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On the number of fibrations transverse to a rational curve in complex surfaces

Sur le nombre de fibrations transverses à une courbe rationnelle dans une surface

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ABSTRACT

We investigate the existence, and lack of uniqueness, of a holomorphic fibration by discs transverse to a rational curve in a complex surface.

RÉSUMÉ

Nous étudions l'existence et le défaut d'unicité de fibrations holomorphes en disques transverses à une courbe rationnelle dans une surface complexe.

Version française abrégée

Soit U une surface complexe contenant une courbe rationnelle lisse C. On s'intéresse à la structure du germe de voisinage (U,C). Lorsque $C^2 \leq 0$, il est bien connu que (U,C) est holomorphiquement équivalent au voisinage de la section nulle dans l'espace total du fibré normal N_C (see [2,7]) ; on dit alors que (U,C) est *linéarisable*. On déduit aisément l'existence de nombreuses fibrations holomorphes par disques transverse à C dans ce cas. D'un autre côté, lorsque $C^2 > 0$, il y a un gros espace de module de germes de tels voisinages (U,C) modulo isomorphismes (voir [3,6]). A contrario, nous montrons qu'il y a très peu de fibrations transverses (en général aucune) dans ce cas. Il y a des familles de dimension infinie de voisinages sans (resp. avec une unique) fibration pour chaque $C^2 > 0$. Lorsque $C^2 = +1$, il existe aussi des familles de dimension infinie de voisinages avec exactement deux fibrations. Notre résultat principal est le suivant.

Théorème A. *Soit (U,C) un germe de surface au voisinage d'une courbe rationnelle C (tout est lisse) avec auto-intersection $C^2 > 0$.*

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– *si (U, C) admet au moins 3 fibrations holomorphes transverses à C, alors $C^2 = 1$ et (U, C) est linéarisable, i.e. holomorphiquement équivalent au voisinage d'une droite dans $\mathbb{P}^2$.*
– *si $C^2 > 1$ et (U, C) admet au moins 2 fibrations holomorphes transverses à C, alors $C^2 = 2$ et (U, C) est holomorphiquement équivalent au voisinage de la diagonale dans $\mathbb{P}^1 \times \mathbb{P}^1$.*

Un analogue non linéaire de la dualité projective entre les points et les droites de $\mathbb{P}^2$ établie par Le Brun (voir [5, §1.3, 1.4]) nous fournit une correspondance bi-univoque entre les germes de voisinages (U, C) avec $C^2 = 1$ et les germes de structures projectives en $(\mathbb{C}^2, 0)$ modulo $\mathrm{Diff}(\mathbb{C}^2, 0)$ (tout est holomorphe). Par une structure projective, on entend une collection de géodésiques (une courbe passant par chaque point et chaque direction) définie par une connexion affine (i.e. une connexion linéaire sur le fibré tangent). De ce point de vue, on a une correspondance bi-univoque entre les fibrations transverses à C et les décompositions de la famille de géodésiques comme pinceau de feuilletages (voir, par exemple, [4]), ou encore de connexions affines à courbure nulle définissant la structure.

Tous les résultats présentés dans cette note seront détaillés dans [1].

1. Introduction

Let U be a smooth complex surface containing a smooth rational curve C. We want to understand the structure of the germ of neighborhood (U, C). When $C^2 \leq 0$, it is known that (U, C) is holomorphically equivalent to the neighborhood of the zero section in the normal bundle N_C (see [2,7]). In this case, we say that (U, C) is *linearizable*, and one can easily deduce the existence of many fibrations by discs transverse to C. On the other hand, when $C^2 > 0$, there is a huge moduli space of germs of neighborhoods (U, C) (see [3,6]). However, we prove that there are very few (in general there are not) transverse fibrations in this latter case. There are infinite dimensional families of neighborhoods without (resp. with a unique) fibration for any $C^2 > 0$. When $C^2 = +1$, there also exist infinite dimensional families of neighborhoods with exactly 2 fibrations. Our main result is the following.

Theorem 1.1. *Let (U, C) be a germ of surface neighborhood of a rational curve C with self-intersection $C^2 > 0$, everything being smooth.*

– *If (U, C) admits at least 3 distinct fibrations by discs transverse to C, then $C^2 = 1$ and (U, C) is linearizable, i.e. holomorphically equivalent to the neighborhood of a line in $\mathbb{P}^2$.*
– *If $C^2 > 1$ and (U, C) admits at least 2 distinct fibrations by discs transverse to C, then $C^2 = 2$ and (U, C) is holomorphically equivalent to the neighborhood of the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$.*

A non-linear analogue of the projective duality between lines and points in $\mathbb{P}^2$ established by Le Brun (see [5, §1.3, 1.4]) provides a one-to-one correspondence between germs of neighborhoods (U, C) with $C^2 = 1$ and germs of projective structure at $(\mathbb{C}^2, 0)$ up to $\mathrm{Diff}(\mathbb{C}^2, 0)$ (everything is holomorphic). By a projective structure, we mean a collection of geodesics (one curve for each point+direction) defined by an affine connection (i.e. a linear connection on the tangent bundle). From this point of view, there is a one-to-one correspondence between fibrations transverse to C and decompositions of the projective structure as a pencil of foliations (see for instance [4]), or equivalently affine connections with vanishing curvature.

2. Normal form

Let us fix a coordinate $x : C \overset{\sim}{\to} \mathbb{C} \cup \{\infty\}$ and decompose it as $C = V_0 \cup V_\infty$, where V_i are disks around $x = i$ with $i = 0, \infty$ overlapping on a neighborhood of the circle $\{|x| = 1\}$. It is easy to see that a germ of surface (U, C) can always be obtained by gluing two open sets $U_0 = V_0 \times \mathbb{D}_\epsilon$ and $U_\infty = V_\infty \times \mathbb{D}_\epsilon$, with coordinates (x_i, y_i) by some analytic diffeomorphism of the form

$$(x_\infty, y_\infty) = \Phi(x_0, y_0) = \left(x_0^{-1} + \sum_{n\geq 1} a_n(x_0) y_0^n, \sum_{n\geq 1} b_n(x_0) y_0^n \right),$$

which we shall call the *cocycle* of the germ (U, C). In restriction to C, we have $x = x_0 = 1/x_\infty$. Of course, different cocycles could give rise to isomorphic germs of surface. In this sense, it is shown in [6] that we can always arrive to an almost unique normal form. In the special case $C^2 = 1$, we obtain the slightly more precise statement below.

Theorem 2.1. *Let (U, C) be a germ of surface with $C^2 = 1$. Then, we can choose the corresponding cocycle in the following* normal form

$$\Phi = \left(\frac{1}{x} + \sum_{n\geq 4} (\sum_{k=3}^{n-1} \frac{a_{k,n}}{x^k}) y^n, \frac{y}{x} + \sum_{n\geq 3} (\sum_{k=2}^{n-1} \frac{b_{k,n}}{x^k}) y^n \right).$$

Moreover, $\Phi^i = (x + \sum a_n^i(x)y^n, \sum b_n^i(x)y^n)$, $i = 0, \infty$, are such that $\Phi^\infty \circ \Phi \circ \Phi^0$ is still in normal form (with possibly different coefficients) if, and only if, there are constants $\alpha, \beta, \gamma \in \mathbb{C}$ and $\theta \in \mathbb{C}^*$ such that

$$a_1^0 = \theta(\alpha x + \beta), a_2^0 = \alpha^2\theta^2 x + \gamma, a_3^0 = \alpha^3\theta^3 x + (\gamma\alpha\theta + \alpha\theta^3 b_{2,3}),$$

$$a_n^0 = \alpha^n\theta^n x + \left(\gamma(\theta\alpha)^{n-2} + \theta^n\left(\sum_{k=1}^{n-2}\binom{n-2}{k-1}\alpha^k b_{2,n-k+1}\right)\right), n \geq 4,$$

$$a_1^\infty = \beta x + \alpha, a_2^\infty = \beta^2 x + \frac{\gamma}{\theta^2},$$

$$a_n^\infty = \beta^n x + \left((n-1)\frac{\beta^{n-2}\gamma}{\theta^2} - (n-2)\alpha\beta^{n-1} - \sum_{k=1}^{n-2} k\beta^k b_{n-k,n-k+1}\right), n \geq 3,$$

$$b_n^0 = \theta^n\alpha^{n-1}, \quad b_n^\infty = \frac{\beta^{n-1}}{\theta}, n \geq 1.$$

The proof of this statement (which will be needed to prove the main result) will be detailed in [1]. Let us just mention that normalizing coordinates (x_0, y_0) (resp. (x_∞, y_∞)) are obtained after blowing-up $\infty \in C$ (resp. $0 \in C$) by the identification of the new germ of neighborhood (with $C^2 = 0$) with the product $C \times \mathbb{D}_\epsilon$. The normal form follows from a good choice of these trivializations.

Remark 1. When in addition U admits a fibration transverse to C, then there exists a normal form compatible with the fibration in the sense that $a_{kn} = 0$ for every k, n (with notation of Theorem 2.1) and the fibration is given by $\{x = \text{const.}\}$.

Remark 2. The classification of germs (U, C) with $C^2 = k > 1$ reduces to the case $C^2 = 1$ since the unique cyclic k-fold cover $\pi : \tilde{U} \to U$ ramifying over C is such that $\tilde{C}^2 = 1$, where $\tilde{C} = \pi^{-1}(C)$.

Remark 3. When $C^2 = 1$, the fourth infinitesimal neighborhood $C(4) = \text{Spec}\left(\frac{\mathcal{O}_U}{\mathcal{I}^5}\right)$ (where $\mathcal{I} \subset \mathcal{O}_U$ is the ideal sheaf defining C) always admits a transverse fibration: the cocycle can always be normalized to $(\frac{1}{x}, \frac{y}{x} + \cdots)$ up to order 4. The first obstruction for having a transverse fibration appears in order five.

Many examples without transverse fibration. Consider the neighborhood U given by the cocycle

$$\Phi = \left(\frac{1}{x} + \frac{y^5}{x^3} + \sum_{n \geq 6}(\sum_{k=3}^{n-1}\frac{a_{k,n}}{x^k})y^n, \frac{y}{x}\right)$$

which is already in normal form and suppose that U admits a transverse fibration. Then by Remark 1 there is a normal form compatible with the fibration, that is, there exists $A = (\alpha, \beta, \gamma, \theta) \in \mathbb{C}^3 \times \mathbb{C}^*$ such that $a_{i,j}^A = 0$ for every i, j, where the notation stands for the coefficients of the normal form after composing with the pair (Φ^0, Φ^∞) associated with A as in Theorem 2.1. On the other hand, we can compute $a_{3,4}^A = -(\gamma - \alpha\beta\theta^2)^2$ and conclude that $\gamma = \alpha\beta\theta^2$. This leads us to $a_{3,5}^A = \theta^5 \neq 0$, which is impossible.

Many examples with exactly one transverse fibration. Consider the neighborhood U given by the cocycle

$$\Phi = \left(\frac{1}{x}, \frac{y}{x} + \sum_{n \geq 5}(\sum_{k=2}^{n-1}\frac{b_{k,n}}{x^k})y^n\right),$$

with $b_{3,5}^2 - b_{2,5} \cdot b_{4,5} \neq 0$, which is in normal form and has a compatible transverse fibration. If there exists another fibration, then we can find some $A = (\alpha, \beta, \gamma, \theta) \in \mathbb{C}^3 \times \mathbb{C}^*$ such that $a_{i,j}^A = 0$ for every i, j. We can also assume that $\theta = 1$. We compute $a_{3,4}^A = -(\gamma - \alpha\beta)^2$, which implies $\gamma = \alpha\beta$. Thus $a_{3,5}^A = \alpha b_{3,5} + \beta b_{2,5} = 0$ and $a_{4,5}^A = \alpha b_{4,5} + \beta b_{3,5} = 0$. By hypothesis, the only solution to this system is $\alpha = \beta = 0$ that also gives $\gamma = 0$. We conclude that $A = (0, 0, 0, 1)$ and the fibration coincides with the initial one.

3. Examples with 2 fibrations transverse to C

Consider (U, C) a germ of surface as before and suppose that there are two fibrations $\mathcal{G}$ and $\mathcal{G}'$ transverse to C. If they are not tangent along C, it is easy to see that their tangency locus is a (smooth) curve transverse to C at one point.

Theorem 3.1. *Let U be a neighborhood of a rational curve C with self-intersection $+1$ and suppose that there are two fibrations $\mathcal{G}$ and $\mathcal{G}'$ transverse to C such that $\mathrm{tang}(\mathcal{G},\mathcal{G}') = C$. Then (U, C) is the ramified covering of degree 2 over a neighborhood of the diagonal curve $\Delta \subseteq V \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ with ramification locus Δ and branching locus C. Moreover, there is no other fibration on U transverse to C.*

Idea of proof. Consider the first integrals $g, g' : U \to C \simeq \mathbb{P}^1$ defined by the two fibrations; then the map $(g, g') : U \to \mathbb{P}^1 \times \mathbb{P}^1$ is the 2-fold cover. See [1] for details.

Suppose now that $T = \mathrm{tang}(\mathcal{G},\mathcal{G}')$ is neither a common fiber of the two fibrations, nor C, and is transverse to $\mathcal{G}$ and $\mathcal{G}'$ along C. Take first integrals $g, g' : U \to \mathbb{P}^1$ such that $g|_C \equiv g'|_C = x$, where x is the global coordinate of $\mathbb{P}^1$, and define

$$\Phi : U \to \mathbb{P}^1 \times \mathbb{P}^1, \;\; p \mapsto (g(p), g'(p)).$$

Observe that $\Phi|_C : C \to \Delta$ is an isomorphism, where Δ is the diagonal curve, and that Φ is a local biholomorphism outside T. It is possible to show that $\Phi(T)$ has a simple tangency with Δ at one point and that Φ is a $2:1$ covering near T, ramifying over $\Phi(T)$. Changing the first integrals (f, g) coinciding along C is the same that considering the diagonal action of $PSL_2(\mathbb{C})$ on $(\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$, that is, $\varphi \mapsto (\varphi(x), \varphi(y))$. Moreover, every curve with a simple tangency with Δ can be constructed by this process.

Theorem 3.2. *The moduli space of neighborhoods (U, C) having two fibrations with a tangency locus T which is neither a common fiber nor C, and which is transverse to $\mathcal{G}$ and $\mathcal{G}$' at C, is in bijection with the space of germs of curves on $(\mathbb{P}^1 \times \mathbb{P}^1, \Delta)$ with a simple tangency with Δ modulo the diagonal action of $PSL_2(\mathbb{C})$.*

4. The case of 3 transverse fibrations

Our goal is to show that the only case having 3 fibrations is the case of the projective plane.

Theorem 4.1. *Let (U, C) be a germ of surface containing a rational curve C with $C^2 = +1$. Assume that U admits 3 regular fibrations $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}''$ transverse to C, then (U, C) is isomorphic to the germ $(\mathbb{P}^2, L_0)$, where L_0 is a line in $\mathbb{P}^2$.*

The detailed proof will be given in [1]; however, we present below a special case in order to show the main ideas. Before proving the theorem, we make some considerations. First of all, observe that there are normal forms

$$\Phi = \left(\frac{1}{x}, \frac{y}{x} + (\frac{b_{2,3}}{x^2})y^3 + \ldots\right), \Phi^i = \left(\frac{1}{x}, \frac{y}{x} + (\frac{b^i_{2,3}}{x^2})y^3 + \ldots\right), \quad i = 1, 2,$$

compatible with $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}''$, respectively, that is, in which the fibration is given by $\{x = \text{const.}\}$ (Remark 1). Denote by $(\alpha_i, \beta_i, \gamma_i, \theta_i)$ the parameter corresponding to the pair of biholomorphisms $(\Phi^0_i, \Phi^\infty_i)$ taking Φ into Φ^i $(i = 1, 2)$ given by Theorem 2.1. Note that coefficients $(a^i_{j,k}, b^i_{j,k})$ of the normal form Φ^i depend of the coefficients $b_{j,k}$ and $(\alpha_i, \beta_i, \gamma_i, \theta_i)$. We clearly have $a^i_{j,k} = 0$ for $i = 1, 2$.

By composing Φ with the change of coordinates associated with $(0, 0, 0, \theta)$, which does not affect the fibrations, we can assume that $b_{2,3} = 1$ or $b_{2,3} = 0$. In a similar way, we can also suppose that $\theta_1 = \theta_2 = 1$.

$$\begin{tikzcd} & \Phi \arrow[dl, "(\alpha_1,\beta_1,\gamma_1,1)"'] \arrow[dr, "(\alpha_2,\beta_2,\gamma_2,1)"] & \\ \Phi^1 & & \Phi^2 \end{tikzcd}$$

We will need the following lemma.

Lemma 4.2. *If $b_{2,3} = 0$, $\alpha_1 \neq 0$, $\beta_1 = \gamma_1 = 0$, and $\beta_2 \neq 0$, $\alpha_2 = \gamma_2 = 0$, then $b_{ij} = 0$ for every i, j.*

Proof of Theorem 4.1. We consider the case $\mathrm{tang}(\mathcal{G},\mathcal{G}') \cap C \neq \mathrm{tang}(\mathcal{G},\mathcal{G}'') \cap C$.

Thus we assume $\mathrm{tang}(\mathcal{G},\mathcal{G}') \cap C = \{0\}$ and $\mathrm{tang}(\mathcal{G},\mathcal{G}'') \cap C = \{\infty\}$. This implies, by the form of the pair $(\Phi^0_i, \Phi^\infty_i)$ associated with $(\alpha_i, \beta_i, \gamma_i, 1)$ given in Theorem 2.1, that $\beta_1 = \alpha_2 = 0$. Observe also that if $\alpha_1 = 0$ then $\mathcal{G}$ and $\mathcal{G}'$ are tangent along C, but Theorem 3.1 implies that this is not the case, thus we can suppose $\alpha_1 \neq 0$. Analogously we also assume $\beta_2 \neq 0$.

If $b_{2,3} = 1$ we consider the equations

$$a^1_{3,4} = a^1_{3,5} = a^1_{4,5} = a^1_{3,6} = a^1_{4,6} = a^1_{5,6} = a^1_{3,7} = a^1_{4,7} = a^1_{5,7} = a^1_{6,7} = 0$$
$$a^2_{3,4} = a^2_{3,5} = a^2_{4,5} = a^2_{3,6} = a^2_{4,6} = a^2_{5,6} = a^2_{3,7} = a^2_{4,7} = a^2_{5,7} = a^2_{6,7} = 0$$

and, using $a^1_{i,j}$, $3 \leq i < j \leq 7$, $a^2_{3,4}, a^2_{3,5}, a^2_{3,6}, a^2_{3,7}$, we find

$$b_{2,4}, b_{3,4}, b_{2,5}, b_{3,5}, b_{4,5}, b_{2,6}, b_{3,6}, b_{4,6}, b_{5,6}, b_{2,7}, b_{3,7}, b_{4,7}, b_{5,7}, b_{6,7}$$

depending on α_1, γ_1, β_2 and γ_2.

We replace them in the remaining equations and use Groebner basis in order to write the ideal

$$\langle a^2_{4,5}, a^2_{4,6}, a^2_{4,7}, a^2_{5,6}, a^2_{5,7}, a^2_{6,7} \rangle = \langle \gamma_2, \alpha_1\beta_2 \rangle$$

but this implies that $\alpha_1\beta_2 = 0$, which is not possible.

If $b_{2,3} = 0$ with a similar argument we arrive in $\lambda_1 = \lambda_2 = 0$ and thus we conclude by Lemma 4.2.

We finish by giving the idea for proving the second part of Theorem 1.1. Assume then that $C^2 = n \geq 2$ and (U, C) admits two fibrations transverse to C. We can prove that the tangency locus is not C. Now, blowing up the curve C at n different points and looking to the tangency locus between the Riccati foliations obtained in the neighborhood of the transform of C, which has zero self-intersection, we see that $C^2 = 2$ and the fibrations have empty tangency locus. We just take first integrals $g, g' : U \to C \simeq \mathbb{P}^1$ defined by the two fibrations and remark that the map $(g, g') : U \to \mathbb{P}^1 \times \mathbb{P}^1$ is a biholomorphism over a neighborhood of the diagonal curve.

All the results of this note will be detailed and completed in [1].

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