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Functional analysis

On the singular values of compact composition operators ☆

Sur les valeurs singulières des opérateurs de composition compacts

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ABSTRACT

Let μ be a positive Borel measure on the unit disc and let T_μ be the associated Toeplitz operator on a standard Bergman space. Under some convexity conditions on a positive function h, we give an upper and lower bounds of the trace of $h(T_\mu)$. As consequence, we give some asymptotic estimates of eigenvalues of T_μ. We also apply these results to composition operators and give some concrete examples.

RÉSUMÉ

Soit μ une mesure de Borel positive sur le disque unité et soit T_μ l'opérateur de Toeplitz associé à μ sur un espace de Bergman standard. Pour une fonction positive h satisfaisant des conditions de convexité, nous donnons des bornes inférieures et supérieures de la trace de $h(T_\mu)$. Ceci nous permet d'obtenir quelques estimations asymptotiques des valeurs propres de T_μ. Nous appliquons ces résultats pour les opérateurs de composition et donnons ensuite quelques exemples concrets.

1. Toeplitz operators on Bergman spaces

1.1. Introduction

Let $H(\mathbb{D})$ be the class of all holomorphic functions on the unit disc $\mathbb{D}$. Let m be the normalized Lebesgue measure on $\mathbb{D}$ and let $\mathrm{d}m_\alpha(z) = (1+\alpha)(1-|z|^2)^\alpha \mathrm{d}m(z)$. The standard weighted Bergman spaces $\mathcal{A}^2_\alpha$, $\alpha > -1$, are given by

$$\mathcal{A}^2_\alpha := \left\{ f \in H(\mathbb{D}) : \|f\|^2_\alpha = \int_{\mathbb{D}} |f(z)|^2 \, \mathrm{d}m_\alpha(z) < \infty \right\}.$$

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Let μ be a positive Borel measure on $\mathbb{D}$. The Toeplitz operator T_μ on $\mathcal{A}^2_\alpha$ is given by

$$T_\mu f(z) = \int_{\mathbb{D}} f(w) K^\alpha(z,w)(1-|z|^2)^\alpha \mathrm{d}\mu(w),$$

where $K^\alpha(z,w) = 1/(1-z\bar{w})^{2+\alpha}$ is the reproducing kernel of $\mathcal{A}^2_\alpha$.

Recall that μ is said to be a Carleson measure for $\mathcal{A}^2_\alpha$ if the embedding operator $J_\mu : \mathcal{A}^2_\alpha \to L^2(\mu)$ is bounded, where $L^2(\mu)$ denotes the space of Borelian functions f on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f(z)|^2 \mathrm{d}\mu(z) < +\infty.$$

Let $\mathrm{d}\mu_\alpha = (1-|z|^2)^\alpha \mathrm{d}\mu$; since $T_\mu = J^*_{\mu_\alpha} J_{\mu_\alpha}$, then T_μ is bounded if and only if μ_α is a Carleson measure for $\mathcal{A}^2_\alpha$, that means by [4],

$$\mu(S(\zeta,h)) = O(h^2) \qquad (h \to 0^+), \quad (\zeta \in \mathbb{T}),$$

where $S(\zeta,h) = \{z \in \mathbb{D} : \ 1-|z| < h, \ |\arg(z\bar{\zeta})| < h\}$.

In the same way, T_μ is compact if and only if

$$\mu(S(\zeta,h)) = o(h^2) \qquad (h \to 0^+), \quad (\zeta \in \mathbb{T}).$$

Let n, j be integers such that $n \geq 1$ and $j \in \{0, 2, .., 2^n - 1\}$. The dyadic square $R_{n,j}$ is given by

$$R_{n,j} = \left\{ z \in \mathbb{D} ; \ 2^{-n-1} < 1-|z| \leq 2^{-n} \text{ and } \frac{2j\pi}{2^n} \leq \arg z < \frac{2(j+1)\pi}{2^n} \right\}.$$

Note that a geometric characterization of positive Borel measures μ such that T_μ belongs to p-Schatten classes, namely $T_\mu \in S_p(\mathcal{A}^2_\alpha)$, has been given by D. Luecking in [7]. Indeed, he proved that $T_\mu \in \mathcal{S}_p(\mathcal{A}^2_\alpha)$ if and only if

$$\sum_{n,j} 2^{2np} \left(\mu(R_{n,j})\right)^p < \infty.$$

1.2. Eigenvalues of Toeplitz operators

First, we will give a generalization of the Luecking characterization of membership in Schatten classes. We will denote by $(\lambda_n(T,\mathcal{H}))_n$ the decreasing sequence of the eigenvalues of the positive compact operator T on a separable Hilbert space $\mathcal{H}$.

Theorem 1.1. *Let μ be a positive Borel measure on the unit disc such that T_μ is compact on $\mathcal{A}^2_\alpha$. Let h be an increasing function such that $h(0) = 0$ and satisfying one of the following conditions:*

- *h is convex.*
- *h is concave and $h(t)/t^\varepsilon$ is increasing for some $\varepsilon > 0$.*

We have

$$B \sum_{n,j} h\left(b 2^{2n} \mu(R_{n,j})\right) \leq \sum_{n,j} h\left(\lambda_n(T_\mu, \mathcal{A}^2_\alpha)\right) \leq A \sum_{n,j} h\left(a 2^{2n} \mu(R_{n,j})\right),$$

where $A, B, a, b > 0$ are constants that depend only on α in the first case and depend on α and ε in the second case.

Note that if $h(2t) \asymp h(t)$, then the trace of $h(T_\mu)$ is finite if and only if $\sum_{n,j} h(2^{2n}\mu(R_{n,j})) < \infty$. In particular, we recover Luecking's theorem, ($h(t) = t^p$, $p > 0$). Note also that this result extends Theorem 1.1 of [1].

Applying Theorem 1.1 with particular functions, $h_\delta(t) = (t-\delta)^+$ and

$$h_{\delta,\varepsilon}(t) = \begin{cases} t & \text{if } t \in [0,\delta] \\ \delta^{1-\varepsilon} t^\varepsilon & \text{if } t \geq \delta, \end{cases}$$

we get the following.

Corollary 1.2. *Let μ be a positive Borel measure on $\mathbb{D}$ such that T_μ is a compact operator on $\mathcal{A}^2_\alpha$. Let $\rho : [0,+\infty] \to [1,+\infty[$ be an increasing function. Suppose that ρ satisfies one of the following conditions:*

- *there exists $\gamma \in (0,1)$ such that $\rho(t)/t^\gamma$ is decreasing.*
- *there are $\beta \in (0,\infty)$, $\gamma \in (1,\infty)$ such that $\rho(t)/t^\beta$ is decreasing and $\rho(t)/t^\gamma$ is increasing.*

Then the following are equivalent:

(1) $\lambda_n(T_\mu, \mathcal{A}^2_\alpha) \asymp 1/\rho(n)$;
(2) $a_n(\mu) \asymp 1/\rho(n)$, *where* $(a_n(\mu))_{n\ge1}$ *is a decreasing enumeration of* $(2^{2n}\mu(R_{n,j}))_{n,j}$.

2. Composition operators

We will consider composition operators on standard weighted analytic spaces on the unit disc $\mathbb{D}$. For $\alpha \ge 0$, $\mathcal{H}_\alpha$ will denote the space of analytic functions $f \in H(\mathbb{D})$ such that

$$\int_{\mathbb{D}} |f'(z)|^2 \,\mathrm{d}m_\alpha(z) < \infty.$$

It becomes a Hilbert space if endowed with the norm $\|.\|_{\mathcal{H}_\alpha}$, given by

$$\|f\|^2_{\mathcal{H}_\alpha} =: |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \,\mathrm{d}m_\alpha(z).$$

By the classical Littlewood–Paley identity, we have $\mathcal{H}_1 = H^2$ the Hardy space. Note also that for $\alpha \in [0,1)$, $\mathcal{H}_\alpha = \mathcal{D}_\alpha$ are the weighted Dirichlet spaces and for $\alpha > 1$, $\mathcal{H}_\alpha$ are the standard weighted Bergman spaces.

Let φ be a holomorphic self-map of $\mathbb{D}$. The composition operator C_φ acting on $\mathcal{H}_\alpha$ with symbol φ is defined by

$$C_\varphi f = f \circ \varphi, \quad f \in \mathcal{H}_\alpha.$$

Several papers gave some general criterion for boundedness, compactness and membership in Schatten classes of composition operators (see, for instance, [10,7,11,3,5,8]).

The Nevanlinna counting function, $N_{\varphi,\alpha}$, of φ associated with $\mathcal{H}_\alpha$ is defined by

$$N_{\varphi,\alpha}(w) = \begin{cases} \displaystyle\sum_{z\in\varphi^{-1}(w)} (1-|z|^2)^\alpha & \text{if } w \in \varphi(\mathbb{D}), \\ 0 & \text{if } w \notin \varphi(\mathbb{D}). \end{cases}$$

In what follows, $\mu_{\varphi,\alpha}$ will denote the measure given by

$$\mathrm{d}\mu_{\varphi,\alpha}(w) = \frac{N_{\varphi,\alpha}(w)}{(1-|w|^2)^\alpha}\,\mathrm{d}m(w), \quad (w \in \mathbb{D}).$$

It is known that the composition operators C_φ are closely related to the Toeplitz operator $T_{\mu_{\varphi,\alpha}}$. More precisely, we have the following result.

Proposition 2.1. *Let φ be an analytic self map of $\mathbb{D}$. Then C_φ is compact on $\mathcal{H}_\alpha$ if and only if $T_{\mu_{\varphi,\alpha}}$ is compact on $\mathcal{A}^2_\alpha$, and*

$$K^{-1}\lambda_{n+1}(T_{\mu_{\varphi,\alpha}}, \mathcal{A}^2_\alpha) \le s^2_{n+1}(C_\varphi, \mathcal{H}_\alpha) \le K\lambda_n(T_{\mu_{\varphi,\alpha}}, \mathcal{A}^2_\alpha),$$

where K is a positive constant that depends only on α and $|\varphi(0)|$.

The asymptotic behavior of singular values of composition operators, for some particular symbols, was considered by several authors; see, for instance, [6,9] and references therein. The following theorem gives us a method to estimate the singular values in some cases, as we will see in the sequel.

Theorem 2.2. *Let φ be an analytic self map of $\mathbb{D}$ and let $h : [0,+\infty) \to [0,+\infty)$ be an increasing function. Let h be an increasing function such that $h(0) = 0$ and satisfying one of the following conditions:*

- *h is convex.*
- *h is concave and $h(t)/t^\varepsilon$ is increasing for some $\varepsilon > 0$.*

We have

$$B\sum_{n,j} h\left(b2^{2n}\mu_{\varphi,\alpha}(W_{n,j})\right) \le \sum_n h\left(s^2_n(C_\varphi,\mathcal{H}_\alpha)\right) \le A\sum_{n,j} h\left(a2^{2n}\mu_{\varphi,\alpha}(W_{n,j})\right),$$

where $A, B, a, b > 0$ depend on $|\varphi(0)|$ and α in the first case, and depend in addition on ε in the second case.

2.1. Composition operators with univalent symbol

Let $\Omega \subset \mathbb{D}$ be a simply connected domain. Let φ be a conformal map from $\mathbb{D}$ onto Ω. Let σ be an automorphism of $\mathbb{D}$. Since C_σ is an invertible operator on $\mathcal{H}_\alpha$, we have $s_n(C_\varphi, \mathcal{H}_\alpha) \asymp s_n(C_{\varphi\circ\sigma}, \mathcal{H}_\alpha)$ $(n \to \infty)$. Then the asymptotic behavior of singular values depends only on Ω. In the sequel, we will suppose that $\varphi(0) = 0$. Let us first introduce the pull-back measure associated with φ. It will be denoted by m_φ and it is the positive Borelian measure defined by

$$m_\varphi(B) = m(\{\zeta \in \mathbb{T} : \varphi(\zeta) \in B\}),$$

where m here is the normalized Lebesgue measure of $\mathbb{T}$.

Theorem 2.3. *Let $\Omega \subset \mathbb{D}$ be a simply connected domain such that $0 \in \Omega$. Let φ be a conformal mapping from $\mathbb{D}$ onto Ω such that $\varphi(0) = 0$. Let h be an increasing function such that $h(0) = 0$ and satisfying one of the following conditions:*

- *h is convex.*
- *h is concave and $h(t)/t^\varepsilon$ is increasing for some $\varepsilon > 0$.*

We have

$$B \sum_{n,j} h\left(b\left(2^n m_\varphi(W_{n,j})\right)^\alpha\right) \le \sum_n h\left(s_n^2(C_\varphi, \mathcal{H}_\alpha)\right) \le B \sum_{n,j} h\left(b\left(2^n m_\varphi(W_{n,j})\right)^\alpha\right),$$

where $A, B, a, b > 0$ depend on α in the first case and on α and ε in the second case.

2.2. Example

Let Ω be a Jordan subdomain of $\mathbb{D}$ that contains 0. Let φ be a conformal map of $\mathbb{D}$ onto Ω such that $\varphi(0) = 0$. By Caratheodory's theorem, φ can be extended continuously from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$. The extension will also be noted by φ. By definition, the harmonic measure $\omega(., E, \Omega)$ is the harmonic extension of χ_E, where E is a closed subset of $\partial\Omega$. By conformal invariance of the harmonic measure, we have:

$$\omega(0, E, \Omega) = m(\varphi^{-1}(E)) = m_\varphi(E).$$

Let Ω be a subdomain of $\mathbb{D}$ such that $0 \in \Omega$, $\partial\Omega \cap \partial\mathbb{D} = \{1\}$ and $\partial\Omega$ has, in a neighborhood of $+1$, a polar equation $1 - r = \gamma(|\theta|)$, where $\gamma : [0, \pi] \to [0, 1]$ is a continuous, increasing function with $\gamma(0) = 0$, and satisfying the following conditions

$$\lim_{t\to 0^+} \frac{\gamma(t)}{t} = 0, \quad \gamma'(t) = O(\gamma(t)/t) \quad (t \to 0^+) \tag{1}$$

and

$$\gamma(t) = O\left(t/\log^\beta(1/t)\right) \quad \text{for some } \beta > 1/2. \tag{2}$$

Let φ be a univalent map of $\mathbb{D}$ onto Ω with $\varphi(0) = 0$ and $\varphi(1) = 1$. Recall that, by Tsuji–Warschawski's theorem, C_φ is compact if and only if

$$\int_0 \frac{\gamma(s)}{s^2}\,\mathrm{d}s = \infty.$$

Recently, it was proved in [2] that the composition operator C_φ on $\mathcal{H}_\alpha$ is in p-Schatten class $(p > 0)$ if and only if

$$\int_0 \frac{\mathrm{e}^{-\frac{p\alpha}{2}\Gamma(t)}}{\gamma(t)}\,\mathrm{d}t < \infty,$$

where

$$\Gamma(t) = \frac{2}{\pi}\int_t^1 \frac{\gamma(s)}{s^2}\,\mathrm{d}s. \tag{3}$$

To take advantage of Theorem 2.3, we need an estimation of the harmonic measure of our domains. To this end, we will use a variant of Ahlfors' and Warschawski's theorems, which is the subject of the following lemma. For the proof, one can exploit the same arguments as those used in [2].

Lemma 2.4. *Under the same hypothesis and notations as above, there exist constants* $C, \varepsilon > 0$ *such that*

(1) $\omega(0, W_{n,j} \cap \partial\Omega, \Omega) \leq \dfrac{C}{2^n} \exp\{-\Gamma\left(\frac{2\pi(j+1)}{2^n}\right)\}$, $(0 \leq j \leq \frac{2^n}{2\pi}\gamma^{-1}(2\pi/2^n))$.

(2) $\text{Card}\left\{j \in \{2^k, .., 2^{k+1}\} : \omega(0, W_{n,j} \cap \partial\Omega, \Omega) \geq \dfrac{\epsilon}{2^n} \exp -\Gamma\left(\frac{2\pi j+1}{2^n}\right)\right\} \asymp 2^k$.

Combining these estimates and Theorem 2.3, we obtain

Theorem 2.5. *Let* γ, Ω *and* φ *as above. Let* $h : [0, +\infty) \to [0, +\infty)$ *be an increasing function. Let h be an increasing function such that* $h(0) = 0$ *and satisfying one of the following conditions*

- *h is convex;*
- *h is concave and* $h(t)/t^{\varepsilon}$ *is increasing for some* $\varepsilon > 0$.

We have

$$B \int_0^1 \frac{h(b\, e^{-\alpha\Gamma(s)})}{\gamma(s)} ds \leq \sum_n h\left(s_n^2(C_\varphi, \mathcal{H}_\alpha)\right) \leq A \int_0^1 \frac{h(a\, e^{-\alpha\Gamma(s)})}{\gamma(s)} ds$$

where $A, B, a, b > 0$ *depend on* α *and* γ *in the first case and on* α, γ *and* ε *in the second case.*

This can be applied to have an asymptotic behavior of singular values of this kind of composition operators. As an illustration, we give the following examples.

Corollary 2.6. *Let* γ, Ω *and* φ *as above. We have*

(1) *If* $\gamma(t) = \frac{ct}{\log(e/t)}$ *with* $c \neq \pi/\alpha$, *then*

$$s_n(C_\varphi, \mathcal{H}_\alpha) \asymp \frac{1}{n^{\frac{\alpha c}{2\pi}}}.$$

(2) *If* $\gamma(t) = \frac{ct}{\log(e/t)\log\log(e/t)}$ *with* $c > 0$, *then*

$$s_n(C_\varphi, \mathcal{H}_\alpha) \asymp \frac{1}{\log^{\alpha c/\pi} n}.$$

The first part of Corollary 2.6 gives, in particular, $C = \frac{2\pi}{\alpha}$, an example of composition operator on $\mathcal{H}_\alpha$ which is in the Dixmier class without being in $S_1(\mathcal{H}_\alpha)$.

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