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Differential geometry

On the blow-up of points on locally conformally symplectic manifolds

Sur les éclatements de points de variétés localement conformément symplectiques

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A R T I C L E I N F O

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A B S T R A C T

We study the blow-up of a locally conformally symplectic manifold at some point, and show that such blow-up also admits a locally conformally symplectic structure.

R É S U M É

Nous étudions l'éclatement d'une variété localement conformément symplectique en un certain point, et montrons que le *blow-up* admet aussi une structure localement conformément symplectique.

1. Introduction

Inspired by the work of McDuff [11] on the blow-up of symplectic manifolds, we consider in this note the blow-up of Lcs manifold at a point. We show that the blow-up of a locally conformally symplectic manifold at some point also admits a locally conformally symplectic structure (see Theorem 1.2). As an application of this result, we obtain many more new examples of locally conformally symplectic manifolds without any locally conformally Kähler metric (see Corollary 2.4).

In [9], Lee introduced the notion of locally conformally symplectic structure, since then it has been studied extensively by Vaisman [14,15], Haller and Rybicki [8], Banyaga [3], Bande and Kotschick [1,2], etc. We refer the reader to Vaisman [14,15] and Banyaga [3] for a more detailed discussion on locally conformally symplectic geometry.

Next, let us recall the definition and the basic notions.

Definition 1.1. A *locally conformally symplectic structure* (for short *Lcs structure*) on a manifold M is a non-degenerate 2-form ω such that there exist an open covering $\{U_i\}_{i\in I}$ of M and a family of smooth functions $f_i : U_i \to \mathbb{R}$ such that $e^{f_i}\omega|_{U_i}$ is a symplectic form on U_i.

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In particular, Lcs manifolds must be orientable and even dimensional. Assume M is a manifold of real dimension $2n \geq 4$. In loc. cit., Lee showed an equivalent definition as follows: a non-degenerate 2-form ω is Lcs if and only if ω satisfies

$$\mathrm{d}\omega = \theta \wedge \omega \tag{1}$$

for some closed 1-form θ, which is uniquely determined by ω since ω is non-degenerate and the wedge product with ω is injective on 1-forms, and θ is called *Lee form*. Assume that M is a complex Hermitian manifold with the Kähler form ω. We say that ω is a *locally conformally Kähler metric* (*LcK metric* for short) if ω satisfies Eq. (1) for some closed 1-form θ; if $\theta = 0$, then ω is called a *Kähler metric*.

However, unlike in the case of compact Kähler manifolds, there is a lack of topological obstructions on compact LcK manifolds, so it is not easy to exclude that a compact manifold carries a LcK metric. Motivated by this fact, in [12], Ornea and Verbitsky proposed a question: *Construct a compact Lcs manifold which admits no LcK metrics* (see [12, Open Problem 1]). After that, Bande and Kotschick [2] obtained the first examples which answered this question (see [2, Section 5.1]). Recently, in [4], Bazzoni and Marrero constructed a compact Lcs nilmanifold which admits no LcK metrics (see [4, Theorem 1]).

We know that there exists no general technique for constructing Lcs manifolds. However, we will show the following result.

Theorem 1.2. *Let M be a Lcs manifold and $\hat{M}_p$ be the blow-up of M at a point $p \in M$. Then $\hat{M}_p$ also admits a Lcs structure.*

This result allows us to construct a new Lcs manifold from a given one. By combining with the examples of Bande and Kotschick [2] and Bazzoni and Marrero [4], we may obtain many more examples of Lcs manifolds which admit no LcK metrics (see Corollary 2.4).

2. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2.

First, we outline the construction of the blow-up of a Lcs manifold at a point in order to fix some notation. We refer the reader to McDuff [11] for a more detailed presentation.

Let M be a Lcs manifold of real dimension $2n \geq 4$ and let p be a point of M. Since M is locally conformally symplectic, we can choose a complex structure on each tangent space T_pM. Then T_pM is a complex bundle over $\{p\}$, and one can form its projective bundle

$$\mathbb{CP}^{n-1} \to \mathbb{CP}^{n-1} \cong \mathbb{P}(T_pM) \to \{p\},$$

and also form the "tautological" line bundle L over $\mathbb{P}(T_pM)$ as follows: L is the subbundle of $\mathbb{P}(T_pM) \times T_pM$ whose fiber is $\{(l, v) \mid v \in l\}$, i.e.

$$L = \{(l, v) \in \mathbb{P}(T_pM) \times T_pM \mid v \in l\}.$$

Then we have the following commutative diagram:

$$\begin{array}{ccccc} L_0 & \longrightarrow & L & \xrightarrow{q_1} & \mathbb{P}(T_pM) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle q_2} \\ T_pM - \{p\} & \longrightarrow & T_pM & \xrightarrow{\varphi} & \{p\} \end{array}$$

where q_1 and π are the projections of L over $\mathbb{P}(T_pM)$ and T_pM respectively, and L_0 is the complement of the zero section in L. It is not difficult to see that L_0 and $T_pM - \{p\}$ are diffeomorphic via π. Moreover, we let V be a disk in the vector space of T_pM, which is symplectic diffeomorphic to a closed tubular neighborhood W of p in M. We denote $\hat{V} := \pi^{-1}(V)$.

Then we can give the following definition of blow-up of a Lcs manifold at a point, and it is similar to define the blow-up at a finite collection of points.

Definition 2.1. The blow-up $\hat{M}_p$ of M at a point $p \in M$ is the smooth manifold

$$\hat{M}_p := \overline{M - W} \cup_{\partial V} \hat{V}$$

where $\partial V \cong \partial \hat{V}$ is identified with ∂W.

Then the map π can be extended to a smooth map $\pi : \hat{M}_p \to M$, which is a diffeomorphism outside of $E := \pi^{-1}(p)$ and $E \cong \mathbb{CP}^{n-1}$.

We are now in the position to prove Theorem 1.2.

Proof of Theorem 1.2. We follow the same strategy as in [16] and [10]. Assume M is a Lcs manifold with Lcs structure ω and its Lee form θ. Let $\pi : \hat{M}_p \to M$ be the blow-up of M at a point $p \in M$. We denote by $E := \pi^{-1}(p)$ the exceptional divisor of the blow-up. Then the pullback form $\pi^*\omega$ is a 2-form on $\hat{M}_p$, which is non-degenerate on $\hat{M}_p - E$. As ω satisfies Eq. (1), $\pi^*\omega$ satisfies

$$\mathrm{d}(\pi^*\omega) = (\pi^*\theta) \wedge (\pi^*\omega). \tag{2}$$

As E is simply-connected, there exists an open neighborhood U of E in $\hat{V}$ with $H^1_{\mathrm{d}R}(U) = 0$. Since $\mathrm{d}(\pi^*\theta) = \pi^*\mathrm{d}\theta = 0$, we can find a smooth function $f : \hat{M}_p \to \mathbb{R}$ such that $\pi^*\theta = -\mathrm{d}f$ on U. We denote $\tilde{\omega} := \mathrm{e}^f \pi^*\omega$, then by Eq. (2) we have

$$\begin{aligned}
\mathrm{d}\tilde{\omega} &= \mathrm{d}(\mathrm{e}^f(\pi^*\omega)) = \mathrm{e}^f \mathrm{d}f \wedge (\pi^*\omega) + \mathrm{e}^f \mathrm{d}(\pi^*\omega) \\
&= \mathrm{e}^f \mathrm{d}f \wedge (\pi^*\omega) + \mathrm{e}^f (\pi^*\theta) \wedge (\pi^*\omega) \\
&= (\mathrm{d}f + \pi^*\theta) \wedge \tilde{\omega},
\end{aligned}$$

and the 1-form $\tilde{\theta} := \mathrm{d}f + \pi^*\theta$ satisfies $\tilde{\theta}|_U = 0$. Since θ is closed, we obtain $\mathrm{d}\tilde{\theta} = \mathrm{d}\mathrm{d}f + \mathrm{d}\pi^*\theta = 0$.

We know that $\tilde{\omega}$ is not non-degenerate on E. From the above construction of blow-up point, we can choose a non-degenerate 2-form α on E such that the support of α is in U (see [11, Section 3] for detailed discussion). In fact, since $\pi : L_0 \to T_pM - \{p\}$ is a diffeomorphism and $H^2_{\mathrm{d}R}(T_pM - \{p\}) = 0$, thus $q_1^*\omega_{\mathrm{FS}}$ is exact on $L_0(= L - E)$ where ω_{FS} is the Fubini–Study form on $\mathbb{P}(T_pM)$. Denote by $q_1^*\omega_{\mathrm{FS}} = \mathrm{d}\beta$ on L_0. Let b be a smooth bump function having support in U and which equals 1 near E. Then we may define α as follows:

$$\alpha = \begin{cases} q_1^*\omega_{\mathrm{FS}}, & \text{on } E \\ \mathrm{d}(b\beta), & \text{on } U - E. \end{cases}$$

Since $q_{1*} : TE \to T\mathbb{P}(T_pM)$ is an isomorphism, hence α is non-degenerate on E and supported in U. Moreover, we have the following.

Claim 2.2. *There exists a small enough $\varepsilon > 0$ such that*

$$\hat{\omega} := \tilde{\omega} + \varepsilon\alpha \tag{3}$$

is a non-degenerate 2-form on $\hat{M}_p$.

Proof. Since π is diffeomorphism on $\hat{M}_p - E$, hence $\tilde{\omega}$ is non-degenerate on $\hat{M}_p - E$. Therefore, there is a small enough $\varepsilon > 0$ such that $\tilde{\omega} + \varepsilon\mathrm{d}(b\beta)$ is non-degenerate on $\hat{M}_p - E$. Because $q_1^*\omega_{\mathrm{FS}}$ is non-degenerate on E, so $\tilde{\omega} + \varepsilon q_1^*\omega_{\mathrm{FS}}$ is non-degenerate on E. Consequently, the 2-form $\tilde{\omega} + \varepsilon\alpha$ is non-degenerate on $\hat{M}_p$. □

Since the support of $\tilde{\theta}$ is disjoint with U and the support of α is in U, then $\tilde{\theta} \wedge \alpha = 0$. By Eq. (3), we get

$$\begin{aligned}
\mathrm{d}\hat{\omega} &= \mathrm{d}\tilde{\omega} = \tilde{\theta} \wedge \tilde{\omega} \\
&= \tilde{\theta} \wedge \tilde{\omega} + \tilde{\theta} \wedge \varepsilon\alpha = \tilde{\theta} \wedge (\tilde{\omega} + \varepsilon\alpha) \\
&= \tilde{\theta} \wedge \hat{\omega}.
\end{aligned}$$

If we define $\hat{\theta} := \tilde{\theta}$, then $\hat{\theta}$ is a closed 1-form such that the non-degenerate 2-form $\hat{\omega}$ satisfies

$$\mathrm{d}\hat{\omega} = \hat{\theta} \wedge \hat{\omega}.$$

This completes the proof of Theorem 1.2. □

Remark 2.3. In fact, one can use the same idea to study locally conformally symplectic orbifolds. In the same spirit as Cavalcanti, Fernández and Muñoz [5, Theorem 3.3] and Fernández and Muñoz [7] for resolving symplectic singularities, one can show that any locally conformally symplectic orbifold has a locally conformally symplectic resolution.

A direct consequence of Theorem 1.2 is the following.

Corollary 2.4. *Assume that M is a Lcs manifold without any LcK metric. Then the blow-up $\hat{M}_p$ of M at a point also admits a Lcs structure without any LcK metric. In particular, the blow-up of M at more points also admits Lcs structure without any LcK metric.*

Proof. Under the hypothesis, Theorem 1.2 implies that $\hat{M}_p$ admits Lcs structure. By Miyaoka's extension theorem (see [13] or cf. [6, Theorem 2.7]), if $\hat{M}_p$ admits a LcK metric, then M also admits a LcK metric. This contradicts the hypothesis and thus the corollary holds. □

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