $\boldsymbol{\varphi}$ of S are respectively denoted and defined by

$$\boldsymbol{G}(\boldsymbol{\varphi}) := \frac{1}{2}\boldsymbol{I}^{-1}(\boldsymbol{I}(\boldsymbol{\varphi}) - \boldsymbol{I}), \quad \boldsymbol{R}(\boldsymbol{\varphi}) := \boldsymbol{I}^{-1}(\boldsymbol{II}(\boldsymbol{\varphi}) - \boldsymbol{II}), \text{ and } \boldsymbol{P}(\boldsymbol{\varphi}) := \frac{1}{2}\boldsymbol{III}^{-1}(\boldsymbol{III}(\boldsymbol{\varphi}) - \boldsymbol{III}).$$

Note that $\boldsymbol{G}(\boldsymbol{\varphi})$, $\boldsymbol{R}(\boldsymbol{\varphi})$ and $\boldsymbol{P}(\boldsymbol{\varphi})$ are second-order mixed (1-covariant and 1-contravariant) tensor fields on the surface S and that $\boldsymbol{P}(\boldsymbol{\varphi})$ is well defined if K is > 0. Note also that, while the classical Koiter nonlinear shell model is defined in terms of $\boldsymbol{G}(\boldsymbol{\varphi})$ and $\boldsymbol{R}(\boldsymbol{\varphi})$, the new nonlinear shell model introduced in this paper is defined in terms of $\boldsymbol{G}(\boldsymbol{\varphi})$ and $\boldsymbol{P}(\boldsymbol{\varphi})$ (see Section 3).

3. Two nonlinear shell models

We describe in this section two nonlinear shell models: the first one is due to W.T. Koiter [7], and the second one is new. We define both models directly on the surface S, without using any local coordinates to describe the middle surface S of the undeformed configuration of the shell. This is not usually the case in the literature, where most shell models, including Koiter's, are defined in local coordinates (see, e.g., [2] and the references therein). But doing so is much more convenient in our case, where S cannot be described by a single local chart.

There are two basic reasons for introducing this new nonlinear shell model: it is both *well posed* (in the sense that there exists a satisfactory existence theory) and *close "asymptotically" to Koiter's* when the undeformed configuration of the shell is almost spherical; cf. Section 4.

In *Koiter's nonlinear shell model*, the unknown deformation of the middle surface S of a shell arising in response to applied forces should minimize the functional J defined by

$$J[\boldsymbol{\varphi}] := \int_S \left\{\varepsilon W(\boldsymbol{G}(\boldsymbol{\varphi})) + \frac{\varepsilon^3}{3} W(\boldsymbol{R}(\boldsymbol{\varphi}))\right\} \mathrm{d}S - L[\boldsymbol{\varphi}]$$

for each $\boldsymbol{\varphi} \in \boldsymbol{\Phi}(S)$, where the set $\boldsymbol{\Phi}(S)$ of *admissible deformations* is defined by

$$\boldsymbol{\Phi}(S) := \{\boldsymbol{\varphi} \in W^{1,4}(S;\mathbb{R}^3) \text{ is an immersion such that } \boldsymbol{a}_3(\boldsymbol{\varphi}) \in W^{1,4}(S;\mathbb{R}^3)\},$$

and

$$W(\boldsymbol{F}) := \frac{2\lambda\mu}{\lambda + 2\mu}(\operatorname{tr}\boldsymbol{F})^2 + 2\mu \operatorname{tr}(\boldsymbol{F}^2)$$

for all second-order mixed tensor fields $\boldsymbol{F}$, and $L : \boldsymbol{\Phi}(S) \to \mathbb{R}$ is the potential of the applied forces acting on S.

In our *new nonlinear shell model*, the unknown deformation of the middle surface S of a shell arising in response to applied forces should minimize the *new* functional

$$\tilde{J}[\boldsymbol{\varphi}] := \int_S \left\{\varepsilon \tilde{W}(\boldsymbol{G}(\boldsymbol{\varphi})) + \frac{\varepsilon^3}{3} K\, \tilde{W}(\boldsymbol{P}(\boldsymbol{\varphi}))\right\} \mathrm{d}S - L[\boldsymbol{\varphi}]$$

for each $\boldsymbol{\varphi} \in \tilde{\boldsymbol{\Phi}}(S)$, where the *new* set $\tilde{\boldsymbol{\Phi}}(S)$ of *admissible deformations* is now defined by

$$\tilde{\boldsymbol{\Phi}}(S) := \{\boldsymbol{\varphi} \in \boldsymbol{\Phi}(S);\ K(\boldsymbol{\varphi}) > 0 \text{ a.e. in } S\},$$

where $K(\boldsymbol{\varphi})$ denotes the Gaussian curvature of the surface $\boldsymbol{\varphi}(S)$,

$$\tilde{W}(\boldsymbol{F}) := \mu(\operatorname{tr}\boldsymbol{F})^2 + \frac{2\mu(\lambda - 2\mu)}{\lambda + 2\mu}\det\boldsymbol{F} + \frac{\mu(3\lambda + 2\mu)}{2(\lambda + 2\mu)}\Big(2\operatorname{tr}\boldsymbol{F} - \log\big(1 + 2\operatorname{tr}\boldsymbol{F} + 4\det\boldsymbol{F}\big)\Big)$$

for all second-order mixed tensor fields $\boldsymbol{F}$, and L is the potential of the applied forces acting on S (the same potential as in Koiter's nonlinear shell model).

Note that, thanks to the factor K (the Gaussian curvature of S) appearing in the definition of $\tilde{J}[\boldsymbol{\varphi}]$, this new functional can be also defined as

$$\tilde{J}[\boldsymbol{\varphi}] = \varepsilon \int_S \tilde{W}(\boldsymbol{G}(\boldsymbol{\varphi}))\,\mathrm{d}S + \frac{\varepsilon^3}{3} \int_\Sigma \tilde{W}(\boldsymbol{G}(\boldsymbol{a}_3(\boldsymbol{\varphi}) \circ \boldsymbol{a}_3^{-1}))\,\mathrm{d}a - L[\boldsymbol{\varphi}]$$

for each $\boldsymbol{\varphi} \in \tilde{\boldsymbol{\Phi}}(S)$, where Σ designates the unit sphere in $\mathbb{R}^3$ and da denotes the area element along Σ.

4. Main results

We are now in a position to state the main results (Theorems 4.1 and 4.2 below) of this Note.

The first theorem shows that, if the undeformed configuration of the shell is "almost spherical" (the precise definition is given in the Introduction), then the total energies of the two nonlinear shell models defined in the previous section are "asymptotically equivalent".

The second theorem shows that, again if the undeformed configuration of the shell is almost spherical, then the minimization problem corresponding to the new nonlinear shell model defined in the previous section possesses a solution (by contrast, it is not known as of now whether Koiter's nonlinear shell model has a solution). Thus the two theorems together show that *this new model can be used instead of Koiter's nonlinear shell model in the case where the undeformed configuration of the shell is almost spherical.*

Theorem 4.1. *Assume that the undeformed configuration of a shell is almost spherical and that the Lamé constants of the elastic material constituting the shell satisfy $\mu > 0$ and $3\lambda + 2\mu > 0$. Let*

$$c_0 := 2\mu\Big(1 - \frac{2|\lambda|}{\lambda + 2\mu}\Big) > 0 \text{ and } C_0 := 2\mu\Big(1 + \frac{2|\lambda|}{\lambda + 2\mu}\Big).$$

Given any smooth enough deformation $\boldsymbol{\varphi} : S \to \mathbb{R}^3$ that satisfies $K(\boldsymbol{\varphi}) > 0$ on S, let the tensor fields $\boldsymbol{G} := \boldsymbol{G}(\boldsymbol{\varphi})$, $\boldsymbol{R} := \boldsymbol{R}(\boldsymbol{\varphi})$, and $\boldsymbol{P} := \boldsymbol{P}(\boldsymbol{\varphi})$, be defined as in Section 2. *Then the stored energy function of Koiter's nonlinear shell model* (see Section 3) *satisfies the inequalities*

$$c_0\Big\{\varepsilon|\boldsymbol{G}|^2 + \frac{\varepsilon^3}{3}|\boldsymbol{R}|^2\Big\} \le \Big\{\varepsilon W(\boldsymbol{G}) + \frac{\varepsilon^3}{3} W(\boldsymbol{R})\Big\} \le C_0\Big\{\varepsilon|\boldsymbol{G}|^2 + \frac{\varepsilon^3}{3}|\boldsymbol{R}|^2\Big\} \text{ on } S,$$

and the stored energy function of the new nonlinear shell model (see Section 3) *is such that*

$$\Big\{\varepsilon \tilde{W}(\boldsymbol{G}) + \frac{\varepsilon^3}{3} K\,\tilde{W}(\boldsymbol{P})\Big\} = \Big\{\varepsilon W(\boldsymbol{G}) + \frac{\varepsilon^3}{3} W(\boldsymbol{R})\Big\}\Big\{1 + \mathcal{O}\big(\varepsilon + |\boldsymbol{G}| + |\boldsymbol{R}|\big)\Big\} \text{on } S.$$

Sketch of proof. The first two inequalities of the theorem follow by noting that

$$W(\boldsymbol{F}) := \frac{2\lambda\mu}{\lambda + 2\mu}(\mathrm{tr}\,\boldsymbol{F})^2 + 2\mu\,\mathrm{tr}\,(\boldsymbol{F}^2) \ge c_0 \mathrm{tr}\,(\boldsymbol{F}^2) = c_0|\boldsymbol{F}|^2$$

for both $\boldsymbol{F} := \boldsymbol{G}$ and $\boldsymbol{F} := \boldsymbol{R}$.

The proof of the second part proceeds as follows: combining the expressions of the functions W and $\tilde{W}$ (Section 3), the polynomial expansion of the function $\log(1 + \cdot)$, and the Cayley–Hamilton theorem, we first deduce that

$$\tilde{W}(\boldsymbol{F}) = W(\boldsymbol{F}) + \mathcal{O}(|\boldsymbol{F})|^3)$$

for all second-order mixed tensor field $\boldsymbol{F}$. It follows that

$$\Big\{\varepsilon \tilde{W}(\boldsymbol{G}) + \frac{\varepsilon^3}{3} K\,\tilde{W}(\boldsymbol{P})\Big\} = \Big\{\varepsilon W(\boldsymbol{G}) + \frac{\varepsilon^3}{3} K\,W(\boldsymbol{P})\Big\} + \varepsilon\mathcal{O}(|\boldsymbol{G}|^3) + \varepsilon^3\mathcal{O}(|\boldsymbol{P}|^3).$$

Using the expression of the third fundamental form of a surface in terms of the first two fundamental forms, we next infer that $\boldsymbol{P}$ can be estimated in terms of $\boldsymbol{G}$ and $\boldsymbol{R}$ as

$$\boldsymbol{P} = \frac{1}{2}(\boldsymbol{S}\boldsymbol{R} + \boldsymbol{R}\boldsymbol{S}) + \mathcal{O}(|\boldsymbol{G}| + |\boldsymbol{R}|^2),$$

where $\boldsymbol{S} := \boldsymbol{I}^{-1}\boldsymbol{II}$ denote the shape operator of S. Hence

$$\mathcal{O}(|\boldsymbol{P}|^3) = \mathcal{O}(|\boldsymbol{G}|^3 + |\boldsymbol{R}|^3),$$

and

$$(\mathrm{tr}\,\boldsymbol{P})^2 = (\mathrm{tr}\,((\boldsymbol{II}^{-1}\boldsymbol{I})\boldsymbol{R}))^2 + \mathcal{O}(|\boldsymbol{G}|^2 + |\boldsymbol{G}||\boldsymbol{R}| + |\boldsymbol{R}|^3),$$
$$\mathrm{tr}\,(\boldsymbol{P}^2) = \frac{1}{2}\Big\{\mathrm{tr}\,\big((\boldsymbol{II}^{-1}\boldsymbol{I})\boldsymbol{R})^2\big) + \mathrm{tr}\,\big((\boldsymbol{III}^{-1}\boldsymbol{I})\boldsymbol{R}^2\big)\Big\} + \mathcal{O}(|\boldsymbol{G}|^2 + |\boldsymbol{G}||\boldsymbol{R}| + |\boldsymbol{R}|^3).$$

Using the assumption that the undeformed configuration of the shell is almost spherical, we next have

$$\boldsymbol{II}^{-1} = K^{-1/2}\boldsymbol{I}^{-1} + \mathcal{O}(\varepsilon) \text{ and } \boldsymbol{III}^{-1} = K^{-1}\boldsymbol{I}^{-1} + \mathcal{O}(\varepsilon).$$

Combined with the previous estimates, this gives

$$(\mathrm{tr}\,\boldsymbol{P})^2 = K^{-1}(\mathrm{tr}\,\boldsymbol{R})^2 + \mathcal{O}(\varepsilon|\boldsymbol{R}|^2 + |\boldsymbol{G}|^2 + |\boldsymbol{G}||\boldsymbol{R}| + |\boldsymbol{R}|^3),$$
$$\mathrm{tr}\,(\boldsymbol{P}^2) = K^{-1}\mathrm{tr}\,(\boldsymbol{R}^2) + \mathcal{O}(\varepsilon|\boldsymbol{R}|^2 + |\boldsymbol{G}|^2 + |\boldsymbol{G}||\boldsymbol{R}| + |\boldsymbol{R}|^3),$$

which in turn implies that

$$K\,W(\boldsymbol{P}) = W(\boldsymbol{R}) + \mathcal{O}(\varepsilon|\boldsymbol{R}|^2 + |\boldsymbol{G}|^2 + |\boldsymbol{G}||\boldsymbol{R}| + |\boldsymbol{R}|^3).$$

The conclusion follows by using this last estimate in the right-hand side of the above estimate of $\Big\{\varepsilon\tilde{W}(\boldsymbol{G}) + \frac{\varepsilon^3}{3}K\,\tilde{W}(\boldsymbol{P})\Big\}$. □

Theorem 4.2. *Assume that the undeformed configuration of a shell is almost spherical and that the potential L of the applied forces is linear, continuous, and translation-invariant* (in the sense that $L[\boldsymbol{\varphi} + \boldsymbol{v}] = L[\boldsymbol{\varphi}]$ for all $\boldsymbol{v} \in \mathbb{R}^3$ and all $\boldsymbol{\varphi} \in \tilde{\boldsymbol{\Phi}}(S)$).
Then the minimization problem: Find $\boldsymbol{\psi} \in \tilde{\boldsymbol{\Phi}}(S)$ such that

$$\tilde{J}(\boldsymbol{\psi}) = \inf_{\boldsymbol{\varphi}\in\tilde{\boldsymbol{\Phi}}(S)} \tilde{J}(\boldsymbol{\varphi})$$

has at least a solution.

Sketch of proof. The proof comprises five stages.

First, we establish the following coerciveness inequality for the functional $\tilde{J}$: there exists a constant $c_1 > 0$ such that, for each $\boldsymbol{\varphi} \in \tilde{\boldsymbol{\Phi}}(S)$,

$$\tilde{J}[\boldsymbol{\varphi}] \geq c_1\varepsilon\Big(\|\mathrm{d}\boldsymbol{\varphi}\|_{L^4(S)} + \|\log(\det(\boldsymbol{I}^{-1}\boldsymbol{I}(\boldsymbol{\varphi})))\|_{L^1(S)} - 1\Big)$$
$$+ c_1\varepsilon^3\Big(\|\mathrm{d}(\boldsymbol{a}_3(\boldsymbol{\varphi}))\|_{L^4(S)} + \|\log(\det(\boldsymbol{III}^{-1}\boldsymbol{III}(\boldsymbol{\varphi})))\|_{L^1(S)} - 1\Big).$$

Second, we deduce from the above coerciveness inequality, combined with the Poincaré–Wirtinger inequality and the assumption that L is translation-invariant, that there exists an infimizing sequence $(\boldsymbol{\psi}_n)_{n\geq 1}$ of $\tilde{J}$ in the set $\tilde{\boldsymbol{\Phi}}(S)$ such that

$$(\boldsymbol{\psi}_n)_{n\geq 1} \text{ and } (\boldsymbol{a}_3(\boldsymbol{\psi}_n))_{n\geq 1} \text{ are bounded sequences in } W^{1,4}(S;\mathbb{R}^3),$$
$$(\log(\det(\boldsymbol{I}^{-1}\boldsymbol{I}(\boldsymbol{\psi}_n))))_{n\geq 1} \text{ and } (\log(\det(\boldsymbol{III}^{-1}\boldsymbol{III}(\boldsymbol{\psi}_n))))_{n\geq 1} \text{ are bounded sequences in } L^1(S).$$

Third, we show that there exists a subsequence $(\boldsymbol{\varphi}_n)_{n\geq 1}$ of the sequence $(\boldsymbol{\psi}_n)_{n\geq 1}$ and two vector fields $\boldsymbol{\varphi} \in W^{1,4}(S;\mathbb{R}^3)$ and $\boldsymbol{\eta} \in W^{1,4}(S;\mathbb{R}^3)$ such that

$$\boldsymbol{\varphi}_n \to \boldsymbol{\varphi} \text{ as } n \to \infty \text{ weakly in } W^{1,4}(S;\mathbb{R}^3) \text{ and strongly in } L^4(S;\mathbb{R}^3),$$
$$\boldsymbol{a}_3(\boldsymbol{\varphi}_n) \to \boldsymbol{\eta} \text{ as } n \to \infty \text{ weakly in } W^{1,4}(S;\mathbb{R}^3), \text{ strongly in } L^4(S;\mathbb{R}^3), \text{ and a.e. in } S,$$
$$\Big(\big\{\det(\boldsymbol{I}^{-1}\boldsymbol{I}(\boldsymbol{\varphi}_n))\big\}^{1/2}\Big)_{n\geq 1} \text{ and } \Big(\big\{\det(\boldsymbol{III}^{-1}\boldsymbol{III}(\boldsymbol{\varphi}_n))\big\}^{1/2}\Big)_{n\geq 1} \text{ are bounded sequences in } L^2(S).$$

Fourth, we show that the vector fields $\boldsymbol{\varphi}$ and $\boldsymbol{\eta}$ found in the previous step are such that (see Section 3 for the notions used below):

$$\boldsymbol{\varphi} \text{ is an } \textit{immersion} \text{ and } \boldsymbol{a}_3(\boldsymbol{\varphi}) = \boldsymbol{\eta} \text{ a.e. on } S,$$

where $\boldsymbol{a}_3(\boldsymbol{\varphi})$ designates the *orientation-preserving unit normal vector field* associated with the immersion $\boldsymbol{\varphi} : S \to \mathbb{R}^3$. To this end, we establish in particular the weak convergences

$$\big\{\det(\boldsymbol{I}^{-1}\boldsymbol{I}(\boldsymbol{\varphi}_n))\big\}^{1/2} \to \big\{\det(\boldsymbol{I}^{-1}\boldsymbol{I}(\boldsymbol{\varphi}))\big\}^{1/2} \text{ as } n \to \infty \text{ weakly in } L^2(S),$$
$$\big\{\det(\boldsymbol{III}^{-1}\boldsymbol{III}(\boldsymbol{\varphi}_n))\big\}^{1/2} \to \big\{\det(\boldsymbol{III}^{-1}\boldsymbol{III}(\boldsymbol{\varphi}))\big\}^{1/2} \text{ as } n \to \infty \text{ weakly in } L^2(S),$$

by using a compensated compactness argument.

Finally, we prove that the vector field $\boldsymbol{\varphi}$ found above minimizes the functional $\tilde{J}$ over the set $\tilde{\boldsymbol{\Phi}}(S)$ by using the reflexivity of the spaces $L^4(S;\mathbb{R}^3)$ and $L^2(S)$ and the weak convergences established in the previous steps. $\square$

Note that the assumption that the undeformed configuration is almost spherical is only needed in Theorem 4.1. By contrast, Theorem 4.2 holds as well under the only assumption that the Gaussian curvature of S be bounded by below by a > 0 constant. But then the corresponding energy is no longer necessarily close to Koiter's.

Acknowledgements

The work described in this paper was substantially supported by a grant from the Research Grants Council of the Hong Kong Special Administration Region, China [Project No. 9042222, CityU 11300315].

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