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Weighted composition operators that are complex symmetric on the Fock space $\mathcal{F}^2(\mathbb{C}^n)$ ☆

Opérateurs de composition à poids qui sont symétriques complexes sur l'espace de Fock $\mathcal{F}^2(\mathbb{C}^n)$

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A B S T R A C T

We investigate which weighted composition operators can be complex symmetric on the Fock space of entire functions of several variables. A general formula of the so-called weighted composition conjugations is given, and a criterion for weighted composition operators to be complex symmetric is obtained.

R É S U M É

Nous étudions les opérateurs de composition à poids sur l'espace de Fock des fonctions entières à plusieurs variables, dans le cas où ces opérateurs sont complexes symétriques. Une formule générale de conjugaison pour ces opérateurs est donnée, et un critère pour que ces opérateurs soient complexes symétriques est obtenu.

1. Introduction

The complex symmetric operators have become important in analysis and its applications. The general study of such operators was initiated in [2,3]. It is now well known that the class of complex symmetric operators is quite large. It includes the Volterra integration operators, normal operators, Hankel operators, compressed Toeplitz operators, etc.

We recall, in a general setting, basic definitions that are needed in this note. Let $\mathcal{H}$ be a complex separable Hilbert space with inner product $\langle ., . \rangle$.

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Definition 1.1. – A mapping $T\colon \mathcal{H} \to \mathcal{H}$ is said to be *anti-linear* (also *conjugate-linear*), if

$$T(\lambda u + \mu v) = \overline{\lambda}\, T(u) + \overline{\mu}\, T(v),\ \forall u, v \in \mathcal{H},\ \forall \lambda, \mu \in \mathbb{C}.$$

– An anti-linear mapping $\mathcal{C}\colon \mathcal{H} \to \mathcal{H}$ is called a *conjugation*, if it is

(1) involutive: $\mathcal{C}^2 = I$, the identity operator;
(2) isometric: $\|\mathcal{C}u\| = \|u\|,\ \forall u \in \mathcal{H}$.

Definition 1.2. A bounded linear operator T on $\mathcal{H}$ is called *complex symmetric* if there exists a conjugation $\mathcal{C}$ on $\mathcal{H}$, such that $T = \mathcal{C}T^*\mathcal{C}$. In this case, T is often called a *$\mathcal{C}$-symmetric operator* (or *CSO*, for short).

For many important function spaces, there is a vast literature of papers studying different properties (such as boundedness, compactness, compact difference, essential norm, etc.) of (weighted) composition operators, whereas for complex symmetric linear (weighted) composition operators, there seems to be very few articles.

In particular, for the latter topic, a structure of such operators acting on the Hardy space $H^2(\mathbb{D})$ for the unit disk was discovered independently in [1] and [6], for a specific conjugation

$$\mathcal{J} f(z) = \overline{f(\overline{z})}.$$

Later, some results were extended to the Hardy space $H^2(\mathbb{B})$ for the unit ball in [8], for the same conjugation $\mathcal{J}$.

In our recent paper [5], we investigated the *general* complex symmetry of linear weighted composition operators on the Fock space $\mathcal{F}^2(\mathbb{C})$ of entire functions. Motivated by defining the conjugation $\mathcal{J}$, we introduced anti-linear weighted composition operators (acting on $\mathcal{F}^2(\mathbb{C})$) of the form

$$\left(\mathcal{A}_{\xi,\eta} f\right)(z) = \xi(z)\overline{f\left(\overline{\eta(z)}\right)},$$

where ξ and η are entire functions on $\mathbb{C}$. This contains as a very particular case, with $\xi(z) \equiv 1$ and $\eta(z) = z$, the conjugation $\mathcal{J}$ above.

We have succeeded to identify all *isometric* as well as all *involutive* anti-linear weighted composition operators on $\mathcal{F}^2(\mathbb{C})$, which together give all the so-called *weighted composition conjugations*. These results allowed us to obtain a complete characterization of the linear weighted composition operators which are complex symmetric with respect to weighted composition conjugations.

The aim of this note is to consider the multi-dimensional case. Namely, we investigate which linear weighted composition operators can be complex symmetric on the Fock space $\mathcal{F}^2(\mathbb{C}^n)$ $(n \geq 2)$. In comparison with [5], the approach is *different*: noticing that an anti-linear operator $\mathcal{C}$ is a conjugation if and only if it is both unitary and selfadjoint, we identify all *selfadjoint* and *unitary* anti-linear weighted composition operators to obtain the structure of weighted composition conjugations on $\mathcal{F}^2(\mathbb{C}^n)$. These results allow us to determine which of linear weighted composition operators are complex symmetric on $\mathcal{F}^2(\mathbb{C}^n)$.

2. Preliminaries

For $n \geq 1$, the Fock space $\mathcal{F}^2(\mathbb{C}^n)$ consists of all entire functions in $\mathbb{C}^n$ for which

$$\|f\| = \left(\frac{1}{\pi^n}\int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^2}\, dV(z)\right)^{1/2} < \infty, \tag{2.1}$$

where dV is the Lebesgue measure on $\mathbb{C}^n$. It is a reproducing kernel Hilbert space, with the inner product given by

$$\langle f, g\rangle = \frac{1}{\pi^n}\int_{\mathbb{C}^n} f(z)\overline{g(z)} e^{-|z|^2}\, dV(z) \tag{2.2}$$

and kernel $K_z(u) = e^{\langle u,z\rangle}$. The norm arising from (2.2) is the same as (2.1).

A holomorphic function ψ in some domain $G \subseteq \mathbb{C}^n$ and a holomorphic self-mapping φ on G induce a *linear weighted composition operator* $W_{\psi,\varphi}$ defined by

$$W_{\psi,\varphi} f = \psi \cdot f \circ \varphi,$$

which acts from some space of holomorphic functions to another.

For the space $\mathcal{F}^2(\mathbb{C})$, characterizations of boundedness of $W_{\psi,\varphi}$ are obtained in [7], whose ideas allow the author of [9] to generalize some results to a multi-dimensional case $\mathcal{F}^2(\mathbb{C}^n)$, under the *additional assumption* that an affine function $\varphi(z) = Az + b$ is injective (which is equivalent to invertibility of A). In particular, putting

$$M_z(\psi,\varphi) = |\psi(z)|^2 e^{|\varphi(z)|^2-|z|^2},\ z\in\mathbb{C}^n \quad \text{and} \quad M(\psi,\varphi)=\sup_{z\in\mathbb{C}^n} M_z(\psi,\varphi),$$

the following results are proved.

Proposition 2.1 *([9]). Let $\psi \not\equiv 0$ be an entire function on $\mathbb{C}^n$, and let φ be a holomorphic self-mapping on $\mathbb{C}^n$.*

(1) *If $W_{\psi,\varphi}$ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$, then $\psi \in \mathcal{F}^2(\mathbb{C}^n)$ and $M(\psi,\varphi)<\infty$, which, in turn, implies that $\varphi(z)=Az+b$, where A is a linear operator on $\mathbb{C}^n$ with $\|A\|\le 1$.*
Moreover, in the case when A is unitary, there exists a constant $s\in\mathbb{C}$ such that

$$\psi(z) = s\,e^{\langle z,c\rangle},\ z\in\mathbb{C}^n, \quad \textit{with} \quad c=-A^*b.$$

(2) *Conversely, if $M(\psi,\varphi)<\infty$, and in addition, φ is injective, then $W_{\psi,\varphi}$ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$.*
In this case, $W_{\psi,\varphi}$ is invertible if and only if $\inf_{z\in\mathbb{C}^n} M_z(\psi,\varphi)>0$. Also, $\|A\|=\|A^{-1}\|=1$.

Notice that a matrix A satisfying the condition $\|A\|=\|A^{-1}\|=1$ must be unitary. This can be generalized for linear operators on general Hilbert spaces.

Lemma 2.2. *A bounded linear operator U acting on a Hilbert space $\mathcal{H}$ is unitary if and only if U is invertible and $\|U\|=\|U^{-1}\|=1$.*

Thanks to Lemma 2.2 and Proposition 2.1 (1), we can obtain a characterization for invertibility of $W_{\psi,\varphi}$, in which the function ψ can be determined precisely.

Proposition 2.3. *The operator $W_{\psi,\varphi}$ is bounded and invertible on $\mathcal{F}^2(\mathbb{C}^n)$ if and only if*

$$\varphi(z)=Az+b, \quad \textit{and} \quad \psi(z)=c\,e^{-\langle Az,b\rangle}, z\in\mathbb{C}^n, \tag{2.3}$$

where A is a linear unitary operator on $\mathbb{C}^n$, $b\in\mathbb{C}^n$, and $c\in\mathbb{C}$.

Remark 2.4. Since an involution is a particular type of invertible operator, the functions ψ,φ that induce an involutive bounded weighted composition operator $W_{\psi,\varphi}$ on $\mathcal{F}^2(\mathbb{C}^n)$ should have the form (2.3). Thus, Proposition 2.3 is very useful in characterizing involutions of both linear and anti-linear weighted composition operators.

Throughout the paper, we always assume that ψ is not identically zero.

3. A general formula for anti-linear weighted composition operators

As noted in the introduction, a complex symmetric structure of $W_{\psi,\varphi}$ was investigated in [5] on $\mathcal{F}^2(\mathbb{C})$ for conjugations of the form of anti-linear weighted composition operators. In this section, we define and study the anti-linear weighted composition operators on $\mathcal{F}^2(\mathbb{C}^n)$.

In what follows, the symbol $\kappa\eta$ is used to indicate the composition $\kappa\circ\eta$, where η is a mapping from $\mathbb{C}^n$ into itself and κ is the standard conjugation on $\mathbb{C}^n$: $\kappa(z)=\kappa(z_1,\dots,z_n)=(\overline{z_1},\dots,\overline{z_n})$.

We would like to mention that boundedness of anti-linear operators is defined similarly as for linear operators, while the adjoint T^* of a bounded anti-linear operator T is defined slightly differently as follows

$$\langle Tx, y\rangle = \langle T^*y, x\rangle, \quad \forall x,y\in\mathcal{H}.$$

By the Riesz Lemma, T^* exists uniquely, and is also bounded anti-linear. As for linear operators, we say that T is *selfadjoint* if $T=T^*$, *unitary* if $T^{-1}=T^*$, and *invertible* if there exists a bounded anti-linear operator S such that $ST=TS=I$.

The following result is a direct generalization of [5, Lemma 3.5] to the multi-dimensional case.

Lemma 3.1. *The operator $\mathcal{K}$ defined by $\mathcal{K}f(z)=\overline{f(\kappa z)}$, is a conjugation on $\mathcal{F}^2(\mathbb{C}^n)$.*

For an entire function $\xi\not\equiv 0$ in $\mathbb{C}^n$ and a holomorphic self-mapping η on $\mathbb{C}^n$, we define an anti-linear weighted composition operator $\mathcal{A}_{\xi,\eta}$ by the rule:

$$\left(\mathcal{A}_{\xi,\eta}f\right)(z)=\xi(z)\overline{f(\kappa\eta(z))}.$$

A boundedness of $\mathcal{A}_{\xi,\eta}$ is given in the following result.

Proposition 3.2. *Let $\xi \not\equiv 0$ be an entire function on $\mathbb{C}^n$, and let η be a holomorphic self-mapping on $\mathbb{C}^n$.*

(1) *If an anti-linear weighted composition operator $\mathcal{A}_{\xi,\eta}$ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$, then $\xi \in \mathcal{F}^2(\mathbb{C}^n)$ and $M(\xi,\eta) < \infty$, which, in turn, implies that $\eta(z) = Az + b$, where A is a linear operator on $\mathbb{C}^n$ with $\|A\| \le 1$.*
(2) *Conversely, if $M(\xi,\eta) < \infty$, and, in addition, η is injective, then $\mathcal{A}_{\xi,\eta}$ is bounded on $\mathcal{F}^2(\mathbb{C}^n)$.*

Here it should be noted that in the papers [7,9], the techniques of adjoint operators in Hilbert spaces play an important role in proving the necessity for the boundedness of linear weighted composition operators. On the other hand, it is possible to prove Proposition 3.2(1) in another way (see, e.g., [4]), which allows us to generalize to the Fock spaces $\mathcal{F}^p(\mathbb{C}^n)$, $p > 0$.

The following result concerns the invertibility of $\mathcal{A}_{\xi,\eta}$ and the explicit determination of the function $\xi(z)$.

Proposition 3.3. *An anti-linear weighted composition operator $\mathcal{A}_{\xi,\eta}$ is invertible on $\mathcal{F}^2(\mathbb{C}^n)$ if and only if*

$$\eta(z) = Az + b, \quad \text{and} \quad \xi(z) = c e^{-\langle Az, b\rangle}, \quad z \in \mathbb{C}^n, \tag{3.1}$$

where $b \in \mathbb{C}^n$, $c \in \mathbb{C}$, and A is a linear operator on $\mathbb{C}^n$ that satisfies

$$A^{-1} = A^*.$$

Thanks to (3.1), we have the following result for unitariness.

Proposition 3.4. *An anti-linear weighted composition operator $\mathcal{A}_{\xi,\eta}$ on $\mathcal{F}^2(\mathbb{C}^n)$ is unitary if and only if*

$$\eta(z) = Az + b, \quad \text{and} \quad \xi(z) = c e^{-\langle Az, b\rangle}, \quad z \in \mathbb{C}^n,$$

where A, b, c satisfy

$$A^{-1} = A^*, \quad |c|^2 e^{|b|^2} = 1.$$

To characterize selfadjointness, we note the fact that the span of kernel functions is dense in $\mathcal{F}^2(\mathbb{C}^n)$, and so our problem is reduced to finding ξ and η such that

$$\mathcal{A}^*_{\xi,\eta} K_z = \mathcal{A}_{\xi,\eta} K_z, \quad \forall z \in \mathbb{C}^n.$$

It turns out that in this case, the matrix A in Proposition 3.2 is complex symmetric.

Proposition 3.5. *An anti-linear weighted composition operator $\mathcal{A}_{\xi,\eta}$ on $\mathcal{F}^2(\mathbb{C}^n)$ is selfadjoint if and only if*

$$\eta(z) = Az + b, \quad \text{and} \quad \xi(z) = c e^{\langle z, \kappa b\rangle}, \quad z \in \mathbb{C}^n,$$

where A, b, c satisfy

$$\kappa A \kappa = A^*, \quad \|A\| \le 1,$$

$$\langle Ax, b + A\kappa b\rangle = 0, \quad \text{whenever } |Ax| = |x|.$$

Combining the results above yields the following important criterion, which is the main result of this section.

Theorem 3.6. *An anti-linear weighted composition operator $\mathcal{A}_{\xi,\eta} : \mathcal{F}^2(\mathbb{C}^n) \to \mathcal{F}^2(\mathbb{C}^n)$ is a conjugation if and only if the functions η and ξ have the following form:*

$$\eta(z) = Az + b, \quad \text{and} \quad \xi(z) = c e^{-\langle Az, b\rangle}, \quad z \in \mathbb{C}^n,$$

where A is a linear operator on $\mathbb{C}^n$, $b \in \mathbb{C}^n$, and $c \in \mathbb{C}$, that satisfy the following conditions

$$A^{-1} = A^* = \kappa A \kappa, \quad A\kappa b + b = 0, \quad \text{and} \quad |c|^2 e^{|b|^2} = 1. \tag{3.2}$$

Remark 3.7. Clearly, Theorem 3.6 gives, in the case $n = 1$, the corresponding result obtained in [4].

4. Linear weighted composition operators

In what follows, the weighted composition conjugation on $\mathcal{F}^2(\mathbb{C}^n)$ is denoted by $\mathcal{C}_{A,b,c}$, that is

$$\mathcal{C}_{A,b,c} f(z) = c e^{\langle z, \kappa b\rangle} \overline{f(\kappa(Az+b))},$$

where A, b, c satisfy conditions (3.2) in Theorem 3.6.

We consider bounded linear weighted composition operators and investigate under what conditions these operators are $\mathcal{C}_{A,b,c}$-symmetric.

The following criterion is the main result in this section.

Theorem 4.1. *Let $\mathcal{C}_{A,b,c}$ be a weighted composition conjugation on $\mathcal{F}^2(\mathbb{C}^n)$, $\psi \not\equiv 0$ an entire function on $\mathbb{C}^n$, and φ a holomorphic self-mapping on $\mathbb{C}^n$. Then the weighted composition operator $W_{\psi,\varphi}$ induced by φ and ψ, is $\mathcal{C}_{A,b,c}$-symmetric on $\mathcal{F}^2(\mathbb{C}^n)$ if and only if*

$$\varphi(z) = Qz + p, \ \psi(z) = \psi(0) e^{\langle z, q\rangle}, \ \psi(0) \neq 0, \text{ and } q = \kappa Ap + \kappa b - Q^*\kappa b, \tag{4.1}$$

where A, Q are linear operators on $\mathbb{C}^n$, and $p, q \in \mathbb{C}^n$, satisfy the following conditions

$$(AQ)^* = \kappa AQ\kappa, \tag{4.2}$$

$$\|Q\| \le 1, \quad \text{and } \langle Qx, p + Qq\rangle = 0 \text{ whenever } |Qx| = |x|. \tag{4.3}$$

Theorem 4.1 allows us to find eigenvalues of symmetric weighted composition operators on the space $\mathcal{F}^2(\mathbb{C}^n)$ as follows. It is well known that if $\sigma(Q) = \{\mu_1, ..., \mu_n\}$ is a spectrum of a linear operator Q in Theorem 4.1, then $\sigma(Q^*) = \{\overline{\mu_1}, ..., \overline{\mu_n}\}$. Suppose that u_k is an eigenvector of Q^* corresponding to the eigenvalue $\overline{\mu_k}$. If $1 \notin \sigma(Q)$, then we put $\alpha = (I_n - Q^*)^{-1} q$ and $\beta = (I_n - Q)^{-1} p$. Below we use the symbol μ^τ instead of $\mu_1^{\tau_1} \cdots \mu_n^{\tau_n}$.

Proposition 4.2. *Let $W_{\psi,\varphi}$ be $\mathcal{C}_{A,b,c}$-symmetric on $\mathcal{F}^2(\mathbb{C}^n)$. Suppose that $1 \notin \sigma(Q)$. Then, $\mu^\tau \psi(\beta)$ is an eigenvalue of $W_{\psi,\varphi}$ corresponding to the eigenvector*

$$h_\tau(z) = \langle z - \beta, u_1\rangle^{\tau_1} \cdots \langle z - \beta, u_n\rangle^{\tau_n} e^{\langle z, \alpha\rangle}.$$

In conclusion, we remark that the class of $\mathcal{C}_{A,b,c}$-symmetric weighted composition operators contains the class of $\mathcal{K}$-symmetric operators as a very particular case.

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