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Higher-order topological sensitivity analysis for the Laplace operator

Analyse de sensibilité topologique d'ordre supérieur pour l'opérateur de Laplace

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ABSTRACT

This paper deals with higher-order topological sensitivity analysis for the Laplace operator with respect to the presence of a Dirichlet geometry perturbation. Two main results are presented in this work. In the first one, we discuss the influence of the considered geometry perturbation on the Laplace solution. The second one is devoted to the higher-order topological derivatives. We derive a higher-order topological sensitivity analysis for a large class of shape functions.

RÉSUMÉ

Dans ce papier, on donne une analyse de sensibilité pour l'opérateur de Laplace par rapport à des perturbations géométriques de type Dirichlet. On pésente deux résultats. Le premier concerne l'influence de la perturbation géométrique sur la solution du problème de Laplace. On dérive une formule de représentation asymptotique d'ordre supérieur décrivant le comportement de la solution perturbée en fonction de la taille de la perturbation. Le deuxième concerne les dérivées d'une fonction de forme par rapport à la modification de la topologie du domaine. On donne un développement asymtotique topologique d'ordre supérieur valable pour une grande classe de fonctions de forme.

1. Introduction

Topological sensitivity analysis has been derived for various operators and has been used for many topology optimization problems, e.g. for the Laplace equation [9], for the Stokes system [1,2,4,12], for the elasticity problem [8,11], for the

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Helmholtz equation [15], for the elastodynamic and acoustic problems [5,6,10], etc. In all these works, the optimization algorithms are based on the first-order topological derivative, which is only valid for small geometry perturbation size. The use of higher-order terms in the topological asymptotic expansion of the shape function may certainly be decisive in improving the topological optimization algorithms without restrictions on the perturbation sizes. This question has been partially addressed by Novotny et al. [13,14] in the particular case of circular holes with an asymptotic expansion limited to order two. The proposed mathematical analysis in [13,14] is based on a restricted approach and cannot be generalized to the higher-order case.

In this work, we consider the three-dimensional case and we derive a higher-order topological sensitivity analysis for the Laplace operator with respect to the presence of Dirichlet geometric perturbations. More precisely, let Ω be a bounded domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. We consider the case in which Ω contains a geometry perturbation $\omega_{z,\varepsilon}$ that is centered at $z \in \Omega$ and has the form $\omega_{z,\varepsilon} = z + \varepsilon\omega$, where $\omega \subset \mathbb{R}^3$ is a given fixed and bounded regular domain containing the origin.

Two main questions are discussed in this paper. The first one concerns the influence of the geometry perturbation on the Laplace equation solution. We derive a higher-order asymptotic expansion for the solution to the perturbed Laplace equation with respect to the geometry perturbation size. This question has been investigated by Ammari and Kang [3] in the inhomogeneity case where the perturbed solution is computed in the entire domain Ω using a continuity condition on the boundary $\partial\omega_{z,\varepsilon}$. In this work, we deal with more singular geometric perturbations. The solution to the perturbed Laplace equation is computed in $\Omega_{z,\varepsilon} = \Omega \setminus \overline{\omega_{z,\varepsilon}}$ with Dirichlet condition on $\partial\omega_{z,\varepsilon}$. As we will show in Section 3, this type of perturbations leads to an asymptotic behavior with respect to ε different from that obtained in [3].

The second one concerns the higher-order topological derivatives. In Section 4, we derive a higher-order topological asymptotic expansion for the Laplace operator. More precisely, we derive an asymptotic expansion of a given shape functional j in the following form:

$$j(\Omega_{z,\varepsilon}) = j(\Omega) + \sum_{k=1}^{N} f_k(\varepsilon)\delta^k j(z) + o(f_N(\varepsilon)), \text{ where}$$

- $(f_k)_{1\le k\le N}$ are positive scalar functions verifying $f_{k+1}(\varepsilon) = o(f_k(\varepsilon))$ and $\lim_{\varepsilon\to 0} f_k(\varepsilon) = 0$.
- $\delta^k j$ denotes the kth topological derivative of the shape function j.

2. Formulation of the problem

Consider a shape function j of the form

$$j(\Omega \setminus \overline{\omega_{z,\varepsilon}}) = J_\varepsilon(u_\varepsilon),$$

where J_ε is defined on $H^1(\Omega \setminus \overline{\omega_{z,\varepsilon}})$ and u_ε is the solution to Laplace problem in the perturbed domain $\Omega_{z,\varepsilon} = \Omega \setminus \overline{\omega_{z,\varepsilon}}$ with homogeneous Dirichlet condition on $\partial\omega_{z,\varepsilon}$

$$\begin{aligned} -\Delta u_\varepsilon &= 0 && \text{in } \Omega_{z,\varepsilon}, \\ \nabla u_\varepsilon \cdot n &= \Phi_n && \text{on } \Gamma_n, \\ u_\varepsilon &= \Phi_d && \text{on } \Gamma_d, \\ u_\varepsilon &= 0 && \text{on } \partial\omega_{z,\varepsilon}, \end{aligned} \tag{1}$$

where $\Phi_n \in H^{-1/2}(\Gamma_n)$ and $\Phi_d \in H^{1/2}(\Gamma_d)$ are two given data, with Γ_n and Γ_d are two parts of the boundary $\partial\Omega$ verifying $\overline{\partial\Omega} = \overline{\Gamma_n} \cup \overline{\Gamma_d}$ and $\Gamma_d \cap \Gamma_n = \emptyset$.

As we have mentioned in the introduction, the aim of this work is to derive a higher-order topological asymptotic expansion for the shape function j with respect to the presence of the geometry perturbation $\omega_{z,\varepsilon}$ in the domain Ω.

3. Sensitivity analysis for the Laplace operator

In this section, we give a sensitivity analysis for the Laplace solution with respect to the presence of a geometry perturbation $\omega_{z,\varepsilon}$ in the domain Ω. More precisely, we derive an asymptotic expansion for the solution u_ε with respect to the perturbation size ε. We start our analysis by the following estimate.

Lemma 3.1. *If the geometry perturbation $\omega_{z,\varepsilon} \subset \Omega$ is not close to the boundary $\partial\Omega$, then the variation $u_\varepsilon - u_0$ admits the following estimate:*

$$u_\varepsilon(x) - u_0(x) = W_0((x-z)/\varepsilon) + O(\varepsilon) \text{ in } \Omega_{z,\varepsilon},$$

where the function $x \longmapsto W_0((x-z)/\varepsilon)$ is the unique solution to the Laplace exterior problem

$$\begin{aligned} -\Delta W_0 &= 0 && \text{in } \mathbb{R}^3 \setminus \overline{\omega}, \\ W_0 &\longrightarrow 0 && \text{at } \infty \\ W_0 &= -u_0(z) && \text{on } \partial\omega. \end{aligned} \tag{2}$$

Proof. Since $\omega_{z,\varepsilon}$ is not close to $\partial\Omega$, one can derive $W_0((x-z)/\varepsilon) = O(\varepsilon)$ near the boundary $\partial\Omega$, see ([1], Proposition 3.1) for a similar proof.

Next, we will extend this result to the higher-order case. The main result of this section is illustrated by the following theorem.

Theorem 3.2. *Let $\omega_{z,\varepsilon} = z + \varepsilon\omega$ be a geometry perturbation inside the domain Ω. If $\omega_{z,\varepsilon} \subset \Omega$ is not close to the boundary $\partial\Omega$, then the perturbed solution u_ε admits the following asymptotic expansion*

$$u_\varepsilon(x) = \sum_{k=0}^{N} \varepsilon^k [U_k(x) + W_k((x-z)/\varepsilon))] + O(\varepsilon^{N+1}) \text{ in } \Omega_{z,\varepsilon}, \text{ where}$$

- *$(U_k)_{1\le k\le N}$ are smooth functions, obtained as the solutions to a sequence of Laplace problems in Ω.*
- *$(W_k)_{1\le k\le N}$ are smooth functions, obtained as the solutions to a sequence of exterior problems in $\mathbb{R}^3 \setminus \overline{\omega}$.*

Proof. The sequences of functions $(U_k)_{0\le k\le N}$ and $(W_k)_{0\le k\le N}$ are constructed using an iterative process with $U_0 = u_0$ and W_0 is the solution to (2).

Using a single layer potential [7], W_k, $0 \le k \le N$ can be written as

$$W_k(y) = \int_{\partial\omega} G(y-t)\, q_k(t)\, \mathrm{d}s(t), \ \forall y \in \mathbb{R}^3 \setminus \overline{\omega},$$

where G is the fundamental solution to the Laplace equation in $\mathbb{R}^3$ and q_k is the solution to a boundary integral equation defined on $\partial\omega$.

In order to present our construction procedure, we start our analysis by studying the variation of the function $x \longmapsto W_k((x-z)/\varepsilon)$ with respect to ε. For each $x \in \mathbb{R}^3 \setminus \overline{\omega_{z,\varepsilon}}$, we have:

$$W_k((x-z)/\varepsilon) = \int_{\partial\omega} G((x-z)/\varepsilon - t)\, q_k(t)\, \mathrm{d}s(t) = \varepsilon \int_{\partial\omega} G((x-z) - \varepsilon\, t)\, q_k(t)\, \mathrm{d}s(t).$$

Using the fact that the perturbation $\omega_{z,\varepsilon}$ is not close to the boundary $\partial\Omega$, one can remark that for all $t \in \partial\omega$ and for all x in a neighborhood of $\Gamma_d \cup \Gamma_n$, the function $\varphi_{x-z,t} : \varepsilon \longmapsto \varphi_{x-z,t}(\varepsilon) = \varepsilon\, G((x-z) - \varepsilon\, t)$ is smooth with respect to ε and satisfies the following behavior

$$\varphi_{x-z,t}(\varepsilon) = \sum_{p=1}^{N} \frac{\varepsilon^p}{p!} \varphi^{(p)}_{x-z,t}(0) + O(\varepsilon^{N+1}),$$

where $\varphi^{(p)}_{x-z,t}(0)$ is the pth derivative of $\varphi_{x-z,t}$ at $\varepsilon = 0$. It depends on the pth derivative of the function G at the point $x - z$. Consequently, the function $x \longmapsto W_k((x-z)/\varepsilon)$ admits the following expansion

$$W_k((x-z)/\varepsilon) = \sum_{p=1}^{N} \varepsilon^p\, W_k^{(p)}(x-z) + O(\varepsilon^{N+1}), \tag{3}$$

where $W_k^{(p)}$ is the smooth function defined in $\mathbb{R}^3 \setminus \overline{\omega}$ by

$$W_k^{(p)}(x-z) = \frac{1}{p!} \int_{\partial\omega} \varphi^{(p)}_{x-z,t}(0) q_k(t)\, \mathrm{d}s(t), \ \forall x \in \mathbb{R}^3 \setminus \overline{\omega}. \tag{4}$$

We are now ready to present the main steps of our construction procedure. Let us assume that we have already calculated the first $k-1$ terms. The kth order term is described by the function $x \longmapsto U_k(x) + W_k((x-z)/\varepsilon)$, $x \in \Omega_{z,\varepsilon}$, which is constructed as follows:

– U_k depends on W_j, $0 \le j \le k-1$ and solves the following interior problem

$$\begin{cases} -\Delta U_k = 0 & \text{in } \Omega, \\ \nabla U_k \cdot n = -\sum_{p=1}^{k} \nabla W_{k-p}^{(p)}(x-z) \cdot n & \text{on } \Gamma_n, \\ U_k = -\sum_{p=1}^{k} W_{k-p}^{(p)}(x-z) & \text{on } \Gamma_d, \end{cases} \tag{5}$$

where $W_j^{(p)}$ is defined by (4).

– W_k depends on U_j, $0 \le j \le k$ and solves the following exterior problem

$$\begin{cases} -\Delta W_k = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\omega}, \\ W_k \longrightarrow 0 & \text{at } \infty \\ W_k = -U_k(z) - \sum_{p=1}^{k} \frac{1}{p!} D^p U_{k-p}(z)(y^p) & \text{on } \partial\omega, \end{cases} \tag{6}$$

where $D^p U_{k-p}(z)$ is the pth derivative of the harmonic function U_{k-p} and $y^p = (y, ..., y) \in (\mathbb{R}^3)^p$.

Finally, we introduce the harmonic function $R_{N,\varepsilon}$ defined in $\Omega_{z,\varepsilon}$ by $R_{N,\varepsilon}(x) = \sum_{k=0}^{N} \varepsilon^k [U_k(x) + W_k((x-z)/\varepsilon))] - u_\varepsilon$, and we prove that $R_{N,\varepsilon}$ satisfies the following boundaries conditions:

– On $\partial\omega_{z,\varepsilon}$: Using the systems (5)–(6), the multi-linearity of $D^p U_{k-p}(z)$, Taylor's Theorem and the fact that $\|x-z\| = O(\varepsilon)$ on $\partial\omega_{z,\varepsilon}$, one can obtain

$$R_{N,\varepsilon}(x) = \sum_{k=0}^{N} \varepsilon^k \Big[U_k(x) - \sum_{p=0}^{N-k} \frac{1}{p!} D^p U_k(z)((x-z)^p) \Big] = O(\varepsilon^{N+1}).$$

– On Γ_d: From (5), (6) and the asymptotic expansion (3), one can derive

$$R_{N,\varepsilon}(x) = \varepsilon^N W_N((x-z)/\varepsilon) + \sum_{k=0}^{N-1} \varepsilon^k \big[W_k((x-z)/\varepsilon) - \sum_{p=1}^{N-k} \varepsilon^p W_k^{(p)}(x-z) \big] = O(\varepsilon^{N+1}).$$

– On Γ_n: By the analysis used in the previous, one can obtain $\nabla R_{N,\varepsilon} \cdot n = O(\varepsilon^{N+1})$. □

4. Higher-order topological asymptotic expansion

This section is focused on the higher-order topological derivatives. It consists in studying the variation of a shape function j with respect to the topology perturbation of the domain. The topology perturbation is described by the hole $\omega_{z,\varepsilon}$ created at an arbitrary point $z \in \Omega$ and having the form $\omega_{z,\varepsilon} = z + \varepsilon\,\omega$. We derive a higher-order topological asymptotic expansion for a large class of shape functions. More precisely, the obtained results are valid for all shape function j having the form

$$j(\Omega_{z,\varepsilon}) = J_\varepsilon(u_\varepsilon),$$

with J_ε being a scalar function defined on $H^1(\Omega_{z,\varepsilon})$ and satisfying the following assumption.

Assumption 4.1. i) The function J_0 is differentiable with respect to u.
ii) There exist real numbers $\delta^1 J(z), ..., \delta^N J(z)$, such that $\forall \varepsilon > 0$

$$J(u_\varepsilon) - J_0(u_0) = D J_0(u_0)(u_\varepsilon - u_0) + \sum_{k=1}^{N} \varepsilon^k \delta^k J(z) + o(\varepsilon^N).$$

In the last equality, the solution u_ε is extended by zero inside the domain $\omega_{z,\varepsilon}$. Its extension will be denoted by u_ε throughout the rest of the paper.

Under the considered assumption, the variation of the shape function j reads

$$j(\Omega_{z,\varepsilon}) - j(\Omega) = \int_{\Omega_{z,\varepsilon}} \nabla(u_0 - u_\varepsilon) \cdot \nabla v_0 \, dx + \sum_{k=1}^{N} \varepsilon^k \delta^k J(z) + o(\varepsilon^N),$$

where v_0 is the solution to the associated adjoint problem.

Using Green formula and Theorem 3.2, the integral term can be decomposed as

$$\int_{\Omega_{z,\varepsilon}} \nabla(u_0 - u_\varepsilon) \cdot \nabla v_0 \, dx = \int_{\omega_{z,\varepsilon}} \nabla u_0 \cdot \nabla v_0 \, dx - \sum_{k=0}^{N} \varepsilon^k \int_{\partial\omega_{z,\varepsilon}} \nabla_x W_k((x-z)/\varepsilon)) \cdot n \, v_0 \, ds$$
$$- \sum_{k=1}^{N} \varepsilon^k \int_{\partial\omega_{z,\varepsilon}} \nabla U_k(x) \cdot n(x) \, v_0(x) \, ds + O(\varepsilon^{N+1}). \tag{7}$$

In the next section, we will derive an estimate for each term on the right-hand-side of the equality (7).

4.1. Preliminary estimates

Lemma 4.2. *The first term in (7) admits the following asymptotic expansion*

$$\int_{\omega_{z,\varepsilon}} \nabla u_0 \cdot \nabla v_0 \, dx = \sum_{k=3}^{N} \varepsilon^k \mathcal{T}^{1,k-3}_{u_0,v_0}(z) + O(\varepsilon^{N+1}),$$

where the functions $z \longmapsto \mathcal{T}^{1,k}_{u_0,v_0}(z)$, $0 \le k \le N$ are defined in Ω by

$$\mathcal{T}^{1,k}_{u_0,v_0}(z) = \sum_{p=0}^{k} \frac{1}{p!(k-p)!} \int_{\omega} \nabla^{(p+1)} u_0(z)(y^p) \cdot \nabla^{(k-p+1)} v_0(z)(y^{k-p}) \, dy, \tag{8}$$

with $y^k = (y, ..., y) \in (\mathbb{R}^3)^k$ and $\nabla^{(k)} w(z)$ denoting the kth derivative of the function w at the point z.

Lemma 4.3. *The second term in (7) admits the following asymptotic expansion*

$$\sum_{k=0}^{N} \varepsilon^k \int_{\partial\omega_{z,\varepsilon}} \nabla_x W_k((x-z)/\varepsilon)) \cdot n \, v_0 \, ds = -\sum_{k=1}^{N} \varepsilon^k \mathcal{T}^{2,k-1}_{W,v_0}(z) + O(\varepsilon^{N+1}),$$

where the functions $z \longmapsto \mathcal{T}^{2,k}_{W,v_0}(z)$, $0 \le k \le N$ are defined in Ω by

$$\mathcal{T}^{2,k}_{W,v_0}(z) = -\sum_{p=0}^{k} \frac{1}{p!} \int_{\partial\omega} \nabla_y W_{k-p}(y) \cdot n(y) [\nabla^{(p)} v_0(z)(y^p)] \, ds(y). \tag{9}$$

Lemma 4.4. *The third term in (7) admits the following asymptotic expansion*

$$\sum_{k=1}^{N} \varepsilon^k \int_{\partial\omega_{z,\varepsilon}} \nabla U_k(x) \cdot n(x) \, v_0(x) \, ds = -\sum_{k=3}^{N} \varepsilon^k \mathcal{T}^{3,k-3}_{U,v_0}(z) + O(\varepsilon^{N+1}),$$

where the functions $z \longmapsto \mathcal{T}^{3,k}_{U,v_0}(z)$, $0 \le k \le N$ are defined in Ω by

$$\mathcal{T}^{3,k}_{U,v_0}(z) = -\sum_{p=0}^{k} \sum_{q=0}^{p} \frac{1}{q!(p-q)!} \int_{\partial\omega} [\nabla^{(q+1)} U_{k-p+1}(z)(y^q)] \cdot n(y) [\nabla^{(p-q)} v_0(z)(y^{p-q})] \, ds(y).$$

4.2. Asymptotic expansion

We are now ready to present the main results of this section. Based on the previous estimates, we derive a higher-order topological asymptotic expansion for all shape function satisfying Assumption 4.1.

Theorem 4.5. *Let $\omega_{z,\varepsilon} = z + \varepsilon\omega$ be a geometry perturbation in Ω. If J_ε satisfies Assumption 4.1, then the associated shape function j admits the following asymptotic expansion*

$$j(\Omega_{z,\varepsilon}) - j(\Omega) = \sum_{k=1}^{N} \varepsilon^k \delta^k j(z) + o(\varepsilon^N),$$

where

$$\delta^k j(z) = \begin{cases} \mathcal{T}^{2,k-1}_{W,v_0}(z) + \delta^k J(z) & \text{if } k = 1, 2 \\ \mathcal{T}^{1,k-3}_{u_0,v_0}(z) + \mathcal{T}^{2,k-1}_{W,v_0}(z) + \mathcal{T}^{3,k-3}_{U,v_0}(z) + \delta^k J(z) & \text{if } 3 \le k \le N. \end{cases}$$

We now discuss Assumption 4.1. We present two examples of shape functions verifying the considered assumption and we calculate their variations $\delta^1 J$, $\delta^2 J$, ..., and $\delta^N J$.

Proposition 4.6. *Let $g \in L^2(\Omega)$ be a given function.*

The function J_ε defined by $J_\varepsilon(u) = \int_{\Omega_{z,\varepsilon}} g\, u \,\mathrm{d}x$, $\forall u \in H^1(\Omega_{z,\varepsilon})$ satisfies the Assumption 4.1 with

$$D J_0(w) = \int_\Omega g w \,\mathrm{d}x, \quad \forall w \in H^1(\Omega), \text{ and } \delta^k J(z) = 0 \text{ in } \Omega \quad k = 1, \ldots, N.$$

Proposition 4.7. *Let U_d be a given desired state, smooth near z.*

The function $J_\varepsilon(u) = \int_{\Omega_{z,\varepsilon}} |\nabla u - \nabla U_d|^2 \,\mathrm{d}x$, $\forall u \in H^1(\Omega_{z,\varepsilon})$ satisfies the Assumption 4.1 with:

$$D J_0(w) = 2 \int_\Omega \nabla(u_0 - U_d) \cdot \nabla w \,\mathrm{d}x, \quad \forall w \in H^1(\Omega),$$

and

$$\delta^k J(z) = \begin{cases} \mathcal{T}^{2,k-1}_{W,u_0}(z) & \text{if } k = 1, 2 \\ \mathcal{T}^{2,k-1}_{W,u_0}(z) + \mathcal{T}^{1,k-3}_{u_0,u_0}(z) + \mathcal{T}^{1,k-3}_{U_d,U_d}(z) + \mathcal{T}^{3,k-3}_{U,u_0}(z) & \text{if } 3 \le k \le N. \end{cases}$$

5. Conclusion

The present work can be considered as a generalization of the topological gradient notion. The obtained results are valid for a large class of shape functions. The mathematical analysis is general and can be easily adapted to other partial differential equations.

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