C. R. Acad. Sci. Paris, Ser. I 354 (2016) 1109–1113

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Algebraic geometry/Differential geometry

Residue formula for Morita–Futaki–Bott invariant on orbifolds ☆

Une formule résiduelle pour l'invariant de Morita–Futaki–Bott sur une orbifold

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A R T I C L E   I N F O

Article history:
Received 9 June 2016
Accepted after revision 7 October 2016
Available online 17 October 2016

Presented by Claire Voisin

A B S T R A C T

In this work, we prove a residue formula for the Morita–Futaki–Bott invariant with respect to any holomorphic vector field, with isolated (possibly degenerated) singularities in terms of Grothendieck's residues.

R É S U M É

On obtient, en utilisant les résidus de Grothendieck, une formule résiduelle pour l'invariant de Morita–Futaki–Bott par rapport à un champ de vecteurs holomorphes avec singularités isolées, dégénérées ou non.

0. Introduction

Let X be a compact complex orbifold of dimension n. That is, X is a complex space endowed with the following property: each point $p \in X$ possesses a neighborhood, which is the quotient $\widetilde{U}/G_p$, where $\widetilde{U}$ is a complex manifold, say of dimension n, and G_p is a properly discontinuous finite group of automorphisms of $\widetilde{U}$, so that locally we have a quotient map $(\widetilde{U}, \tilde{p}) \xrightarrow{\pi_p} (\widetilde{U}/G_p, p)$. See [1].

Let $\eta(X)$ be the complex Lie algebra of all holomorphic vector fields of X. Choose any Hermitian metric h on X and let ∇ and Θ be the Hermitian connection and the curvature form with respect to h, respectively. Let ξ be a global holomorphic vector field on X and consider the operator

$$L(\xi) := [\xi, \cdot\,] - \nabla_\xi(\,\cdot\,) : T^{1,0}X \longrightarrow T^{1,0}X.$$

☆ This work was partially supported by CNPq, CAPES, FAPEMIG and FAPESP-2015/20841-5.
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http://dx.doi.org/10.1016/j.crma.2016.10.006
1631-073X/

Let ϕ be an invariant polynomial of degree $n+k$; the *Futaki–Morita integral invariant* is defined by

$$f_\phi(\xi) = \int_X \bar{\phi}\big(\underbrace{L(\xi), \ldots, L(\xi)}_{k\text{ times}}, \underbrace{\frac{\mathrm{i}}{2\pi}\Theta, \ldots, \frac{\mathrm{i}}{2\pi}\Theta}_{n\text{ times}}\big),$$

where $\bar{\phi}$ denotes the polarization of ϕ. Morita and Futaki proved in [6] that the definition of $f_\phi(\xi)$ does not depend on the choice of the Hermitian metric h. It is well known that the Futaki–Morita integral invariant can be calculated via a Bott-type residue formula for non-degenerated holomorphic vector fields, see [5–7] and [4] in the orbifold case. In this work, we prove a residue formula for holomorphic vector fields with isolated and possibly degenerated singularities in terms of Grothendieck's residues (see [8, Chapter 5]).

Theorem 1. *Let $\xi \in \eta(X)$ a holomorphic vector field with only isolated singularities, then*

$$\binom{n+k}{n} f_\phi(\xi) = (-1)^k \sum_{p\in \mathrm{Sing}(\xi)} \frac{1}{\# G_p} \mathrm{Res}_{\tilde{p}} \left\{ \frac{\phi(J\tilde{\xi})\, \mathrm{d}\tilde{z}_1 \wedge \cdots \wedge \mathrm{d}\tilde{z}_n}{\tilde{\xi}_1 \ldots \tilde{\xi}_n} \right\},$$

where, given p such that $\xi(p) = 0$ and $(\widetilde{U}, \tilde{p}) \xrightarrow{\pi_p} (\widetilde{U}/G_p, p)$ denotes the projection: $\tilde{\xi} = \pi_p^* \xi$, $J\tilde{\xi} = \left(\frac{\partial \tilde{\xi}_i}{\partial \tilde{z}_j}\right)_{1\le i,\, j\le n}$ and $\mathrm{Res}_{\tilde{p}} \left\{ \frac{\phi(J\tilde{\xi})\, \mathrm{d}\tilde{z}_1 \wedge \cdots \wedge \mathrm{d}\tilde{z}_n}{\tilde{\xi}_1 \ldots \tilde{\xi}_n} \right\}$ is Grothendieck's point residue and $(\tilde{z}_1, \ldots, \tilde{z}_n)$ is a germ of the coordinate system on $(\widetilde{U}, \tilde{p})$.

We note that such residue can be calculated using Hilbert's Nullstellensatz and Martinelli's integral formula. In fact, if $\tilde{z}_i^{a_i} = \sum_{j=1}^n b_{ij} \tilde{\xi}_j$, then (see [11])

$$\mathrm{Res}_{\tilde{p}} \left\{ \frac{\phi(J\tilde{\xi})\, \mathrm{d}\tilde{z}_1 \wedge \cdots \wedge \mathrm{d}\tilde{z}_n}{\tilde{\xi}_1 \ldots \tilde{\xi}_n} \right\} = \frac{1}{\prod_{i=1}^n (a_i - 1)!} \left(\frac{\partial^n}{\partial \tilde{z}_1^{a_1}, \ldots, \tilde{z}_n^{a_n}} \left(\mathrm{Det}(b_{ij}) \phi(J\tilde{\xi})\right) \right)(\tilde{p}). \tag{1}$$

Moreover, note that if $p \in \mathrm{Sing}(\xi)$ is a non-degenerated singularity of ξ, then

$$\mathrm{Res}_{\tilde{p}} \left\{ \frac{\phi(J\tilde{\xi})\, \mathrm{d}\tilde{z}_1 \wedge \cdots \wedge \mathrm{d}\tilde{z}_n}{\tilde{\xi}_1 \ldots \tilde{\xi}_n} \right\} = \frac{\phi(J\tilde{\xi}(\tilde{p}))}{\mathrm{Det}(J\tilde{\xi}(\tilde{p}))}.$$

Theorem 1 allows us to calculate the Morita–Futaki invariant for holomorphic vector fields with possible degenerated singularities. For instance, in a recent work [9], the Futaki–Bott residue for vector fields with degenerated singularities, on the blowup of Kähler surfaces, was calculated by Li and Shi. Compare the equation (1) with Lemma 3.6 of [9].

Futaki showed in [5] that if X admits a Kähler–Einstein metric, then $f_{C_1^{n+1}} \equiv 0$, where $C_1 = Tr$ denotes the trace, i.e., the first elementary symmetric polynomial. Taking $\phi = C_1^{n+1} = Tr^{n+1}$, we obtain the following corollary of Theorem 1.

Corollary 2. *Let $\xi \in \eta(X)$ with only isolated singularities, then*

$$f_{C_1^{n+1}}(\xi) = \frac{-1}{(n+1)^2} \sum_{p\in \mathrm{Sing}(\xi)} \frac{1}{\# G_p} \mathrm{Res}_{\tilde{p}} \left\{ \frac{Tr^{n+1}(J\tilde{\xi})\, \mathrm{d}\tilde{z}_1 \wedge \cdots \wedge \mathrm{d}\tilde{z}_n}{\tilde{\xi}_1 \ldots \tilde{\xi}_n} \right\}.$$

This result generalizes the Proposition 1.2 of [4]. It is well known that projective planes are Kähler–Einstein. However, the non-existence of Kähler–Einstein metrics on singular weighted projective planes was shown in previous works; see, for example, [12]. As an application of Theorem 1, we will give, in Section 1, a new proof of this fact.

1. Non-existence of Kähler–Einstein metric on weighted projective planes

Here we consider weighted complex projective planes with only isolated singularities, which we briefly recall.

Let w_0, w_1, w_2 be positive integers two by two co-primes, set $w := (w_0, w_1, w_2)$ and $|w| := w_0 + w_1 + w_2$. Define an action of $\mathbb{C}^*$ in $\mathbb{C}^3 \setminus \{0\}$ by

$$\begin{array}{ccc} \mathbb{C}^* \times \mathbb{C}^3 \setminus \{0\} & \longrightarrow & \mathbb{C}^3 \setminus \{0\} \\ \lambda.(z_0, z_1, z_2) & \longmapsto & (\lambda^{w_0} z_0, \lambda^{w_1} z_1, \lambda^{w_2} z_2) \end{array}$$

and let $\mathbb{P}^2_w := \mathbb{C}^3 \setminus \{0\}/\sim$. The weights are chosen to be pairwise co-primes in order to assure a finite number of singularities and to give $\mathbb{P}^2_w$ the structure of an effective, Abelian, compact orbifold of dimension 2. The singular locus is:

$$\mathrm{Sing}(\mathbb{P}^2_w) = \{[1:0:0]_\omega, [0:1:0]_\omega, [0:0:1]_\omega\}.$$

We have the canonical projection

$$\begin{array}{rcl} \pi : \mathbb{C}^3 \setminus \{0\} & \longrightarrow & \mathbb{P}^2_w \\ (z_0, z_1, z_2) & \longmapsto & [z_0^{w_0} : z_1^{w_1} : z_2^{w_2}]_w \end{array}$$

and the natural map

$$\begin{array}{rcl} \varphi_w : \mathbb{P}^n & \longrightarrow & \mathbb{P}^n_w \\ [z_0 : z_1 : z_2] & \longmapsto & [z_0^{w_0} : z_1^{w_1} : z_2^{w_2}]_w \end{array}$$

of degree $\deg \varphi_w = w_0 w_1 w_2$. The map φ_w is *good* in the sense of [1, section 4.4], which means, among other things, that V-bundles behave well under pullback. It is shown in [10] that there is a line V-bundle $\mathcal{O}_{\mathbb{P}^2_w}(1)$ on $\mathbb{P}^2_w$, unique up to isomorphism, such that

$$\varphi_w^* \mathcal{O}_{\mathbb{P}^2_w}(1) \cong \mathcal{O}_{\mathbb{P}^2}(1)$$

and, by naturality, $c_1(\varphi_w^* \mathcal{O}_{\mathbb{P}^2_w}(1)) = c_1(\mathcal{O}_{\mathbb{P}^2}(1)) = \varphi_w^* c_1(\mathcal{O}_{\mathbb{P}^2_w}(1))$, from which we obtain the Chern number

$$[\mathbb{P}^2_w] \frown \left(c_1(\mathcal{O}_{\mathbb{P}^2_w}(1))\right)^n = \int_{\mathbb{P}^n_w} \left(c_1(\mathcal{O}_{\mathbb{P}^2_w}(1))\right)^2 = \frac{1}{w_0 w_1 w_2}$$

since

$$1 = \int_{\mathbb{P}^2} (c_1(\mathcal{O}_{\mathbb{P}^2}(1)))^2 = \int_{\mathbb{P}^2} \varphi_w^* \left(c_1(\mathcal{O}_{\mathbb{P}^2_w}(1))\right)^2 = (\deg \varphi_w) \int_{\mathbb{P}^2_w} \left(c_1(\mathcal{O}_{\mathbb{P}^2_w}(1))\right)^2 .$$

There exist an Euler type sequence on $\mathbb{P}^n_w$

$$0 \longrightarrow \underline{\mathbb{C}} \longrightarrow \bigoplus_{i=0}^{2} \mathcal{O}_{\mathbb{P}^2_w}(w_i) \longrightarrow T\mathbb{P}^2_w \longrightarrow 0,$$

where

(i) $1 \longmapsto (w_0 z_0, w_1 z_1, w_2 z_2)$.
(ii) $(P_0, P_1, P_2) \longmapsto \pi_* \left(\sum_{i=0}^2 P_i \frac{\partial}{\partial z_i}\right)$.

It is well known that the non-singular weighted projective planes admit Kähler–Einstein metrics. On the other side, singular weighted projective spaces do not admit Kähler–Einstein metrics, see [12]. We give a simple proof of the non-existence of Kähler–Einstein metrics on singular $\mathbb{P}^2_\omega$ by using Corollary 2.

Theorem 3. The singular weighted projective space $\mathbb{P}^2_\omega$ does not admit any Kähler–Einstein metric.

Proof. Choose $a_0, a_1, a_2 \in \mathbb{C}^*$ such that $a_i w_j \neq a_j w_i$, for all $i \neq j$. Suppose, without loss of generality, that $1 \leq w_0 \leq w_2 < w_1$. Consider the holomorphic vector field on $\mathbb{P}^2_\omega$ given by

$$\xi_a = \sum_{k=0}^{2} a_k Z_k \frac{\partial}{\partial Z_k} \in H^0(\mathbb{P}^2_\omega, T\mathbb{P}^2_\omega),$$

where (Z_0, Z_1, Z_3) denotes the homogeneous coordinate system.

The local expression of ξ over $U_i = \{[Z_0 : Z_1 : Z_3] \in \mathbb{P}^2;\ Z_i \neq 0\}$ is given by

$$\xi_a|_{U_i} = \sum_{\substack{k=0 \\ k \neq i}}^{2} \left(a_k - a_i \frac{w_k}{w_i}\right) Z_k \frac{\partial}{\partial Z_k}.$$

Therefore, the singular set $\mathrm{Sing}(\xi|_{U_i})$ is reduced to $\{0\}$ and it is nondegenerate. In general,

$$\mathrm{Sing}(\xi_a) = \{[1:0:0]_\omega, [0:1:0]_\omega, [0:0:1]_\omega\} = \mathrm{Sing}(\mathbb{P}^2_\omega).$$

It follows from Corollary 2 that

$$f(\xi_a) = \frac{-1}{3^2} \sum_{i=0}^{2} \frac{1}{w_i^2} \frac{\left(\sum_{k\neq i}(a_k w_i - a_i w_k)\right)^3}{\prod_{k\neq i}(a_k w_i - a_i w_k)}.$$

Thus

$$\zeta(a_0,a_1,a_2) = -3^2 w_0^2 w_1^2 w_2^2 \prod_{0\le i<j\le 2} (a_i w_j - a_j w_i) f(\xi_a) =$$

$$(3w_1^5 w_2^2 w_0 - 3w_1^4 w_2^3 w_0 + 3w_1^3 w_2^4 w_0 + 3w_1^2 w_2^5 w_0 - 3w_0^4 w_2^2 w_1^2 + 3w_0^3 w_2^3 w_1^2 + 6w_0^2 w_2^4 w_1^2 + \\ + 3w_0^4 w_1^2 w_2^2 - 3w_0^3 w_1^3 w_2^2 - 6w_0^2 w_1^4 w_2^2) \cdot a_1 a_2 a_0^2 + \cdots$$

is a homogeneous polynomial of degree 4 in the variables a_0, a_1, a_2. Suppose by contradiction that $\zeta(a_0,a_1,a_2) \equiv 0$. In particular, the coefficient of the monomial $a_0^2 a_1 a_2$ is zero. Thus, we have the following equation

$$w_2(w_1 w_2 + w_2^2 + w_0^2 + 2w_0 w_2) = w_1(w_1 w_2 + w_1^2 + w_0^2 + 2w_0 w_1).$$

This contradicts $1 \le w_0 \le w_2 < w_1$. Thus the non-vanishing of $\zeta(a_0,a_1,a_2)$ implies that $f(\xi_a)$ is not zero. Therefore, $\mathbb{P}^2_\omega$ does not admit Kähler–Einstein metrics. □

2. Proof of Theorem 1

For the proof, we will use Bott–Chern's transgression method, see [2] and [3].

Let $p_1,\ldots,p_m$ be the zeros of ξ. Let $\{U_\beta\}$ be an open cover orbifold of X ($\varphi_\beta : \widetilde{U}_\beta \to U_\beta \subset X$ coordinate map). Suppose that $\{U_\beta\}$ is a trivializing neighborhood for the holomorphic tangent orbibundle TX (see [1, section 2.3]) of X and that we have disjoint neighborhoods coordinates U_α with $p_\alpha \in U_\alpha$ and $p_\alpha \notin U_\beta$ if $\alpha \neq \beta$. On each $\widetilde{U}_\alpha$, take local coordinates $\tilde{z}^\alpha = (\tilde{z}_1^\alpha,\ldots,\tilde{z}_n^\alpha)$ and the holomorphic frame $\{\frac{\partial}{\partial \tilde{z}_1^\alpha},\ldots,\frac{\partial}{\partial \tilde{z}_n^\alpha}\}$ of TX. Thus, we have a local representation

$$\tilde{\xi}^\alpha = \sum \tilde{\xi}_i^\alpha \frac{\partial}{\partial \tilde{z}_i^\alpha},$$

where $\tilde{\xi}_i^\alpha$ are holomorphic functions in $\widetilde{U}_\alpha$, $1 \le i \le n$. Let $\tilde{h}_\alpha$ the Hermitian metric in $\widetilde{U}_\alpha$ defined by $\langle \partial/\partial \tilde{z}_i^\alpha, \partial/\partial \tilde{z}_j^\alpha \rangle = \delta_j^i$. Also consider $\widetilde{U}'_\alpha \subset \widetilde{U}_\alpha$ and $U'_\alpha = \varphi_\alpha(\widetilde{U}'_\alpha)$ for each α. Take a Hermitian metric h_0 in any $X \backslash \cup_\alpha \{p_\alpha\}$ and $\{\rho_0, \rho_\alpha\}$ a partition of unity subordinate to the cover $\{X \backslash \cup_\alpha \overline{U'_\alpha}, U_\alpha\}_\alpha$. Define a Hermitian metric $h = \rho_0 h_0 + \sum \rho_\alpha h_\alpha$ in X. Then we have that for every α, the metric curvature $\Theta \equiv 0$ in U'_α.

Consider the matrix of the metric connection ∇ in the open $\widetilde{U}^\beta$ given by $\theta^\beta = (\sum_k \Gamma_{ik}^{\beta j} d\tilde{z}_k^\beta)$.

The local expression of $L(\xi)$ is given by $\tilde{E}^\beta = (\tilde{E}_{ij}^\beta)$ such that

$$\tilde{E}_{ij}^\beta = -\frac{\partial \tilde{\xi}_i^\beta}{\partial \tilde{z}_j^\beta} - \sum_s \Gamma_{js}^{\beta i} \tilde{\xi}_s^\beta,$$

see [2] and [8]. We indicate by $\mathcal{A}^{p,q}(X)$ the vector space of complex-valued $(p+q)$-forms on X of type (p,q). Define

$$\phi_r := \binom{n+k}{r} \bar{\phi}(\underbrace{E,\ldots,E}_{n+k-r}, \underbrace{\Theta,\ldots,\Theta}_{r}) \in \mathcal{A}^{r,r}(X) \quad r = 0,\ldots,n.$$

Let $\omega \in \mathcal{A}^{1,0}(X)$ in $X \backslash \mathrm{Sing}(\xi)$, with $\omega(\xi) = 1$. Following Bott's idea (see [2]), it is sufficient to show that there exists ψ such that $i(\xi)(\bar{\partial}\psi + \phi_n) = 0$ on $X \backslash \mathrm{Sing}(\xi)$. We take $\psi = \sum_{r=0}^{n-1} \psi_r$ such that

$$\psi_r = \omega \wedge (\bar{\partial}\omega)^{n-r-1} \wedge \phi_r \in \mathcal{A}^{n,n-1}(X) \quad r = 0,\ldots,n-1.$$

The following formulas hold (see [2] or [8]):

a) $\bar{\partial}\Theta = 0$, $\bar{\partial}E = i(\xi)\Theta$;
b) $\bar{\partial}\phi_r = i(\xi)\phi_{r+1}$, $r = 0,\ldots,n+1$;
c) $i(\xi)\bar{\partial}\omega = 0$.

Let us prove b): since $\bar{\partial}\Theta = 0$ and $\bar{\partial}E = i(\xi)\Theta$, we have

$$\bar{\partial}\phi_r = \binom{n+k}{r} \sum_{i=1}^{n+k-r} \bar{\phi}(E,\ldots,i(\xi)\Theta,\ldots,E,\Theta,\ldots,\Theta) = i(\xi)\phi_{r+1}.$$

Therefore, a), b) and c) implies that on $X \backslash \mathrm{Sing}(\xi)$ we get

$$i(\xi)(\bar{\partial}\,\psi + \phi_n) = 0.$$

Therefore, $\mathrm{d}\psi = \bar{\partial}\psi = -\phi_n$ on $X \backslash \mathrm{Sing}(\xi)$. Thus, by the Satake–Stokes Theorem, we have

$$\begin{aligned}\binom{n+k}{n} f_\phi(\xi) &= \left(\frac{\mathrm{i}}{2\pi}\right)^n \int_X \phi_n = \left(\frac{\mathrm{i}}{2\pi}\right)^n \lim_{\epsilon \to 0} \int_{X \backslash \cup_\alpha B_\epsilon(p_\alpha)} \phi_n \\ &= -\left(\frac{\mathrm{i}}{2\pi}\right)^n \lim_{\epsilon \to 0} \int_{X \backslash \cup_\alpha B_\epsilon(p_\alpha)} \mathrm{d}\psi = \left(\frac{\mathrm{i}}{2\pi}\right)^n \lim_{\epsilon \to 0} \sum_\alpha \int_{\partial B_\epsilon(p_\alpha)} \psi^\alpha, \end{aligned} \tag{2}$$

where is $B_\epsilon(p_\alpha) = B_\epsilon(\tilde{p}_\alpha)/G_{p_\alpha}$ and $B_\epsilon(\tilde{p}_\alpha)$ is an Euclidean ball centered at $\tilde{p}_\alpha$ such that $\overline{B_\epsilon(\tilde{p}_\alpha)} \subset U'_\alpha$. Since our metric is Euclidean in $B_\epsilon(\tilde{p}_\alpha)$, its connection is zero and

$$\tilde{E}^\alpha_{ij} = -\frac{\partial \tilde{\xi}^\alpha_i}{\partial \tilde{z}^\alpha_j}.$$

Now, by our choice of metric, Θ and hence ϕ_r, for $r > 0$, vanishes identically in $B_\epsilon(\tilde{p}_\alpha)$. Then, we have

$$\tilde{\psi}^\alpha = \tilde{\psi}^\alpha_0 = \omega \wedge (\bar{\partial}\,\omega)^{n-1} \phi(\tilde{E}^\alpha) = (-1)^{n+k} \omega \wedge (\bar{\partial}\,\omega)^{n-1} \phi(J\tilde{\xi}^\alpha)$$

on $B_\epsilon(\tilde{p}_\alpha)$. Therefore,

$$\tilde{\psi}^\alpha = (-1)^k \omega \wedge (\bar{\partial}\,\omega)^{n-1} \phi(J\tilde{\xi}^\alpha). \tag{3}$$

Consider the map $\Phi : \mathbb{C}^n \to \mathbb{C}^{2n}$ given by $\Phi(\tilde{z}) = (\tilde{z} + \tilde{\xi}(\tilde{z}), \tilde{z})$. There is a $(2n, 2n-1)$ closed form β_n in $\mathbb{C}^{2n} \backslash \{0\}$ (the Bochner–Martinelli kernel) such that

$$\Phi^* \beta_n = \left(\frac{\mathrm{i}}{2\pi}\right)^n \omega \wedge (\bar{\partial}\,\omega)^{n-1}. \tag{4}$$

Finally, if we substitute (3) and (4) into (2), and by using Martinelli's formula ([8, p. 655])

$$\int_{\partial B_\epsilon(\tilde{p}_\alpha)} \phi(J\tilde{\xi}^\alpha)\, \Phi^* \beta_n = \mathrm{Res}_{\tilde{p}_\alpha} \left\{ \frac{\phi\big(J\tilde{\xi}^\alpha\big)\, \mathrm{d}\tilde{z}_1 \wedge \cdots \wedge \mathrm{d}\tilde{z}_n}{\tilde{\xi}_1 \ldots \tilde{\xi}_n} \right\}$$

we obtain

$$\binom{n+k}{n} f_\phi(\xi) = (-1)^k \sum_\alpha \frac{1}{\# G_{p_\alpha}} \mathrm{Res}_{\tilde{p}_\alpha} \left\{ \frac{\phi\big(J\tilde{\xi}^\alpha\big)\, \mathrm{d}\tilde{z}_1 \wedge \cdots \wedge \mathrm{d}\tilde{z}_n}{\tilde{\xi}_1 \ldots \tilde{\xi}_n} \right\}.$$

Acknowledgements

We are grateful to Marcio Soares and Marcos Jardim for many stimulating conversations on this subject.

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