C. R. Acad. Sci. Paris, Ser. I 354 (2016) 1032–1036

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Dynamical systems/Mathematical physics

Ergodicity of the Ehrenfest wind–tree model

Ergodicité du modèle vent–arbre des Ehrenfest

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ARTICLE INFO

Article history:
Received 3 March 2016
Accepted after revision 31 August 2016
Available online 14 September 2016

Presented by Claire Voisin

ABSTRACT

We consider aperiodic wind–tree models, and show that, for a generic (in the sense of Baire) configuration, the wind–tree dynamics is ergodic in almost every direction.

RÉSUMÉ

Nous considérons une modèle apériodique de vent dans des arbres et nous montrons que, pour une configuration générique (dans le sens de Baire), la dynamique de vent–arbre est ergodique dans presque toutes les directions.

1. Introduction

In 1912, Paul et Tatyana Ehrenfest wrote a notable encyclopedia article on the foundations of statistical mechanics [7]. The first chapter of this article "The older formulation of statistico-mechanical investigations (Kineto-statistics of the molecule)" discusses the work of Boltzmann and Maxwell on gas dynamics. In the Appendix to Section 5 of this chapter, the Ehrenfests say that "it seems advisable to explain" the work of Maxwell–Boltzmann "on a much simplified model". This model, now known as the Ehrenfest wind–tree model, is the subject of this article. In the Ehrenfest wind–tree model, a point particle (the "wind") moves freely on the plane and collides with the usual law of geometric optics with an infinite number of irregularly placed identical square scatterers (the "trees").

The second chapter of the Ehrenfests' article "The modern formulation of statistico-mechanical investigation (Kineto-statistics of the gas model)" can be viewed as the birth place of the term "ergodic".[1] Our main result is the study of the wind–tree model in the framework of (infinite) ergodic theory. There is a natural first integral of the model, if we fix the initial direction, then orbits take only four directions. We show that for almost every value of this first integral, the Ehrenfest wind–tree is ergodic for generic (in the sense of Baire) configurations. This continues our previous work where we had

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[1] A nice discussion of the history of the birth of ergodic theory is given in the first chapter of the book [10].

http://dx.doi.org/10.1016/j.crma.2016.08.008
1631-073X/

shown that the generic in the sense of Baire wind–tree model is recurrent [23], minimal in almost every direction, and has a dense set of periodic points [19].

The wind–tree model has been intensively studied by physicists, see for example [3,6,9,12,24,25], and the references therein. From the mathematically rigorous point of view, there have been many recent results about the dynamical properties of a periodic version of wind–tree models: scatterers are identical square obstacles, one obstacle centered at each lattice point. The periodic wind–tree model has been shown to be recurrent [11,15,1], to have abnormal diffusion [5,4], and to have an absence of ergodicity in almost every direction [8]; furthermore the periodic wind–tree model can not have a minimal direction.[2] Periodic wind–tree models naturally yield infinite periodic translation surfaces; ergodicity in almost every direction for such surfaces has been obtained only in a few situations [13,14,21].

We work on the following setup: the plane is tiled by one-by-one cells with corners on the lattice $\mathbb{Z}^2$; inside each cell we place a square tree of a fixed size with the center chosen arbitrarily. Our main result is that for the generic in the sense of Baire wind–tree configuration, for almost all directions the wind–tree model is ergodic; this is in stark contrast to the situation for the periodic wind–tree model. This result can be viewed as the confirmation of the ergodic hypothesis in the framework of the Ehrenfest wind–tree model.

Our proofs hold in a more general setting than the one described above: for example, we can vary the size of the square, or use certain other polygonal trees; the class of possible extensions are the same as those discussed in [19] in the framework of generic minimality in almost every direction.

The method of proof is by approximation by finite wind–tree models where the dynamics is well understood. There is a long history of proving results about billiard dynamics by approximation, which began with the article of Katok and Zemlyakov [16]. This method was used in several of the results on wind–tree models mentioned above [15,1,23], see [22] for a survey of some other usages in billiards. The idea of approximating infinite measure systems by compact systems was first studied in [18].

2. Definitions and main result

Consider the plane $\mathbb{R}^2$ tiled by one-by-one closed square *cells* with corners on the lattice $\mathbb{Z}^2$. Fix $r \in [1/4, 1/2)$. We consider the set of $2r$ by $2r$ squares, with vertical and horizontal sides, centered at (a, b) contained in the unit cell $[0, 1]^2$; this set is naturally parametrized by

$$\mathcal{A} := \{t = (a, b) : r \le a \le 1 - r,\ r \le b \le 1 - r\}$$

with the usual topology inherited from $\mathbb{R}^2$. Our parameter space is $\mathcal{A}^{\mathbb{Z}^2}$ with the product topology. It is a Baire space. Each parameter $g = (a_{i,j}, b_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \mathcal{A}^{\mathbb{Z}^2}$ corresponds to a wind–tree table in the plane in the following manner: the tree inside the cell corresponding to the lattice point $(i, j) \in \mathbb{Z}^2$ is a $2r$-by-$2r$ square with its center at position $(a_{i,j}, b_{i,j}) + (i, j)$. The *wind–tree table* B^g is the plane $\mathbb{R}^2$ with the interiors of the union of these trees removed. Note that trees can intersect only at the boundary of cells.

The *billiard flow* ϕ_t is the unit speed free motion on the interior of B^g with elastic collision from the boundary of B^g (the boundary of the union of the trees). The phase space Ω^g of the billiard flow is thus the Cartesian product of $B^g \times \mathbb{S}^1$ with inwards and outwards pointing unit vectors identified according to the elastic collision rule at the boundary of B^g. The billiard flow ϕ_t on the phase space preserves the volume measure $\mu \times \lambda$, where μ is the area measure on B^g and λ the length measure on $\mathbb{S}^1$. Note that μ is an infinite measure. See [20] for more details on billiards.

For each θ, let $[\theta]$ be the set of all possible directions under the billiard flow starting in direction θ, i.e. $[\theta] = \{\pm\theta, \pm(\pi - \theta)\}$. We will refer to the billiard flow restrict to the set $\{(x, \psi) \in \Omega^g : \psi \in [\theta]\}$ as the *billiard flow* ϕ_t^θ *in the direction* θ; it preserves the measure $\mu \times d$ where d is the discrete measure on $[\theta]$.

A flow ψ_t preserving a measure m is called *ergodic* if for each Borel measurable A, $m(\psi_t(A)\triangle A) = 0\ \forall t \in \mathbb{R}$ implies that $m(A) = 0$ or $\mu(A^c) = 0$. A map T preserving a measure n is called *ergodic* if for each Borel measurable set A, $n(T^{-1}A\triangle A) = 0$ implies that $n(A) = 0$ or $n(A^c) = 0$.

Theorem 2.1. *There is a dense G_δ subset $\mathcal{G}$ of parameters $\mathcal{A}^{\mathbb{Z}^2}$ such that for each $g \in \mathcal{G}$ there is a dense G_δ subset of directions $\mathcal{H} \subset \mathbb{S}^1$ of full measure such that the billiard flow on Ω_g in the direction θ is ergodic for every $\theta \in \mathcal{H}$.*

Our methods do not produce explicit results. We do not have explicit examples of configurations that are ergodic in almost every direction, not even an explicit configuration that is ergodic in a single fixed direction. We also do not know if a given explicit direction, such as the direction $\pi/4$, which interested the Ehrenfests, is ergodic for our configurations.

Note that we will use different cross sections than those used in [19]. Let $\mathcal{D}_n := \{(x, y) \in \mathbb{R}^2 : \max(|x|, |y|) = n\}$ and $\mathcal{D} := \bigcup_{n\ge 1} \mathcal{D}_n$. We will consider various first return maps of the billiard flow on the wind–tree table associated with a parameter g,

[2] K. Frączek explained to us that this follows from arguments close to those in the article [2].

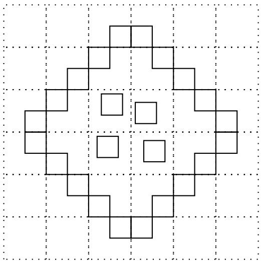

Fig. 1. A 2-ringed configuration.

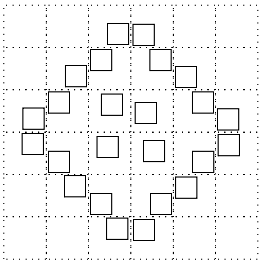

Fig. 2. A small perturbation.

$$T^g : \mathcal{D} \times \mathbb{S}^1 \to \mathcal{D} \times \mathbb{S}^1 \quad T^g_\theta : \mathcal{D} \times [\theta] \to \mathcal{D} \times [\theta]$$
$$T^g_n : \mathcal{D}_n \times \mathbb{S}^1 \to \mathcal{D}_n \times \mathbb{S}^1 \quad T^g_{n,\theta} : \mathcal{D}_n \times [\theta] \to \mathcal{D}_n \times [\theta].$$

We will apply these definitions only in the cases where the D_n do not intersect any tree.

The advantage of these cross-sections is that the maps T^g_θ and $T^g_{n,\theta}$ have natural invariant length measures, which do not depend on the parameter g: ν and respectively ν_n. The measure ν is infinite, while the measures ν_n are finite, and we think of them as normalized.

For any positive integer N, we define $R_{\overline{N}}$ to be the closed rhombus $\{(x, y) : |x| + |y| \leq N + \frac{1}{2}\}$. Suppose that N is an integer satisfying $N \geq 2$. We will call a parameter f *N-tactful* if for each cell inside the rhombus $R_{\overline{N}}$, the corresponding tree is contained in the interior of its cell. We will call an N-tactful parameter f *N-ringed*, if the boundary of $R_{\overline{N}}$ is completely covered by trees (see Fig. 1). Let E_N the set of pairs (i, j), so that the interior of the (i, j)-th cell is contained in $R_{\overline{N}}$.

For each tree $t \in \mathcal{A}$, let $U(t, \varepsilon)$ be the standard ε-neighborhood in $\mathbb{R}^2$ intersected with the interior of $\mathcal{A}$ in $\mathbb{R}^2$. For any parameter $g = (t_{i,j}) \in \mathcal{A}^{\mathbb{Z}^2}$, consider the open cylinder set $U_N(g, \varepsilon) = \prod_{(i,j)\in E_N} U(t_{i,j}, \varepsilon)$.

3. Proofs

We start by reducing the question of the ergodicity of the flow to the ergodicity of first return maps $T^g_{n,\theta}$. This is done in the following lemma.

Lemma 3.1. *Let g be a parameter that is N-tactful for all $N \geq 3$. For θ irrational, the following conditions are equivalent:*

(i) $\left(\phi^g_\theta\right)_t$ *is ergodic*
(ii) T^g_θ *is ergodic*
(iii) $T^g_{n,\theta}$ *is ergodic for all $n \geq 1$.*

Proof. (i $\Longrightarrow$ ii) If B is a non-trivial T^g_θ-invariant set, then $\bigcup_{t\in\mathbb{R}} \left(\phi^g_\theta\right)_t(B)$ is a non-trivial $\left(\phi^g_\theta\right)_t$-invariant set.

(ii $\Longrightarrow$ i) Consider a non-trivial $\left(\phi^g_\theta\right)_t$ invariant set C. Consider $C' := \bigcup_{t\in\mathbb{R}} \left(\phi^g_\theta\right)_t C$, clearly $\mu(C' \triangle C) = 0$ and C' is a $\left(\phi^g_\theta\right)_t$-invariant set. Since the flow is a flow built under a piece-wise continuous bounded function, $C' \cap \mathcal{D}$ is a non-trivial T^g_θ-invariant set.

(ii $\Longrightarrow$ iii) If A is a non-trivial $T^g_{n,\theta}$-invariant set, then $\bigcup_{k\in\mathbb{Z}} (T^g_\theta)^k(A)$ is a non-trivial T^g_θ-invariant set.

(iii $\Longrightarrow$ ii) Suppose B is a non-trivial T^g_θ-invariant set. We claim that $B_n := B \cap \mathcal{D}_n$ must be a non-trivial $T^g_{n,\theta}$-invariant set for some n. If not, then for each n, $\nu_n(B_n) = 0$ or $\nu_n(B^c_n) = 0$. Let $I := \{n \in \mathbb{N}^* : \nu_n(B^c_n) = 0\}$. Clearly I is non-empty since $0 < \nu(B) = \sum_{n\in\mathbb{N}^*} \nu_n(B_n)$. Suppose $n \in I$, then since B_n has full measure in $\mathcal{D}_n \times [\theta]$, and since a set of positive measure of $\mathcal{D}_n \times [\theta]$ is mapped to $\mathcal{D}_{n+1} \times \theta$, we must have $n + 1 \in I$. If $n \geq 2$ then the same holds for $n - 1$, thus $I = \mathbb{N}^*$. This implies $\nu(B^c) = 0$, a contradiction. Thus B_n is a non-trivial $T^g_{n,\theta}$-invariant set for some n, i.e. $T^g_{n,\theta}$ is not ergodic for some n. $\square$

Proof of Theorem 2.1. By Baire's theorem, the set of configurations that are N-tactful for all N is dense since for each N the set of all N-tactful configurations is an open dense set. Thus we can consider a countable dense, set of parameters that are N-tactful for all N. By modifying the parameters, we can assume that each one is N-ringed for a certain N still maintaining the density. Call this countable dense set $\{f_i\}$, with f_i being N_i-ringed. We also assume $N_{i+1} > N_i$. Suppose δ_i are strictly positive. Let

$$\mathcal{G} := \bigcap_{m\geq 1} \bigcup_{i\geq m} U_{N_i}(f_i, \delta_i).$$

See Fig. 2 for a configuration close to a 2-ringed configuration. Clearly $\mathcal{G}$ is a dense G_δ. We will show that the δ_i can be chosen in such a way that all the configurations in $\mathcal{G}$ are ergodic in almost every direction.

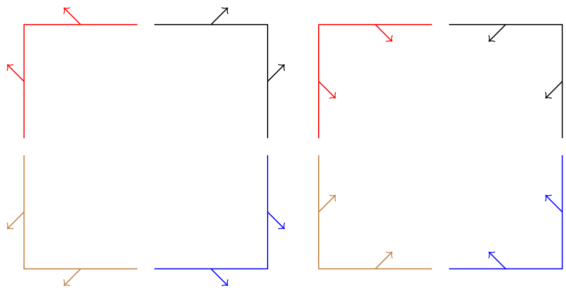

Fig. 3. X_n^θ decomposes into eight "intervals".

For each $n \geq 1$ let $X_n := \mathcal{D}_n \times \mathbb{S}^1$, $X_n^\theta := \mathcal{D}_n \times [\theta]$, $\hat{\nu}_n := \nu_n \times \lambda$ and $\hat{\nu}_n^\theta := \nu_n$. Let $\{h_j\}_{j\geq 1}$ be a countable collection of continuous functions in $L^1(X_1, \hat{\nu}_1)$, for which the restriction to $L^1(X_1^\theta, \hat{\nu}_1^\theta)$ is dense for all θ. For each θ, the measure space $(X_n^\theta, \hat{\nu}_n)$ is the union of eight isometric copies of the interval $[0, 2n(|\sin(\theta)| + |\cos(\theta)|)]$ (cf. Fig. 3).

For any n we define $h_j^n \in L^1(X_n, \hat{\nu}_n)$ by $h_j^n(s,\theta) := h_j(s/n, \theta)$ for $s \in [0, 2n(|\sin(\theta)| + |\cos(\theta)|)]$. For the sake of simplicity, we will drop the dependency on n and note $h_j = h_j^n$.

Now fix $g \in \mathcal{A}^{\mathbb{Z}^2}$ and let

$$S_{n,\ell}^g h_j(s,\theta) = \frac{1}{\ell}\sum_{k=0}^{\ell-1} h_j\big((T_{n,\theta}^g)^k(s,\theta)\big).$$

By the Birkhoff ergodic theorem, the maps $T_{n,\theta}^g$ are ergodic for all n and for almost every θ, if and only if for all n and for almost all θ, we have

$$S_{n,\ell}^g h_j(s,\theta) \to \int\limits_{X_n^\theta} h_j(t)\,\mathrm{d}\hat{\nu}_n(t)$$

as ℓ goes to infinity for all $j \geq 1$. Here the integral is over the set X_n^θ, thus we drop the θ dependence of the functions h_j from the notation.

Now fix i. Recall that f_i is an N_i-ringed parameter. Let $n_i = \lfloor \frac{N_i - 1}{2} \rfloor$. For any $1 \leq n \leq n_i$, D_n is included inside the ring, and since f_i is N_i tactful, D_n does not intersect the boundary of any tree. Because of the Kerckhoff–Masur–Smillie theorem [17], the billiard inside the ring is ergodic for almost every direction, thus $T_{n,\theta}^{f_i}$ is ergodic for almost every θ. Thus we can find a positive integer ℓ_i, an open set $H_i \subset \mathbb{S}^1$ and sets $B_{n,i}^\theta \subset X_n^\theta$ so that $\hat{\nu}_n^\theta(B_{n,i}^\theta) > 1 - \frac{1}{i}$, $\lambda(H_i) > 1 - \frac{1}{i}$ and

$$\Big| S_{n,\ell_i}^{f_i} h_j(s,\theta) - \int\limits_{X_n^\theta} h_j(t)\,\mathrm{d}\hat{\nu}_n(t) \Big| < \frac{1}{i}$$

for all $s \in B_{n,i}^\theta$, $\theta \in H_i$, $n \leq n_i$, $1 \leq j \leq i$.

Now we would like to extend these estimates to the neighborhood $U_{N_i}(f_i, \delta_i)$ for a sufficiently small strictly positive δ_i. By the triangular inequality we have:

$$\Big| S_{n,\ell_i}^{g} h_j(s,\theta) - \int\limits_{X_n^\theta} h_j(t)\,\mathrm{d}\hat{\nu}_n(t) \Big| \leq \Big| S_{n,\ell_i}^{g} h_j(s,\theta) - S_{n,\ell_i}^{f_i} h_j(s,\theta) \Big| + \Big| S_{n,\ell_i}^{f_i} h_j(s,\theta) - \int\limits_{X_n^\theta} h_j(t)\,\mathrm{d}\hat{\nu}_n(t) \Big|.$$

For any point (s,θ) of continuity of $(T_{n,\theta}^{f_i})^{\ell_i}$, the point $(T_{n,\theta}^{g})^{\ell_i}(s,\theta)$ varies continuously with g in a small neighborhood of f_i; thus we can find $\delta_i > 0$, an open set $\hat{H}_i \subset H_i$ and a set $\hat{B}_{n,i}^\theta \subset B_{n,i}^\theta$ so that if $g \in U_{N_i}(f_i, \delta_i)$, then

$$\Big| S_{n,\ell_i}^{g} h_j(s,\theta) - \int\limits_{X_n^\theta} h_j(t)\,\mathrm{d}\hat{\nu}_n(t) \Big| < \frac{2}{i}$$

for all $s \in \hat{B}_{n,i}^\theta$, $\theta \in \hat{H}_i$, $n \leq n_i$, $1 \leq j \leq i$; and $\hat{B}_{n,i}^\theta$ and $\hat{H}_i$ are both of measure larger than $1 - \frac{2}{i}$.

Suppose $g \in \mathcal{G} := \cap_{m=1}^\infty \cup_{i=m}^\infty U_{N_i}(f_i, \delta_i)$. Since $\lambda(\hat{H}_i) > 1 - 2/i$, the G_δ set $\mathcal{H} = \cap_{M=1}^\infty \cup_{i=M}^\infty \hat{H}_i$ has measure 1. Fix $\theta \in \mathcal{H}$, then there is an infinite sequence k_i such that $\theta \in \hat{H}_{k_i}$. Consider $\mathcal{B}_n^\theta = \cap_{M=1}^\infty \cup_{i=M}^\infty \hat{B}_{n,k_i}^\theta$. Since $\hat{\nu}_n^\theta(\hat{B}_{n,k_i}^\theta) > 1 - \frac{1}{k_i}$, it follows that $\hat{\nu}_n^\theta(\mathcal{B}^\theta) = 1$.

We can thus conclude that for λ-a.e. θ, for $\hat{\nu}_n$-a.e. s, for each $j \geq 1$, for each $n \geq 1$

$$\lim_{k\to\infty} S^g_{n,\ell_{k_i}}(h_j(s,\theta)) \to \int_{X_n^\theta} h_j(t)\,\mathrm{d}\hat{\nu}_n(t) \tag{1}$$

as $i \to \infty$. Since the h_j^θ are dense in $L^1(X_n^\theta, \hat{\nu}_n^\theta)$, Equation (1) together with the Birkhoff ergodic theorem imply that, for each $n \geq 1$, $T^g_{n,\theta}$ is ergodic for all $\theta \in \mathcal{H}$. The ergodicity of the billiard flow in every direction in $\mathcal{H}$ follows immediately from Lemma 3.1. $\square$

Acknowledgements

We gratefully acknowledge the support of ANR Perturbations (ANR-10-BLAN 0106) and ANR GeoDyM (ANR-11-BS01-0004) as well as the grant APEX PAD attributed by the region PACA. This work has been carried out in the framework of the Labex Archimède (ANR-11-LABX-0033) and of the A*MIDEX project (ANR-11-IDEX-0001-02), funded by the "Investissements d'avenir" program of the French Government, managed by the French National Research Agency (ANR)".

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