C. R. Acad. Sci. Paris, Ser. I 354 (2016) 887–890

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Lie algebras

A substitution theorem for the Borcherds–Weyl semigroup

Théorème de substitution pour le semi-groupe de Borcherds–Weyl

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ARTICLE INFO

Article history:
Received 10 May 2016
Accepted 30 June 2016
Available online 1 August 2016

Presented by Michèle Vergne

ABSTRACT

A result concerning Bruhat sequences for a Borcherds–Kac–Moody algebra is established. It is needed for the Littelmann path model. For a Kac–Moody Lie algebra, it is a consequence of the exchange lemma. In the present framework, the proof is more complex.

RÉSUMÉ

Un résultat pour les suites de Bruhat est établi dans le cadre d'une algèbre de Borcherds–Kac–Moody. Il est nécessaire au modèle des chemins de Littelmann. Pour une algèbre de Kac–Moody, c'est une conséquence du lemme de substitution. Dans le cadre actuel, la démonstration est plus complexe.

Version française abrégée

Soit T le semi-groupe de Borcherds–Weyl associé à une algèbre de Borcherds–Kac–Moody [1]. Soient λ, μ des poids dominants et $\tau \in T$. Soit $\alpha^\vee$ une coracine positive réelle telle que $\alpha^\vee(\tau\lambda) < 0$. Il est relativement facile d'en déduire [3, Lemma 2.2.7] que $\alpha^\vee(\tau\mu) \leq 0$.

Le résultat principal (voir Théorème 3.1) de cette note est de montrer que $\alpha^\vee(\tau\mu) < 0$, lorsque τ est donné par une suite de Bruhat associée à μ, c'est-à-dire par les formules (1), (2). Ceci est utilisé dans le modèle des chemins de Littelmann [3, 7.3.8]. Les ingrédients principaux de la preuve sont le lemme de substitution [2, 5.3.2] pour T et une décomposition réduite dominante [2, 2.2.6] pour τ. On peut trouver une esquisse de preuve de ce théorème dans la version ameliorée [3, Lemma 7.3.7] non publiée de [2].

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[1] The first author was supported in part by the Binational Science Foundation, Grant No. 711628.

http://dx.doi.org/10.1016/j.crma.2016.06.004

1. Introduction

1.1. The Borcherds–Weyl semi-group

In a recent paper [2], we extended the Littelmann path model to the case of integrable highest weight modules for a Borcherds–Kac–Moody algebra $\mathfrak{g}$.

Recall [1] that $\mathfrak{g}$ is constructed from a vector space $\mathfrak{h}$ and a Cartan matrix expressed in the form $\alpha_i^\vee(\alpha_j)$, where the $\alpha_i^\vee$ are linearly independent elements of $\mathfrak{h}$ and the α_j are linearly independent elements of $\mathfrak{h}^*$. The latter form the set Π of simple roots for the pair $(\mathfrak{g}, \mathfrak{h})$. The set Π is a disjoint union of a subset of Π_{re} of "real" simple roots defined by the condition that $\alpha^\vee(\alpha) = 2$ and a subset of "imaginary" simple roots Π_{im} defined by the condition that $\alpha^\vee(\alpha) \in -\mathbb{N}$.

For all $\alpha \in \Pi$, one defines a linear automorphism of $\mathfrak{h}^*$ by $r_\alpha\lambda = \lambda - \alpha^\vee(\lambda)\alpha$.

We call the semigroup T generated by the $r_\alpha : \alpha \in \Pi$, the Borcherds–Weyl semigroup. The $r_\alpha : \alpha \in \Pi_{\text{re}}$ are reflections and are so involutive. They generate a Weyl subgroup W of T. The $r_\alpha : \alpha \in \Pi_{\text{im}}$ are of infinite order (yet invertible). In Section 2, we denote by r_i a simple reflection and more generally, in Section 3, an element of $\{r_\alpha : \alpha \in \Pi\}$.

1.2. The Littelmann path model

The extension of the Littelmann path model to the Borcherds case introduced in [2] requires a fairly extensive study of T. This was carried out in [2]. In this, there were some points not fully attended to. This led to a corrected version of [2] which appeared only in the arXiv [3].

The aim of this note is to give a detailed proof of [3, Lemma 7.3.7], which we believe is a quite non-trivial result of independent interest and which is as yet unpublished. It results from the exchange lemma when $W = T$. Otherwise, the proof is significantly more difficult using several results from [2] concerning T. We also note in 3.7 the independence of the second joining condition [2, 7.3.3 (2)] on changes of paths.

1.3. Distance

Let Δ (resp. Δ^+) denote the set of non-zero (resp. positive) roots [2, 2.1.2, 2.1.4] and P^+ the set of dominant weights [2, 2.1.3] of $\mathfrak{g}$ relative to Π. Set $\Delta_{\text{re}} := W\Pi_{\text{re}}$, $\Delta_{\text{re}}^+ := \Delta_{\text{re}} \cap \Delta^+$, $\Delta_{\text{im}} = W\Pi_{\text{im}}$. One has $\Delta_{\text{im}} \subset \Delta^+$.

For all $\beta \in W\Pi$, let $\beta^\vee$ denote the corresponding coroot [2, 2.1.9].

Fix $\lambda \in P^+$. For certain pairs $\mu, \nu \in T\lambda$ one may define [3, 5.1.1] the distance $\text{dist}(\mu, \nu)$ between them. If $\text{dist}(\mu, \nu) = 1$, then there exists $\beta \in W\Pi \cap \Delta^+$ such that $\beta^\vee(\mu) > 0$ with $\nu = r_\beta\mu$. In this case, we write $\mu \overset{\beta}{\leftarrow} \nu$. If $\text{dist}(\mu, \nu) = t \in \mathbb{N}$, then we may write

$$\mu := \mu_t \overset{\beta_t}{\leftarrow} \mu_{t-1} \overset{\beta_{t-1}}{\leftarrow} \cdots \overset{\beta_2}{\leftarrow} \mu_1 \overset{\beta_1}{\leftarrow} \mu_0 =: \nu, \tag{1}$$

and moreover there are no strictly longer such sequences. We denote such a sequence, called a Bruhat sequence, by (β).

Set

$$\tau_{(\beta)} = r_{\beta_1} r_{\beta_2} \cdots r_{\beta_t}, \tag{2}$$

written simply as τ.

One has $\nu = \tau\mu$. Set $\tau_i := r_{\beta_i} r_{\beta_{i+1}} \cdots r_{\beta_t} : i = 1, 2, \dots, t, \tau_{t+1} = \text{Id}$. Notice that (1) implies that $\beta_i^\vee(\tau_{i+1}\mu) > 0$, for all $i = 1, 2, \dots, t$.

Given $\tau \in T$, let $\ell(\tau)$ denote its reduced length.

2. The substitution theorem for *W*

In this section we assume $W = T$.

2.1. If $\mu \in P^+$, one can give an interpretation of $\text{dist}(\mu, \nu)$ in terms of τ defined by (1), (2) above. Note first that $\nu \in W\mu$ with $\mu \in P^+$, implies that $\mu - \nu \in \mathbb{N}\Pi$. Let $o(\mu - \nu)$ denote the sum of the coefficients of $\mu - \nu$ written as a sum of simple roots.

Lemma. *Assume $\mu \in P^+$ and $\nu \in W\mu$. Then there exist $t' \in \mathbb{N}$ and a sequence of simple roots $\alpha_1, \alpha_2, \dots, \alpha_{t'}$ such that*

$$\alpha_i^\vee(r_{i-1} \dots r_1\nu) < 0, \tag{3}$$

with $r_{t'} r_{t'-1} \dots r_1\nu = \mu$. Moreover $\tau' := r_1 r_2 \dots r_{t'}$ is a reduced decomposition.

Proof. Since $\nu \in W\mu$, then $\nu = \mu$ if $\nu \in P^+$. Otherwise there is a simple root α_1 such that $\alpha_1^\vee(\nu) < 0$. Then $o(\mu - r_1\nu) < o(\mu - \nu)$ and the proof of the first part proceeds by induction on $o(\mu - \nu)$. The last part follows from (3). □

2.2. Retain the notation of 2.1.

Proposition. *Assume $\mu \in P^+$ and $\nu \in W\mu$. Suppose dist$(\mu, \nu) = t$ and define $\tau \in W$ as in (2) and τ' following (3). Then $\tau = \tau'$ and $\ell(\tau) = t' = t$.*

Proof. Adopt the notation of (1) and of Lemma 2.1.

By (3) and the definition of dist it follows that $t' \leq t$.

Observe that

$$\tau\mu = r_{\beta_1} r_{\beta_2} \dots r_{\beta_t}\mu = r_1 r_2 \dots r_{t'}\mu = \tau'\mu. \tag{4}$$

Then by (1) we obtain $\beta_1^\vee(\nu) = \beta_1^\vee(r_1 r_2 \dots r_{t'}\mu) < 0$ and so $r_{t'} \dots r_1\beta_1 \in -\Delta^+$. Yet $\beta_1 \in \Delta^+$, so there exists $i|t' > i \geq 1$ maximal such that $r_i \dots r_1\beta_1 \in \Delta^+$. This forces $r_i \dots r_1\beta_1 = \alpha_{i+1}$. Thus $r_{\beta_1} = r_1 r_2 \dots r_i r_{i+1} r_i \dots r_1$. Substitution in $\nu = r_{\beta_1} \dots r_{\beta_\ell}\mu$ gives

$$r_{\beta_2} r_{\beta_3} \dots r_{\beta_t}\mu = r_{\beta_1}\nu = r_{\beta_1} r_1 r_2 \dots r_{t'}\mu = r_1 r_2 \dots r_i r_{i+2} r_{i+3} \dots r_{t'}\mu. \tag{4'}$$

In the passage from (4) to (4′), we did not use (3) (only (1)). Thus we may repeat this step to obtain $r_{\beta_{t-t'}} \dots r_{\beta_t}\mu = \mu$.

Yet the left-hand side must be dominant and so this forces $t - t' = 0$. Then $\tau' = \tau$, through induction on t using (1), (4′) and formula for r_{β_1}. On the other hand, $\ell(\tau') = t'$ by the last part of Lemma 2.1. □

2.3. Retain the notation of 2.1 and 2.2. For all $w \in W$, set $S(w) := \{\beta \in \Delta^+ | w\beta \in -\Delta^+\}$.

Corollary. *Assume $\mu \in P^+$ and $\nu \in W\mu$. Define $\tau \in W$ as in (2). Take $\lambda \in P^+$ and $\alpha \in \Delta^+$ such that $\alpha^\vee(\tau\lambda) < 0$. Then $\alpha^\vee(\tau\mu) < 0$.*

Proof. By the hypothesis and Proposition 2.2, one obtains $\alpha \in S(\tau^{-1}) = S(\tau'^{-1})$. Thus there exists $i \in \{1, 2, \dots, t\}$ such that $\alpha = r_1 r_2 \dots, r_{i-1}\alpha_i$. Then $\alpha^\vee(\tau\mu) = \alpha_i^\vee(r_i \dots r_t\mu) = \alpha_i^\vee(r_{i-1} \dots r_1\nu)$ and so the assertion follows from (3). □

3. The substitution theorem in the general case

3.1.

Theorem. *Take λ, $\mu \in P^+$ and $\nu \in T\mu$. Suppose dist$(\mu, \nu) = t$ and adopt the notation of (1), (2). If $\alpha^\vee(\tau\lambda) < 0$ for some $\alpha \in \Delta^+_{\text{re}}$, then $\alpha^\vee(\tau\mu) < 0$.*

The proof of the theorem is given in the following sections. One may already remark that conclusion $\alpha_i^\vee(\tau\mu) \leq 0$ results from [3, Lemma 2.2.7]. This is a more general (and easy) result, which does not need that the special form of τ by taking it to be given by (1), (2).

3.2. Take $\tau \in T$. It is clear that τ may be written in the form

$$\tau = w_0 r_{i_1} w_1 \cdots w_{k-1} r_{i_k} w_k : w_j \in W, \alpha_{i_j} \in \Pi_{\text{im}}, \forall j \in \{0, 1, \dots, k\}. \tag{5}$$

We say that (5) is a reduced expression for τ if $\ell(\tau) = k + \sum_{i=0}^k \ell(w_i)$.

We say that (5) is a dominant reduced expression if τ is reduced and successively the $\ell(w_k), \ell(w_{k-1}), \dots, \ell(w_0)$ take their minimal values. As noted in [2, 2.2.4, 2.2.6] this is attained by successively taking for $j = k, k-1, \dots, 1$, the simple reflections r_u for which $\ell(r_u w_i) < \ell(w_i)$ and $r_{i_j} r_u = r_u r_{i_j}$, to the left.

Observe that a dominant reduced expression defines an ordered set of simple imaginary roots, namely $(\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k})$.

By [2, Lemma 2.2.3] a dominant reduced expression has the property that for all $\mu \in P^+$ and for all $i = 1, 2, \dots, k$, the sub-expression $r_{i_i} w_i \cdots w_{k-1} r_{i_k} w_k \mu$ of $w_0 r_{i_1} w_1 \cdots w_{k-1} r_{i_k} w_k \mu$ lies in P^+.

3.3. Retain the above notation and hypotheses. Let us write $\tau^* = r_{i_1} w_1 \cdots w_{k-1} r_{i_k} w_k$ and w_0 simply as w. Then $\mu^* := \tau^*\mu \in P^+$ and $\nu = w\mu^*$.

Lemma. *Either the conclusion of the theorem holds or it may be reduced to showing that its conclusion holds when $w \in \text{Stab}_W \mu^*$.*

Proof. The proof is analogous to that of Proposition 2.2.

By Lemma 2.1 we may choose $\alpha_1, \alpha_2, \dots, \alpha_{t'} \in \Pi_{\text{re}}$ such that $w' = r_1 r_2 \dots r_{t'}$ satisfies $\nu = w'\mu^* = w\mu^*$ and that (3) holds. Then $w'' := w'^{-1} w \in \text{Stab}_W \mu^*$. Set $w'_i = r_1 r_2 \dots r_{i-1}$. Then $\gamma_i := w'_i \alpha_i \in \Delta^+$ and $w'^{-1}\gamma_i = r_{t'} \dots r_i \alpha_i \in -\Delta^+$ by the last part of Lemma 2.1. Thus $S(w'^{-1}) = \{\gamma_i : i = 1, 2, \dots, t'\}$.

On the other hand, $\alpha_1^\vee(\tau\mu) < 0$ and $\alpha_1^\vee(\mu) \geq 0$. By the exchange lemma [2, Lemma 5.3.2] for T, it follows that α_1 is equal to β_k for some k with $1 \leq k \leq t$. We can assume k minimal with this property.

Then

$$\alpha_1^\vee(\mu_k) > 0, \alpha_1^\vee(\mu_{k-1}) < 0.$$

Set $\beta_s' := r_1\beta_s$, for all $s = 1, 2, \dots, k-1$; they are positive roots by the minimality of k. Then $(\beta_s')^\vee(r_{\beta_{s+1}'}\cdots r_{\beta_k'}\mu_k) = \beta_s^\vee(\mu_s) > 0$ and $\tau\mu = r_1 r_{\beta_1'}\cdots r_{\beta_{k-1}'} r_{\beta_{k+1}}\cdots r_{\beta_\ell}\mu = r_1 r_2 \cdots r_{t'}\mu^*$. Thus we may cancel r_1 and obtain

$$r_{\beta_1'}\cdots r_{\beta_{k-1}'} r_{\beta_{k+1}}\cdots r_{\beta_\ell}\mu = r_2\cdots r_{t'}\mu^*.$$

Repeating this constructive gives $\nu = \mu^* = w''\mu^*$, with $\mathrm{dist}(\mu,\nu) = t - t'$.

Now take $\alpha \in \Delta_{\mathrm{re}}^+$ as in the hypothesis of the theorem.

If $\alpha \in S(w'^{-1})$, then there exists $i \in \{1, 2, \dots, t'\}$ such that $\alpha = \gamma_i$. Then $\alpha^\vee(\nu) = (w_i'\alpha_i)^\vee(\nu) < 0$ by (3), and we are done.

If $\alpha \notin S(w'^{-1})$, then $\alpha' := w'^{-1}\alpha \in \Delta_{\mathrm{re}}^+$ and $\alpha'^\vee(w''\tau^*\lambda) = \alpha^\vee(\tau\lambda) < 0$, whilst $w'' \in \mathrm{Stab}_W\,\mu^*$. □

3.4. Retain the notation and hypotheses of the first paragraph of 3.3.

Lemma. *Suppose $\nu = w\mu^* = \mu^*$. Then $\beta_1 \in \Pi_{\mathrm{im}}$. Moreover the hypothesis of the theorem holds with τ replaced by $\tau_2 := r_{\beta_2}\cdots r_{\beta_t}$.*

Proof. Since $\nu \in P^+$, the first part follows from [2, Lemma 6.3.4]. On the other hand, $\alpha^\vee(\tau\lambda) < 0$ and so $\tau\lambda \notin P^+$. Yet $r_{\beta_1}P^+ \subset P^+$ by [2, 2.2.3], since $\beta_1 \in \Pi_{\mathrm{im}}$. Thus $\tau_2\lambda \notin P^+$, so the reduced dominant expression of τ_2 can be written as $\tau_2 = w_0\tau^{**}$ with $\tau^{**}\eta \in P^+$ for all $\eta \in P^+$ and $w_0 \in W \setminus \{\mathrm{Id}\}$. Then by [2, Lemma 2.2.3] we can write $w_0 = w_0' w_0''$ where lengths add, $w_0' \in W \setminus \{\mathrm{Id}\}$, commutes with r_{β_1} and $r_{\beta_1}w_0''\tau^{**}\eta \in P^+$, for all $\eta \in P^+$. Finally $\tau = w_0' r_{\beta_1} w_0''\tau^{**}$.

It follows that $w_0' \in \mathrm{Stab}_W\,\mu^*$ and since $\alpha^\vee(\tau\lambda) < 0$, that $\alpha \in S(w_0'^{-1})$. This last inclusion implies that α can be written as a sum of the roots in Π_{re} such the corresponding reflections occur in w_0'. However, each of these reflections commutes with r_{β_1} and so the corresponding coroots vanish on β_1. In particular $\alpha^\vee(\beta_1) = 0$. We conclude that $\alpha^\vee(\tau_2\lambda) = \alpha^\vee(\tau\lambda) < 0$. This completes the proof of the lemma. □

3.5. The proof of the theorem is completed by induction on t. It is trivial for $t = 0$ as $\lambda \in P^+$ and so there can be no $\alpha \in \Delta_{\mathrm{re}}^+$ satisfying its hypothesis. Then by Lemma 3.3 either the conclusion of the theorem holds for t or by the same lemma and the next, it is reduced to the case when t is decreased by 1.

3.6. The conclusion of Theorem 3.1 is used in [3, Lemma 7.3.8].

3.7. The proof of the theorem shows (as in the case $W = T$) that $\mathrm{dist}(\mu,\nu) = \ell(\tau)$. More interestingly, we can replace (β) by a Bruhat sequence (β') of the same length with only simple roots and indeed those given by a dominant reduced expression for τ. In this, let $i_1 < i_2 < \ldots < i_k$ be such that $\{\beta_{i_s}\}_{s=1}^k$ are the positive imaginary roots occurring in $\{\beta_i\}_{i=1}^t$; in particular, $\beta_{i_s} \in W\alpha_{i_s}$, for some unique $\alpha_{i_s} \in \Pi_{\mathrm{im}}$. Then $r_{i_s} = r_{\alpha_{i_s}}$, for all $s = 1, 2, \dots, k$, in the dominant reduced expression for τ.

Observe that in the passage from (β) to (β') the values of $\beta_{i_s}^\vee(\tau_{i_s+1}\mu) : s = 1, 2, \dots, k$ do not change. This shows that the joining condition [2, 7.3.3 (2)] is independent of the choice of the Bruhat sequence linking the pair (μ,ν), given that the order of the imaginary roots is not changed. We do not need more than this because the operations on Bruhat sequences performed in [2,3] involve only changing (β) by W translates of its elements.

Yet it is an open question as to whether two dominant reduced expressions of a given element of T can admit differing sequences of imaginary roots (apart from interchanging adjacent simple imaginary roots α, α' for which $\alpha^\vee(\alpha') = 0$). This is a delicate question involving the nature of the relations in T.

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