C. R. Acad. Sci. Paris, Ser. I 354 (2016) 685–689

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial differential equations/Calculus of variations

Minimizing movements along a sequence of functionals and curves of maximal slope

Mouvements minimisants le long d'une séquence de fonctionnelles et courbes de pente maximale

Andrea Braides [a], Maria Colombo [b], Massimo Gobbino [c], Margherita Solci [d]

[a] *Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica 1, 00133 Roma, Italy*
[b] *ETH Institute for Theoretical Studies, Clausiusstrasse 47, CH-8092 Zürich, Switzerland*
[c] *Dipartimento di Matematica, Via Filippo Buonarroti 1c, 56127 Pisa, Italy*
[d] *DADU, Università di Sassari, Piazza Duomo 6, 07041 Alghero (SS), Italy*

A R T I C L E   I N F O

Article history:
Received 26 February 2016
Accepted after revision 25 April 2016
Available online 17 May 2016

Presented by Haïm Brézis

A B S T R A C T

We prove that a general condition introduced by Colombo and Gobbino to study limits of curves of maximal slope allows us to characterize also minimizing movements along a sequence of functionals as curves of maximal slope of a limit functional.

© 2016 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

R É S U M É

Nous montrons qu'une condition générale présentée par Colombo et Gobbino pour étudier les limites des courbes de pente maximale permet également de caractériser les mouvements minimisants le long d'une séquence de fonctionelles comme des courbes de pente maximale de la fonctionnelle limite.

© 2016 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

Following a vast earlier literature, the notion of minimizing movement has been introduced by De Giorgi to give a general framework for Euler schemes in order to define a gradient-flow-type motion also for a non-differentiable energy ϕ. It consists in introducing a time scale τ, defining a time-discrete motion by an iterative minimization procedure in which the distance from the previous step is penalized in a way depending on τ, and then obtaining a time-continuous limit as $\tau \to 0$. This notion is at the base of modern definitions of variational motion and has been successfully used to construct

E-mail addresses: braides@mat.uniroma2.it (A. Braides), maria.colombo@eth-its.ethz.ch (M. Colombo), massimo.gobbino@unipi.it (M. Gobbino), Margherita@uniss.it (M. Solci).

http://dx.doi.org/10.1016/j.crma.2016.04.011
1631-073X/© 2016 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

a theory of gradient flows in metric spaces by Ambrosio, Gigli, and Savaré [3]. In particular, under suitable assumptions, it can be shown that a minimizing movement is a curve of maximal slope for ϕ.

When a sequence of energies ϕ_ε parameterized by a (small positive) parameter ε has to be taken into account, in order to define an effective motion, one may examine the minimizing movements of ϕ_ε and take their limit as $\varepsilon \to 0$, or, depending on the problem at hand, instead compute the minimizing movement of the (Γ-)limit ϕ of ϕ_ε. In general, these two motions are different. This is due to the trivial fact that the energy landscape of ϕ may not carry enough information to describe the energy landscapes of ϕ_ε, since local minimizers may appear or disappear in the limit process; as easy examples, show [4].

A general approach is to proceed in the minimizing-movement scheme, letting the parameter ε and the time scale τ tend to 0 together. This gives a notion of *minimizing movement along ϕ_ε at a given scale τ*. In this way, we can detect fine phenomena due to the presence of local minima. The limit of the minimizing movements and the minimizing movement of the limit are recovered as extreme cases. A first example of this approach has been given in [6] for spin energies converging to a crystalline perimeter, in which case the extreme behaviours are complete pinning and flat flow [1]. For a general choice of the parameters, the limit motion is neither of the two, but depends on the ratio between ε and τ and is a degenerate motion by crystalline curvature with pinning only of large sets. More examples can be found in [4]. Note that in general the functions that we obtain as minimizing movements along a sequence cannot be easily rewritten as minimizing movements of a single functional.

In another direction, conditions have been exhibited that ensure that the limit of gradient flows for a family ϕ_ε, or of curves of maximal slope, be the gradient flow, or a curve of maximal slope, for their Γ-limit ϕ (see [8] and [7]). This suggests that under such conditions, all minimizing movements along ϕ_ε, at whatever scale, may converge to minimizing movements of ϕ. In this paper, we prove a result in that direction, showing that if a lower-semicontinuity inequality holds for the descending slope of ϕ_ε then any minimizing movement is a curve of maximal slope for ϕ (Theorem 2.3). This property holds in particular in the case of convex energies (see [7], and also [2] and [4]). Note that the limit curve of maximal slope may still depend on the way ε and τ tend to 0, and that in the extreme case of ε tending to 0 fast enough with respect to τ, it is also a minimizing movement for the limit (Theorem 2.5b). In the case where all curves of maximal slope are minimizing movements for ϕ, this is a kind of 'commutativity result' between minimizing movements and Γ-convergence. This again holds if ϕ is convex, which is automatic if also ϕ_ε are convex. Nevertheless, in some cases it has been possible to directly prove that minimizing movements along a sequence are minimizing movements for the limit, also for some non-convex energies as scaled Lennard–Jones ones [5].

The result presented in this note is suggested by the analog result for curves of maximal slope in [7]: it makes the analysis in [4] more precise, and we think it will be a useful reference for future applications. Its proof follows modifying the arguments of [3], which show that the minimizing movements for a single functional are curves of maximal slope, and is briefly presented at the end of Section 2.

2. The limit result

In what follows, (X, d) is a complete metric space.

Definition 2.1 *(Minimizing movements along ϕ_ε at scale τ_ε).* For all $\varepsilon > 0$ let $\phi_\varepsilon : X \to (-\infty, +\infty]$, and let $u^0_\varepsilon \in X$. Suppose that there exists some $\tau^* > 0$ such that for every $\varepsilon > 0$ and $\tau \in (0, \tau^*)$, there exists a sequence $\{u^i_{\varepsilon,\tau}\}$ that satisfies $u^0_{\varepsilon,\tau} = u^0_\varepsilon$ and $u^{i+1}_{\varepsilon,\tau}$ is a solution to the minimum problem

$$\min\Big\{\phi_\varepsilon(v) + \frac{1}{2\tau} d^2(v, u^i_{\varepsilon,\tau}) : v \in X\Big\}. \tag{1}$$

Let $\tau = \tau_\varepsilon$ be a family of positive numbers such that $\lim_{\varepsilon\to 0} \tau_\varepsilon = 0$ and define the piecewise-constant functions $\bar{u}_\varepsilon = \bar{u}_{\varepsilon,\tau_\varepsilon} : [0, +\infty) \to X$ as $\bar{u}_\varepsilon(t) = u^{i+1}_{\varepsilon,\tau}$ for $t \in (i\tau, (i+1)\tau]$. A *minimizing movement along ϕ_ε at scale τ_ε* with initial data u^0_ε is any pointwise limit of a subsequence of the family $\bar{u}_\varepsilon$.

From now on, we will make the following hypotheses, which ensure the existence of minimizing movements along the sequence ϕ_ε at any given scale τ_ε [4]: ($u^* \in X$ is an arbitrary given point):

(i) for all $\varepsilon > 0$, ϕ_ε is lower semicontinuous;
(ii) there exist $C^* > 0$ and $\tau^* > 0$ such that $\inf\big\{\phi_\varepsilon(v) + \frac{1}{2\tau^*} d(v, u^*) : v \in X\big\} \geq C^* > -\infty$ for all $\varepsilon > 0$;
(iii) for all $C > 0$, there exists a compact set K such that $\{u : d^2(u, u^*) \leq C, |\phi_\varepsilon(u)| \leq C\} \subset K$ for all $\varepsilon > 0$.

Remark 1. (a) Alternatively, in the definition above, we can suppose $\tau \to 0$ and choose $\varepsilon = \varepsilon_\tau \to 0$. In this way, we define *minimizing movements along ϕ_{ε_τ} at scale τ*;

(b) if $\phi_\varepsilon = \phi$ for all ε, then a minimizing movement along ϕ_ε at any scale is a (generalized) *minimizing movement for ϕ* as defined in [3].

For $\phi\colon X \to (-\infty, +\infty]$ we define the *(descending) slope* of ϕ as

$$|\partial\phi|(x) = \begin{cases} \limsup\limits_{y\to x} \dfrac{(\phi(x)-\phi(y))^+}{d(x,y)} & \text{if } \phi(x) < +\infty \text{ and } x \text{ is not isolated} \\ 0 & \text{if } \phi(x) < +\infty \text{ and } x \text{ is isolated} \\ +\infty & \text{if } \phi(x) = +\infty. \end{cases} \tag{2}$$

We say that $v\colon [0,T] \to X$ belongs to $AC^2([0,T];X)$ if there exists $A \in L^2(0,T)$ such that

$$d(v(s), v(t)) \le \int_s^t A(r)\,\mathrm{d}r \quad \text{for any } 0 \le s \le t \le T. \tag{3}$$

The smallest such A is the *metric derivative* of v and it is denoted by $|v'|$.

Definition 2.2 *(Curve of maximal slope).* A *curve of maximal slope* for $\phi\colon X \to (-\infty,+\infty]$ in $[0,T]$ is $u \in AC^2([0,T];X)$ such that there exist a non-increasing function $\varphi\colon [0,T] \to \mathbb{R}$ such that

$$\varphi(s) - \varphi(t) \ge \frac{1}{2}\int_s^t |u'|^2(r)\,\mathrm{d}r + \frac{1}{2}\int_s^t |\partial\phi|^2(u(r))\,\mathrm{d}r \quad \text{for any} \quad 0 \le s \le t \le T \tag{4}$$

and a Lebesgue-negligible set E such that $\phi(u(t)) = \varphi(t)$ in $[0,T] \setminus E$.

Theorem 2.3 *(Minimizing movements and curves of maximal slope). Let $\phi_\varepsilon, \phi\colon X \to (-\infty,+\infty]$ satisfy* (i)–(iii) *and the following condition:*

(H) *for all subsequences ϕ_{ε_n} and $v_n \to v$ with $\sup_n\{|\phi_{\varepsilon_n}(v_n)| + |\partial\phi_{\varepsilon_n}|(v_n)\} < +\infty$, we have*

$$\lim_{n\to+\infty} \phi_{\varepsilon_n}(v_n) = \phi(v) \quad \textit{and} \quad \liminf_{n\to+\infty} |\partial\phi_{\varepsilon_n}|(v_n) \ge |\partial\phi|(v). \tag{5}$$

Let $u^0_{\varepsilon,\tau}$ be such that there exist S', S for which $d^2(u^0_{\varepsilon,\tau}, u^*) \le S' < +\infty$ and $|\phi_\varepsilon(u^0_{\varepsilon,\tau})| \le S < +\infty$. Then for all τ_ε any minimizing movement along ϕ_ε at scale τ_ε with initial data $u^0_{\varepsilon,\tau_\varepsilon}$ is a curve of maximal slope for ϕ.

Remark 2. (a) Note that we do not suppose that ϕ_ε Γ-converge to ϕ. This is usually deduced at points with finite slope for ϕ;

(b) the results can be proved under the more general assumption that hypotheses (i) and (iii) hold with respect to a weaker topology compatible with the distance d (see [3, Sect. 2.1]);

(c) if the initial slopes satisfy the additional bounds $|\partial\phi_\varepsilon|(u^0_{\varepsilon,\tau}) \le S < +\infty$, then any limit curve u as in the statement of Theorem 2.3 satisfies $\phi(u(t)) \le \phi(u(0))$ for almost all $0 \le t \le T$.

Corollary 2.4 *(Curves of maximal slope and Γ-convergence). Let ϕ_ε satisfy* (i)–(iii) *and*

$$\phi_\varepsilon(y) \ge \phi_\varepsilon(x) - d(x,y)|\partial\phi_\varepsilon|(x) \quad \textit{for any } y \in X \tag{6}$$

for any x such that $\phi_\varepsilon(x)$ and $|\partial\phi_\varepsilon|(x)$ are finite (Slope Cone Property [7])*. Let ϕ_ε Γ-converge to ϕ_0 with respect to d. Then the claim of* Theorem 2.3 *holds with $\phi = \phi_0$.*

The proof of the corollary follows from [7, Proposition 3.4]. Note that Corollary 2.4 holds even if assumption (6) is weakened by subtracting in the right-hand side a lower-order term with respect to $d(x,y)$, as described in [7, Remark 3.5].

Remark 3 *(Convex energies).* If ϕ_ε are convex, then they satisfy condition (6).

We note that in general the claim of Theorem 2.3 does not hold under the only hypothesis of Γ-convergence of ϕ_ε. More precisely, the following theorem gives a connection between minimizing movements along a sequence and the (generalized) minimizing movements of the limits.

Theorem 2.5 *(Γ-convergence and minimizing movements). Let ϕ_ε satisfy* (i)–(iii)*. Then*

(a) *there exists $\overline{\tau} = \overline{\tau}_\varepsilon$ such that if $\tau_\varepsilon \le \overline{\tau}_\varepsilon$ then each minimizing movement along ϕ_ε at scale τ_ε is (up to subsequences) the limit as $\varepsilon \to 0$ of the (generalized) minimizing movements for ϕ_ε;*

(b) *if ϕ_ε Γ-converge to ϕ with respect to d, then there exists $\overline{\varepsilon} = \overline{\varepsilon}_\tau$ such that if $\varepsilon_\tau \le \overline{\varepsilon}_\tau$ then each minimizing movement along ϕ_{ε_τ} at scale τ is (up to subsequences) a (generalized) minimizing movement for ϕ.*

In [4, Theorem 8.1] it is proved the existence of families $\overline{\tau}_\varepsilon$ and $\overline{\varepsilon}_\tau$ such that the conclusions of (a) and (b) hold for minimizing movement along ϕ_ε exactly at scale $\overline{\tau}_\varepsilon$ and for minimizing movement exactly along $\phi_{\overline{\varepsilon}_\tau}$ at scale τ respectively. Theorem 2.5 follows by noticing that the arguments of that proof imply that we may define $\overline{\tau}_\varepsilon$ and $\overline{\varepsilon}_\tau$ such that this holds for $\tau_\varepsilon \le \overline{\tau}_\varepsilon$ and $\varepsilon_\tau \le \overline{\varepsilon}_\tau$, respectively.

In case (b), note that if the limit ϕ satisfies some additional differentiability hypotheses (see [3, Theorems 2.3.1 and 2.3.3]), then each minimizing movement is a curve of maximal slope for the limit ϕ.

Proof of Theorem 2.3. In order to prove a priori uniform estimates, we introduce some definitions in analogy to those in [3, Section 3.2]. For any $\varepsilon > 0$ and $\delta \in (0, \tau^*)$, given $u \in X$ we denote by $J_{\varepsilon,\delta}(u)$ the set of v minimizing $\phi_\varepsilon(\cdot) + \frac{1}{2\delta} d^2(\cdot, u)$. Then $\tilde{u}_{\varepsilon,\tau} : [0, +\infty) \to X$ is any interpolation of the values of $\overline{u}_{\varepsilon,\tau}$ in $\tau\mathbb{N}$ satisfying $\tilde{u}_{\varepsilon,\tau}(t) \in J_{\varepsilon,t-i\tau}(u^i_{\varepsilon,\tau})$ for $t \in (i\tau, (i+1)\tau]$, and

$$G_{\varepsilon,\tau}(t) = \frac{\sup\{d(v, u^i_{\varepsilon,\tau}) : v \in J_{\varepsilon,t-i\tau}(u^i_{\varepsilon,\tau})\}}{t - i\tau} \quad \text{for} \quad t \in (i\tau, (i+1)\tau]. \tag{7}$$

Note that for $t > 0$

$$G_{\varepsilon,\tau}(t) \ge |\partial\phi_\varepsilon|(\tilde{u}_{\varepsilon,\tau}(t)). \tag{8}$$

Furthermore, $|u'_{\varepsilon,\tau}|$ denotes the piecewise-constant function defined by $|u'_{\varepsilon,\tau}|(t) = \tau^{-1} d(u^{i+1}_{\varepsilon,\tau}, u^i_{\varepsilon,\tau})$ for $t \in (i\tau, (i+1)\tau)$.

The following lemma is the analog of [3, Lemma 3.2.2] for a sequence ϕ_ε.

Lemma 2.6 *(A priori uniform estimates). Let $\varepsilon, \tau > 0$ and $\tau < \tau^*$. Let ϕ_ε satisfy* (i)–(iii)*, and $u^0_{\varepsilon,\tau}$ be such that there exist S', S for which $d^2(u^0_{\varepsilon,\tau}, u^*) \le S' < +\infty$ and $|\phi_\varepsilon(u^0_{\varepsilon,\tau})| \le S < +\infty$. Let $\{u^i_{\varepsilon,\tau}\}_i$ be defined as in Definition 2.1 with initial data $u^0_{\varepsilon,\tau}$. Then for any $i, j \in \mathbb{N}$ with $i < j$*

$$\phi_\varepsilon(u^i_{\varepsilon,\tau}) - \phi_\varepsilon(u^j_{\varepsilon,\tau}) = \frac{1}{2}\int_{i\tau}^{j\tau} |u'_{\varepsilon,\tau}|^2 \, dt + \frac{1}{2}\int_{i\tau}^{j\tau} |G_{\varepsilon,\tau}|^2 \, dt. \tag{9}$$

Moreover, fixed $T > 0$, there exists a constant C depending only on T, S, S', τ^*, u^* such that, for $\tau < \tau^*/8$, for any N with $N\tau \le T$.

$$d^2(u^N_{\varepsilon,\tau}, u^*) \le C, \quad |\phi_\varepsilon(u^N_{\varepsilon,\tau})| \le C \tag{10}$$

$$d^2(\tilde{u}_{\varepsilon,\tau}(t), \overline{u}_{\varepsilon,\tau}(t)) \le C\tau \quad \textit{in} \ [0, T] \tag{11}$$

$$\int_0^{N\tau} |u'_{\varepsilon,\tau}|^2(r)\, dr \le \phi_\varepsilon(u^0_\varepsilon) - \phi_\varepsilon(u^N_{\varepsilon,\tau}) \le C, \qquad \int_0^{N\tau} |G_{\varepsilon,\tau}|^2(r)\, dr \le \phi_\varepsilon(u^0_\varepsilon) - \phi_\varepsilon(u^N_{\varepsilon,\tau}) \le C. \tag{12}$$

The proof follows that of [3, Lemma 3.2.2], where a single ϕ is considered. The uniform bounds are ensured by the condition (ii) for the sequence ϕ_ε and by the uniform bounds on the initial data, which allow us to prove that for any i

$$d^2(u^i_{\varepsilon,\tau}, u^*) \le 2S' + 2\tau^* S - 2\tau^* C^* + \frac{4}{\tau^*}\sum_{j=1}^{i} \tau\, d^2(u^j_{\varepsilon,\tau}, u^*). \tag{13}$$

An application of the Gronwall Lemma to this inequality gives the required uniform bounds (see [3, pp. 67–68]).

With the aid of this lemma, we can prove Theorem 2.3 by showing that, up to subsequences, $v_n = \tilde{u}_{\varepsilon_n,\tau_{\varepsilon_n}}(t)$ satisfies the hypotheses of condition (H) for almost all t. The first estimate in (12) implies the weak convergence in $L^2_{\mathrm{loc}}[0, +\infty)$ of a subsequence $|u'_{\varepsilon_n,\tau_n}| \rightharpoonup A$. Since each $\phi_{\varepsilon_n}(\tilde{u}_{\varepsilon_n,\tau_n}(\cdot))$ is not increasing (see [3, Lemma 3.1.2]), we may apply Helly's Lemma (e.g., [3, Lemma 3.3.3]), obtaining (up to a further subsequence)

$$\varphi(t) = \lim_{n\to+\infty} \phi_{\varepsilon_n}(\tilde{u}_{\varepsilon_n,\tau_n}(t)) \quad \text{for any} \ t \in [0, T]; \tag{14}$$

in particular, the family $|\phi_{\varepsilon_n}(\tilde{u}_{\varepsilon_n,\tau_n}(t))|$ is equibounded for any $t \in [0, T]$. From (8), the second estimate in (12) and Fatou's Lemma

$$\int_0^T \left(\liminf_{n\to+\infty} |\partial\phi_{\varepsilon_n}|(\tilde{u}_{\varepsilon_n,\tau_n}(t))\right) dt \le \liminf_{n\to+\infty} \int_0^T |\partial\phi_{\varepsilon_n}|^2(\tilde{u}_{\varepsilon_n,\tau_n}(t))\, dt < +\infty,$$

we get that there exists a Lebesgue-negligible set E such that

$$\liminf_{n\to+\infty}|\partial\phi_{\varepsilon_n}|(\tilde u_{\varepsilon_n,\tau_n}(t))<+\infty \qquad \forall t\in[0,T]\setminus E. \tag{15}$$

The estimates in (10) and the compactness hypothesis (iii) imply that the family $\bar u_{\varepsilon_n,\tau_n}$ is contained in a compact subset of X. We set $s(n)=i_n\tau_n$ with $s(n)\le s$ and $s(n)\to s$, and $t(n)=j_n\tau_n$ with $t(n)\ge t$ and $t(n)\to t$. By the definition of $|u'_{\varepsilon_n,\tau_n}|$ and by weak convergence we get:

$$\limsup_{n\to+\infty} d(\bar u_{\varepsilon_n,\tau_n}(s),\bar u_{\varepsilon_n,\tau_n}(t))\le\limsup_{n\to+\infty}\int_{s(n)}^{t(n)}|u'_{\varepsilon_n,\tau_n}|(r)\,\mathrm{d}r\le\int_s^t A(r)\,\mathrm{d}r \tag{16}$$

for any $s,t\in[0,T]$. This estimate and the pre-compactness of the family $\bar u_{\varepsilon_n,\tau_n}$ allow us to use a Ascoli–Arzelà argument (see, e.g., [3, Proposition 3.3.1]) to obtain the existence of a function u pointwise limit of a further subsequence of $\bar u_{\varepsilon_n,\tau_n}(t)$. Since (11) holds, also $\tilde u_{\varepsilon_n,\tau_n}(t)$ converges to $u(t)$ for any $t\in[0,T]$. Since (15) is in force, we can extract for any $t\in[0,T]\setminus E$ a (t-dependent) sequence $n_k\to+\infty$ such that

$$\lim_{k\to+\infty}|\partial\phi_{\varepsilon_{n_k}}|(\tilde u_{\varepsilon_{n_k},\tau_{n_k}}(t))=\liminf_{n\to+\infty}|\partial\phi_{\varepsilon_n}|(\tilde u_{\varepsilon_n,\tau_n}(t))<+\infty.$$

Applying (H) to $\tilde u_{\varepsilon_{n_k},\tau_{n_k}}(t)$, for $t\in[0,T]\setminus E$, we get

$$\phi(u(t))=\lim_{k\to+\infty}\phi_{\varepsilon_{n_k},\tau_{n_k}}(\tilde u_{\varepsilon_{n_k},\tau_{n_k}}(t))=\lim_{n\to+\infty}\phi_{\varepsilon_n,\tau_n}(\tilde u_{\varepsilon_n,\tau_n}(t))=\varphi(t) \tag{17}$$

$$|\partial\phi|(u(t))\le\lim_{k\to+\infty}|\partial\phi_{\varepsilon_{n_k}}|(\tilde u_{\varepsilon_{n_k},\tau_{n_k}}(t))=\liminf_{n\to+\infty}|\partial\phi_{\varepsilon_n}|(\tilde u_{\varepsilon_n,\tau_n}(t)). \tag{18}$$

It remains to prove that u is a curve of maximal slope for ϕ. Passing to the limit in (16) it follows that $u\in AC^2([0,T];X)$ and that for almost all $t\in[0,T]$ $|u'|(t)\le A(t)$. The weak convergence ensures that

$$\liminf_{n\to+\infty}\int_{s(n)}^{t(n)}|u'_{\varepsilon_n,\tau_n}|^2(r)\,\mathrm{d}r\ge\int_s^t A^2(r)\,\mathrm{d}r\ge\int_s^t|u'|^2(r)\,\mathrm{d}r. \tag{19}$$

Estimates (8) and (18), and an application of Fatou's Lemma imply

$$\liminf_{n\to+\infty}\int_{s(n)}^{t(n)}|G_{\varepsilon_n,\tau_n}|^2(r)\,\mathrm{d}r\ge\liminf_{n\to+\infty}\int_{s(n)}^{t(n)}|\partial\phi_{\varepsilon_n}|^2(\tilde u_{\varepsilon_n,\tau_n}(r))\,\mathrm{d}r\ge\int_s^t|\partial\phi|^2(u(r))\,\mathrm{d}r. \tag{20}$$

By the non-increasing monotonicity of $\phi_{\varepsilon_n}(\tilde u_{\varepsilon_n,\tau_n}(\cdot))$, we have the inequalities $\phi_{\varepsilon_n}(u^{i_n}_{\varepsilon_n,\tau_n})\ge\phi_{\varepsilon_n}(\tilde u_{\varepsilon_n,\tau_n}(s))$ and $\phi_{\varepsilon_n}(u^{j_n}_{\varepsilon_n,\tau_n})\le\phi_{\varepsilon_n}(\tilde u_{\varepsilon_n,\tau_n}(t))$, so that, also using (19), (20) and (17), from (9) we obtain:

$$\varphi(s)-\varphi(t)\ge\frac12\int_s^t|u'|^2\,\mathrm{d}r+\frac12\int_s^t|\partial\phi|^2(u(r))\,\mathrm{d}r, \tag{21}$$

where $\phi(u(t))=\varphi(t)$ in $[0,T]\setminus E$. $\square$

Acknowledgements

The authors thank the anonymous referee for her/his useful comments. The second author has been partially supported by Dr. Max Rössler, the Walter Haefner Foundation and the ETH Zurich Foundation.

References

[1] F. Almgren, J.E. Taylor, Flat flow is motion by crystalline curvature for curves with crystalline energies, J. Differ. Geom. 42 (1995) 1–22.
[2] L. Ambrosio, N. Gigli, A user's guide to optimal transport, in: B. Piccoli, M. Rascle (Eds.), Modelling and Optimisation of Flows on Networks, in: Lecture Notes in Mathematics, Springer, Berlin, 2013, pp. 1–155.
[3] L. Ambrosio, N. Gigli, G. Savaré, Gradient Flows in Metric Spaces and in the Space of Probability Measures, Lectures in Mathematics ETH, Zürich, Birkhäuser, Basel, Switzerland, 2008.
[4] A. Braides, Local Minimization, Variational Evolution and Γ-convergence, Springer Verlag, Berlin, 2014.
[5] A. Braides, A. Defranceschi, E. Vitali, Variational evolution of one-dimensional Lennard–Jones systems, Netw. Heterog. Media 9 (2014) 217–238.
[6] A. Braides, M.S. Gelli, M. Novaga, Motion and pinning of discrete interfaces, Arch. Ration. Mech. Anal. 95 (2010) 469–498.
[7] M. Colombo, M. Gobbino, Passing to the limit in maximal slope curves: from a regularized Perona–Malik equation to the total variation flow, Math. Models Methods Appl. Sci. 22 (2012) 1250017.
[8] E. Sandier, S. Serfaty, Gamma-convergence of gradient flows and application to Ginzburg–Landau, Commun. Pure Appl. Math. 57 (2004) 1627–1672.