Differential geometry

Remarks on shrinking gradient Kähler–Ricci solitons with positive bisectional curvature

Remarques sur les solitons de Kähler–Ricci évanescents à courbure bisectionnelle positive

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A R T I C L E   I N F O

A B S T R A C T

In this short note, using an argument by Munteanu and Wang, we provide an alternative proof of the fact first obtained by Lei Ni that shrinking gradient Kähler–Ricci solitons with positive bisectional curvature must be compact.

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R É S U M É

Dans cette Note, en utilisant un argument de Munteanu et Wang, nous donnons une démonstration alternative du fait, déjà obtenu par Lei Ni, que les solitons de Kähler–Ricci évanescents de courbure bisectionelle positive sont compacts.

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1. Introduction

A gradient Ricci soliton is a self-similar solution to the Ricci flow that flows by diffeomorphism and homothety. The study of solitons has become increasingly important in both the study of the Ricci flow and metric measure theory. In Perelman's proof of Poincaré conjecture, one issue he needs to prove is that a three-dimensional shrinking gradient Ricci soliton with positive sectional curvature is compact. It is natural to ask whether this holds in high dimension. Last year, Munteanu–Wang [4] proved that this is true.

Theorem 1.1 *(Munteanu–Wang [4]). Any shrinking gradient Ricci soliton with nonnegative sectional curvature and positive Ricci curvature is compact.*

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In this paper, we consider the gradient Kähler–Ricci soliton, namely a triple (M^n, g, f) associated with a Kähler manifold (M, g) such that

$$R_{i\bar{j}} + \nabla_i \nabla_{\bar{j}} f = \beta g_{i\bar{j}}, \qquad \text{and} \qquad \nabla_i \nabla_j f = 0, \tag{1}$$

for some constant $\beta \in \mathbb{R}$. It is called shrinking, steady or expanding, if $\beta > 0$, $\beta = 0$ or $\beta < 0$ respectively. Using similar method as in [4], we prove the following theorem.

Theorem 1.2. *Shrinking gradient Kähler–Ricci solitons with nonnegative bisectional curvature and positive Ricci curvature are compact.*

Actually, L. Ni [5] proved the above theorem using a different method. He firstly proved that the scalar curvature has a uniform positive lower bound, and used the result that the average of scalar curvature on a noncompact Kähler manifold with positive bisectional curvature has a decay estimate of the form $\frac{C}{1+r}$; in this way he [5] proved the following theorem.

Theorem 1.3 (Ni [5]). *Let (M^n, g) be a shrinking gradient Kähler–Ricci soliton with nonnegative bisectional curvature, then we have:*
(i) *if the bisectional curvature of M is positive then M must be compact and isometric-biholomorphic to $\mathbb{C}P^n$;*
(ii) *if M has nonnegative bisectional curvature then the universal cover $\tilde{M}$ splits as $\tilde{M} = N_1 \times N_2 \times \cdots \times N_l \times \mathbb{C}^k$ isometric-biholomorphically, where N_i are compact irreducible Hermitian Symmetric Spaces.*

Next section we will give the proof the Theorem 1.2. The idea follows from Munteanu–Wang [4]: we assume the manifold is noncompact, then we have the asymptotic behavior of potential f, the key is to get some lower bound of the Ricci curvature. For this, we introduce a quantity involving the Ricci curvature and the potential f as in [3] and [4], then using the maximum principle, we can get the Ricci curvature's lower bound, which is related to f. Once we have this, it is easy to obtain that the average of the scalar curvature on the geodesic ball could be sufficiently large; however, it is impossible, so M must be compact.

2. Proof of Theorem 1.2

For simplicity, we assume $\beta = 1$. Firstly, we obtain a few formulas for gradient Kähler–Ricci solitons.

Lemma 2.1. (a) $R + \Delta f = n$;
(b) $R + |\nabla f|^2 - f = \text{constant}$;
(c) $\Delta_f R_{i\bar{j}} = R_{i\bar{j}} - R_{i\bar{j}k\bar{l}}R_{l\bar{k}}$;
(d) $\Delta_f R = R - |\text{Ric}|^2$.
Here $|\nabla f|^2 = g^{i\bar{j}}\nabla_i f \nabla_{\bar{j}} f$, $\Delta_f R_{i\bar{j}} = \Delta R_{i\bar{j}} - g^{k\bar{l}}\nabla_k f \nabla_{\bar{l}} R_{i\bar{j}}$.

Proof. Both (a) and (b) follow from the soliton equation (1) and the Bochner formula. For the convenience of the reader, the proof of (c) is given in the appendix. Taking trace over (c), we get (d). □

Theorem 2.2. *Suppose that (M^n, g, f) is a complete shrinking gradient Kähler–Ricci soliton, if we assume the bisectional curvature is nonnegative and the Ricci curvature is positive, then it must be compact.*

Proof. Assume that (M, g, f) is noncompact. From Lemma 2.1, we know that $R + |\nabla f|^2 - f = C$. By adding a constant to f if necessary, we can assume that $R + |\nabla f|^2 = f$.

Concerning the potential f, Cao and Zhou [1] proved that

$$\frac{1}{2}(d(x, p) - C_1)^2 \le f(x) \le \frac{1}{2}(d(x, p) + C_2)^2, \tag{2}$$

where p is a fixed point, C_1 and C_2 are positive constants depending only on the dimension of the manifold and the geometry of the unit ball $B_p(1)$.

Denote $\lambda(x)$ as the minimal eigenvalue of the Ricci curvature at x, suppose v is the eigenvector corresponding to $\lambda(x)$, then

$$R_{i\bar{j}k\bar{l}}R_{\bar{k}l}v^i v^{\bar{j}} = R(v, \bar{v}, \frac{\partial}{\partial z^k}, \frac{\partial}{\partial z^{\bar{l}}})R_{\bar{k}l}.$$

Diagonalizing Ric at x so that $R_{\bar{k}l} = \lambda_k \delta_{kl}$. Since the bisectional curvature is nonnegative, we have

$$R_{i\bar{j}k\bar{l}}R_{\bar{k}l}v^i v^{\bar{j}} = R(v, \bar{v}, \frac{\partial}{\partial z^k}, \frac{\partial}{\partial z^{\bar{k}}})\lambda_k \ge 0. \tag{3}$$

Hence λ satisfies the following differential inequality in the sense of barrier,

$$\Delta_f \lambda \le \lambda. \tag{4}$$

Actually, this means that at any point x, we can find a smooth function h such that $h(x) = \lambda(x)$, $h \ge \lambda$ on $B(x, \delta)$ and $\Delta_f h(x) \le h(x)$, where δ is a small positive constant, h can be constructed as follows.

Suppose $v \in T_x^{1,0}M$ is the unit eigenvector corresponding to $\lambda(x)$, taking parallel translation of v along any unit speed geodesic starting from x, then in a small neighborhood $B(x, \delta)$, we get a smooth vector field V with $V(x) = v$. Define $h(y) = Ric(y)(V(y), \overline{V(y)})$, then $h \ge \lambda$ for $y \in B(x, \delta)$ and $h(x) = \lambda(x)$, moreover, using $\nabla V(x) = 0$ and (3), we get

$$\Delta_f h(x) = (\Delta_f Ric)(V(x), \overline{V(x)}) = \Delta_f R_{i\bar{j}} v^i v^{\bar{j}} = (-R_{i\bar{j}k\bar{l}} R_{\bar{k}l} + R_{i\bar{j}}(x)) v^i v^{\bar{j}} \le Ric(x)(v, \overline{v}) = h(x).$$

Choose a geodesic ball $B(p, r)$ of radius r large enough, because the Ricci curvature is positive, then

$$a = \min_{\partial B(p,r)} \lambda > 0.$$

We define

$$U = \lambda - \frac{a}{f} - \frac{2na}{f^2}.$$

From the growth rate of f, it follows that if r is large enough, $U > 0$ on $\partial B(p, r)$.

$$\Delta_f U = \Delta_f \lambda - \Delta_f \left(\frac{a}{f}\right) - \Delta_f \left(\frac{2na}{f^2}\right) \le \lambda - a(\frac{1}{f} - \frac{n}{f^2}) - 2na \cdot \frac{3}{2f^2} = U.$$

We have now proved $\Delta_f U \le U$ on $M \backslash B(p, r)$ if r is large enough.

Next we prove $U \ge 0$ on $M \backslash B(p, r)$. If there is a point $y_0 \in M \backslash B(p, r)$ such that $U(y_0) < 0$, then there must exist a point $x_0 \in M \backslash B(p, r)$ such that $U(x_0) = \min_{y \in \{M \backslash B(p,r)\}} U(y) < 0$, this is due to $\underline{\lim}_{d(x,p) \to \infty} U(x) \ge 0$ and $U > 0$ on $\partial B(p, r)$.

At x_0, suppose that v is the unit eigenvector corresponding to $\lambda(x_0)$. Taking parallel translation along all the unit speed geodesic starting from x_0, then in a small ball $B(x_0, \delta)$ we get a smooth vector field V with $V(x_0) = v$. Define $\widetilde{U} = Ric(V(y), V(y)) - \frac{a}{f} - \frac{2na}{f^2}$, then for any $y \in B(x_0, \delta)$, $\widetilde{U}(y) \ge U(y)$ and $\widetilde{U}(x_0) = U(x_0)$, so

$$0 \le \Delta_f \widetilde{U}(x_0) \le \widetilde{U}(x_0) < 0.$$

Contradiction, so $U \ge 0$ on $M \backslash B(p, r)$, i.e. $Ric \ge \frac{a}{f}$. As the argument in [4], we get $R \ge b \cdot \log f$ for some $b > 0$.

From the soliton equation (1), we get $\Delta f + R = n$. Consider the sublevel set $\{f \le c\}$ of f, integrate, we get

$$\int_{\{f \le c\}} n = \int_{\{f \le c\}} (\Delta f + R) = \int_{\{f = c\}} |\nabla f| + \int_{\{f \le c\}} R \ge \int_{\{f \le c\}} R.$$

So the average of R over $\{f \le c\}$ is less than n, then it is easy to see that the average of R over $B(p, r)$ is less than some $A > n$.

On the other hand, using the argument in [4], picking q with $d(p, q) = \frac{3r}{4}$, by (2), we have

$$\int_{B(p,r)} R \ge \int_{B(q, \frac{r}{4})} R \ge b \cdot \log \left(\frac{1}{2}(\frac{r}{2} - C_1)^2\right) \text{Vol} \left(B(q, \frac{r}{4})\right).$$

Applying the Bishop–Gromov relative volume comparison, we get

$$\text{Vol} \left(B(q, \frac{r}{4})\right) \ge c(n) \text{Vol} \left(B(q, 2r)\right) \ge c(n) \text{Vol} \left(B(p, r)\right).$$

Hence

$$\int_{B(p,r)} R \ge b \cdot c(n) \cdot \log \left(\frac{1}{2}(\frac{r}{2} - C_1)^2\right) \text{Vol} \left(B(p, r)\right),$$

this means that the average of R over $B(p, r)$ is greater that $A > n$ if r is sufficiently large. Contradiction. □

Acknowledgements

Appendix A

In this appendix, we give a proof of (c) in Lemma (2.1). It is easy to obtain this identity using Ricci flow, for the evolution of Ricci curvature, refer to (Cor. 2.83 in [2]). Using the equation of the Kähler–Ricci soliton (1), we have

$$\Delta_f R_{i\bar{j}} = -\Delta \nabla_i \nabla_{\bar{j}} f - \nabla_k f \nabla_{\bar{k}} R_{i\bar{j}}$$

$$= -\nabla_{\bar{j}} \nabla_i \Delta f + R_{i\bar{j}k\bar{l}} \nabla_l \nabla_{\bar{k}} f - \frac{1}{2} R_{i\bar{k}} \nabla_k \nabla_{\bar{j}} f - \frac{1}{2} R_{k\bar{j}} \nabla_i \nabla_{\bar{k}} f - \nabla_k f \nabla_{\bar{k}} R_{i\bar{j}}$$

$$= \nabla_{\bar{j}} (R_{i\bar{k}} \nabla_k f) - \nabla_k f \nabla_{\bar{k}} R_{i\bar{j}} + R_{i\bar{j}k\bar{l}} \nabla_l \nabla_{\bar{k}} f - \frac{1}{2} R_{i\bar{k}} \nabla_k \nabla_{\bar{j}} f - \frac{1}{2} R_{k\bar{j}} \nabla_i \nabla_{\bar{k}} f$$

$$= \nabla_{\bar{j}} R_{i\bar{k}} \nabla_k f + R_{i\bar{k}} \nabla_k \nabla_{\bar{j}} f - \nabla_k f \nabla_{\bar{k}} R_{i\bar{j}} + R_{i\bar{j}k\bar{l}} \nabla_l \nabla_{\bar{k}} f - \frac{1}{2} R_{i\bar{k}} \nabla_k \nabla_{\bar{j}} f - \frac{1}{2} R_{k\bar{j}} \nabla_i \nabla_{\bar{k}} f$$

$$= R_{i\bar{j}k\bar{l}} \nabla_l \nabla_{\bar{k}} f + \frac{1}{2} R_{i\bar{k}} \nabla_k \nabla_{\bar{j}} f - \frac{1}{2} R_{k\bar{j}} \nabla_i \nabla_{\bar{k}} f$$

$$= R_{i\bar{j}k\bar{l}} (g_{l\bar{k}} - R_{l\bar{k}}) + \frac{1}{2} R_{i\bar{k}} (g_{k\bar{j}} - R_{k\bar{j}}) - \frac{1}{2} R_{k\bar{j}} (g_{i\bar{k}} - R_{i\bar{k}}) = R_{i\bar{j}} - R_{i\bar{j}k\bar{l}} R_{l\bar{k}},$$

where we have used $\nabla_{\bar{j}} R_{i\bar{k}} = \nabla_{\bar{k}} R_{i\bar{j}}$ in the fifth equality. Therefore we obtain formula (c).

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