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Koebe sets for certain classes of circularly symmetric functions

Ensembles de Koebe pour certaines classes de fonctions circulairement symétriques

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ABSTRACT

A function f analytic in $\Delta \equiv \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$, is said to be circularly symmetric if the intersection of the set $f(\Delta)$ and a circle $\{\zeta \in \mathbb{C} : |\zeta| = \varrho\}$ has one of three forms: the empty set, the whole circle, an arc of the circle which is symmetric with respect to the real axis and contains ϱ. By X we denote the class of all circularly symmetric functions, and by Y the subclass of X consisting of univalent functions.

The main concern of the paper is to determine two Koebe sets: for the class $Y \cap K(i)$ of circularly symmetric functions that are convex in the direction of the imaginary axis and for the class $Y \cap S^*$ of circularly symmetric and starlike functions, i.e. sets of the form $K_{Y\cap K(i)} = \bigcap_{f\in Y\cap K(i)} f(\Delta)$ and $K_{Y\cap S^*} = \bigcap_{f\in Y\cap S^*} f(\Delta)$. In the last section of the paper, we consider a similar problem for the class $Y \cap S^* \cap K(i)$.

RÉSUMÉ

Une fonction f analytique dans $\Delta \equiv \{\zeta \in \mathbf{C} : |\zeta| < 1\}$, normalisée par $f(0) = f'(0) - 1 = 0$, est dite circulairement symétrique si l'intersection de l'ensemble $f(\Delta)$ et d'un cercle $\{\zeta \in \mathbf{C} : |\zeta| = \rho\}$ est, soit l'ensemble vide, soit le cercle complet, soit un arc de cercle symétrique par rapport à l'axe réel et contenant ρ. Nous notons X la classe des fonctions circulairement symétriques et Y la sous-classe de X des fonctions univalentes.

L'objet de cette Note est de déterminer les ensembles de Koebe pour la classe $Y \cap K(i)$ des fonctions circulairement symétriques qui sont convexes dans la direction de l'axe imaginaire et pour la classe $Y \cap S^*$ des fonctions circulairement symétriques qui sont étoilées, c'est-à-dire de déterminer les ensembles $K_{Y\cap K(i)} = \bigcap_{f\in Y\cap K(i)} f(\Delta)$ et $K_{Y\cap S^*} = \bigcap_{f\in Y\cap S^*} f(\Delta)$. Dans la dernière section, nous considérons ce problème pour la sous-classe $Y \cap S^* \cap K(i)$.

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1. Introduction

In 1955, Jenkins published an article [3], in which he introduced the idea of a circularly symmetric function. Namely, an analytic function f, normalized by $f(0) = f'(0) - 1 = 0$, is said to be circularly symmetric if the set $f(\Delta)$, where $\Delta \equiv \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, is a circularly symmetric set. Further, a set D is called circularly symmetric when, for each $\varrho \in \mathbb{R}^+$, a set $D \cap \{\zeta \in \mathbb{C} : |\zeta| = \varrho\}$ has one of three forms: the empty set, the whole circle, an arc of the circle which is symmetric with respect to the real axis and contains ϱ. Let us denote by X the class of all circularly symmetric functions, and by Y the subclass of X consisting of these functions in X that are univalent.

In his paper, Jenkins gave some geometric properties of circularly symmetric functions. We need two of them. Firstly, for each $f \in X$, a function $F(\varphi) \equiv |f(r\,\mathrm{e}^{\mathrm{i}\varphi})|$ is nonincreasing for $\varphi \in (0, \pi)$ and nondecreasing for $\varphi \in (\pi, 2\pi)$. Secondly, each function $f \in X$ has real coefficients. This property results in the symmetry of the set $f(\Delta)$ with respect to the real axis.

From the time of the publication of Jenkins's paper onwards, circularly symmetric functions have been considered only in a few papers. It is worth recalling the paper of M. and W. Szapiel [7]. They gave two representation formulae: for circularly symmetric functions that are additionally locally univalent and for circularly symmetric starlike functions. Deng in papers [1,2] discussed the logarithmic coefficients of $f \in Y$. The authors of [5] solved a few coefficient problems and obtained some distortion theorems for certain subclasses of X.

At the end of this overview, we would like to recall the paper that inspired us to further research in this direction. In 1967, Krzyż and Reade [4] found the set $K_Y = \bigcap_{f\in Y} f(\Delta)$, i.e. the Koebe set for Y. It is worth noting that the structural formula for a function in Y was not then (and still is not) known. However, it was possible to determine the Koebe set in this class.

In this paper, we shall determine two other Koebe sets: for the class $Y \cap K(i)$ of circularly symmetric functions that are convex in the direction of the imaginary axis and for the class $Y \cap S^*$ of circularly symmetric and starlike functions. The representation formula for $Y \cap S^*$ is known. Namely [7],

$$f \in Y \cap S^* \Leftrightarrow \frac{zf'(z)}{f(z)} \in \tilde{T} \cap P\,, \tag{1}$$

where $\tilde{T}$ is the class of typically real functions, i.e. functions satisfying $\mathrm{Im}\, z\, \mathrm{Im}\, f(z) \ge 0$, $z \in \Delta$, and P is the class of functions p with positive real part, $p(0) = 1$. No analogous formula exists for functions in $Y \cap K(i)$.

Similarly to [4], the results in this paper are obtained using a geometric method. First, the extremal sets will be proposed. Next, applying the technique of subordination, we will find $K_{Y\cap K(i)} = \bigcap_{f\in Y\cap K(i)} f(\Delta)$ and $K_{Y\cap S^*} = \bigcap_{f\in Y\cap S^*} f(\Delta)$.

2. Koebe set for $Y \cap K(i)$

For any $\varrho > 0$, we denote by $\tilde{D}_{\varrho,\theta}$ the set of the form

$$\tilde{D}_{\varrho,\theta} = \begin{cases} \Delta_\varrho \cup \{w : \mathrm{Re}\, w > \varrho\cos\theta\}\,, & \theta \in (0, \pi] \\ \Delta_\varrho\,, & \theta = 0\,. \end{cases}$$

If $\theta \in (0, \pi)$, then the boundary of $\tilde{D}_{\varrho,\theta}$ consists of an arc of the circle centered in the origin with radius ϱ and two vertical rays emanating from $\varrho\,\mathrm{e}^{\mathrm{i}\theta}$ and $\varrho\,\mathrm{e}^{-\mathrm{i}\theta}$. It is easily seen that the measure of the external angles between the rays and the circular arc is equal to θ. In the limiting case, $\tilde{D}_{\varrho,\theta}$ becomes Δ_ϱ for $\theta = 0$ or a half-plane $\{w : \mathrm{Re}\, w > -\varrho\}$ when $\theta = \pi$.

According to the Riemann theorem, there exists a univalent function $\tilde{f}_{\varrho,\theta}$, such that $\tilde{f}_{\varrho,\theta}(\Delta) = \tilde{D}_{\varrho,\theta}$, with $\tilde{f}_{\varrho,\theta}(0) = 0$ and $\tilde{f}'_{\varrho,\theta}(0) > 0$. We define $f_\theta = \tilde{f}_{\varrho,\theta}/\tilde{f}'_{\varrho,\theta}(0)$ and $D_\theta = f_\theta(\Delta)$.

From the description of $\tilde{D}_{\varrho,\theta}$, it follows that $\tilde{f}_{\varrho,\theta}$ is circularly symmetric and convex in the direction of the imaginary axis. Moreover, $f_\theta \in Y \cap K(i)$.

The sets $\tilde{D}_{\varrho,\theta}$ are the image domains of Δ under functions of the form $f_4 \circ f_3 \circ f_2 \circ f_1$, where $f_1(z) = \arctan z$, $f_3(z) = \tan z$, and f_2, f_4 are affine functions. Let us denote by h a function

$$h(z) = \tan(a \cdot \arctan z + b)\ ,\ a, b \in \mathbb{R}\,. \tag{2}$$

Since the image set of Δ under $\arctan z$ is a vertical strip $\{\zeta \in \mathbb{C} : |\mathrm{Im}\,\zeta| < \frac{\pi}{4}\}$, choosing

$$a = 2 - \theta/\pi \quad \text{and} \quad b = \theta/4 \tag{3}$$

we obtain the function

$$z \mapsto (2 - \theta/\pi)\arctan z + \theta/4$$

mapping the disk Δ onto the set $\{\zeta \in \mathbb{C} : -\frac{\pi}{2} + \frac{\theta}{2} < \mathrm{Im}\,\zeta < \frac{\pi}{2}\}$. This function is typically real. Furthermore, the semicircles that lie in the right and in the left half-planes correspond to straight lines $\mathrm{Im}\,\zeta = \frac{\pi}{2}$ and $\mathrm{Im}\,\zeta = -\frac{\pi}{2} + \frac{\theta}{2}$, respectively.

Observe now that the function

$$\tan\zeta = \frac{1}{i}\frac{1-\mathrm{e}^{-2\mathrm{i}\zeta}}{1+\mathrm{e}^{-2\mathrm{i}\zeta}}$$

maps vertical straight lines $\zeta = k\frac{\pi}{2} + \mathrm{i}\,t$, $t \in \mathbb{R}$, where k is a fixed real number, $k \in [-1, 1]$, onto sets Ω_k:

$$\begin{aligned}
\Omega_{-1} &= \left\{\mathrm{i}\,\varrho \;:\; \varrho \in (-\infty, -1] \cup [1, \infty)\right\} \\
\Omega_k &= T\left(-\cot k\pi\,, -\frac{1}{\sin k\pi}\right) \cap \{w : \operatorname{Re} w \le 0\} \text{ for } k \in (-1, 0) \\
\Omega_0 &= \left\{\mathrm{i}\,\varrho \;:\; \varrho \in [-1, 1]\right\} \\
\Omega_k &= T\left(-\cot k\pi\,, \frac{1}{\sin k\pi}\right) \cap \{w : \operatorname{Re} w \ge 0\} \text{ for } k \in (0, 1) \\
\Omega_1 &= \left\{i\varrho \;:\; \varrho \in (-\infty, -1] \cup [1, \infty)\right\} \,.
\end{aligned}$$

The symbols $T(w_0, r)$ and $\Delta(w_0, r)$ stand for $|w - w_0| = r$ and $|w - w_0| < r$, respectively. For every fixed $k \in (-1, 0) \cup (0, 1)$, the set Ω_k is a circular arc with endpoints in $-\mathrm{i}$ and i. From the above, Lemma 1 follows.

Lemma 1. *For every fixed $k \in (-1, 0)$, the vertical strip $\{\zeta \in \mathbb{C} : k\frac{\pi}{2} < \operatorname{Im}\zeta < \frac{\pi}{2}\}$ is univalently mapped by* $\tan z$ *onto*

$$\Delta\left(-\cot k\pi\,, -\frac{1}{\sin k\pi}\right) \cup \{w : \operatorname{Re} w > 0\}\,. \tag{4}$$

It can be easily checked that the external angles between the vertical rays and the circular arc are equal to $\theta = \pi(1+k)$. Because of this correspondence, from now on, we will use θ as the parameter instead of k. The following relation holds $k \in (-1, 0) \Leftrightarrow \theta \in (0, \pi)$.

The above facts lead to

$$h(\Delta) = \Delta\left(-\cot\theta, \frac{1}{\sin\theta}\right) \cup \{w : \operatorname{Re} w > 0\}\,. \tag{5}$$

Now, composing h and a Möbius transformation, we obtain

$$H(z) = \frac{h\left(\frac{z+x}{1+xz}\right) - h(x)}{(1-x^2)h'(x)}\;,\; x \in (-1, 1)\,. \tag{6}$$

Certainly, $H(0) = 0$, $H'(0) = 1$.

Hence, $H(\Delta)$ coincides with the image set of (5) under a translation and a homothetic transformation. Taking x that $h(x) = -\cot\theta$, the boundary of $H(\Delta)$ contains a circle arc centered on the origin. Hence

$$\tan\left((2-\theta/\pi)\arctan x + \theta/4\right) = -\cot\theta\,.$$

Simple calculation leads to

$$x = \tan\left(\frac{\pi}{4}\cdot\frac{3\theta - 2\pi}{2\pi - \theta}\right)\,. \tag{7}$$

For this x there is

$$h'(x) = (2-\theta/\pi)\cos^2\left(\frac{\pi}{4}\cdot\frac{3\theta - 2\pi}{2\pi - \theta}\right) / \sin^2\theta \tag{8}$$

and

$$1 - x^2 = \cos\left(\frac{\pi}{2}\cdot\frac{3\theta - 2\pi}{2\pi - \theta}\right) / \cos^2\left(\frac{\pi}{4}\cdot\frac{3\theta - 2\pi}{2\pi - \theta}\right)\,. \tag{9}$$

The final form of H is the following

$$H(z) = \frac{\sin^2\theta}{(2-\theta/\pi)\cos\left(\frac{\pi}{2}\cdot\frac{3\theta-2\pi}{2\pi-\theta}\right)}\left(\tan\left((2-\theta/\pi)\arctan\left(\frac{z+x}{1+xz}\right) + \theta/4\right) + \cot\theta\right)\,. \tag{10}$$

Since H depends on the parameter θ, we can write H_θ instead of H. We have proved Lemma 2.

Lemma 2. *For every fixed* $\theta \in (0, \pi)$ *and x given by (7), the function* H_θ *is in* $Y \cap K(i)$.

Moreover, translating (5) by a vector $\cot\theta$ and applying homothety with a scale factor $s = 1/(1-x^2)h'(x)$, we obtain $H_\theta(\Delta)$. From (8), (9) and (5) one can conclude with Lemma 3.

Lemma 3. *For every fixed* $\theta \in (0, \pi)$ *we have*

$$H_\theta(\Delta) = \Delta_{R(\theta)} \cup \{w : \operatorname{Re} w > R(\theta)\cos\theta\} ,$$

where

$$R(\theta) = \frac{\pi \sin\theta}{(2\pi - \theta)\sin\frac{\pi\theta}{2\pi-\theta}} . \tag{11}$$

Now we are ready to establish the main result.

Theorem 1. *The Koebe set* $K_{Y\cap K(i)}$ *is a bounded domain, symmetric with respect to the real axis. Its boundary is given by the polar equation* $w = \varrho(\theta)\,\mathrm{e}^{\mathrm{i}\theta}$, $\theta \in (-\pi, \pi]$, *where*

$$\varrho(\theta) = \begin{cases} 1 & \text{for} \quad \theta = 0 \\ R(|\theta|) & \text{for} \quad \theta \in (-\pi, 0) \cup (0, \pi) \\ 1/2 & \text{for} \quad \theta = \pi \ . \end{cases} \tag{12}$$

Proof. Let K denote the Koebe set for $Y \cap K(i)$ that we are looking for. Because of the real coefficients of functions in $Y \cap K(i)$, the set K is symmetric with respect to the real axis.

At the beginning, we shall show that $K \cap \mathbb{R} = (-1/2, 1)$. According to McGregor [6], the Koebe set for the class $K_R(i)$ of functions with real coefficients convex in the direction of the imaginary axis coincides with $\Delta_{1/2}$. Since $Y \cap K(i) \subset K_R(i)$, we have $\Delta_{1/2} \subset K$. What is more, $f(z) = \frac{z}{1-z}$ also belongs to the class $Y \cap K(i)$ and $f(-1) = -1/2$. Thus $-1/2 \in \partial K$.

On the other hand, if $f(1)$ for some $f \in Y \cap K(i)$ were less than 1, then $|f(\mathrm{e}^{\mathrm{i}\varphi})|$ would be less than 1 for each $\varphi \in [0, 2\pi]$. It would indicate that f is subordinated to the identity function. But this is not possible. It means that for any circularly symmetric function f, there is $f(1) \geq 1$ and equality holds only for $f(z) = z$. Hence $1 \in \partial K$.

Let $w = \varrho\,\mathrm{e}^{\mathrm{i}\theta}$ be a point from the boundary of K and let $\theta \in (0, \pi)$, $\varrho > 0$. It means that there exists a function $f \in Y \cap K(i)$ such that $w \in \partial f(\Delta)$.

The convexity of f in the direction of the imaginary axis implies that for $t \geq 0$ we have

$$f(z) \neq \varrho\cos\theta + \mathrm{i}(\varrho\sin\theta + t)$$

and

$$f(z) \neq \varrho\cos\theta - \mathrm{i}(\varrho\sin\theta + t) .$$

Additionally, f is circularly symmetric. Consequently, $f(\Delta)$ is disjoint from the arc of the circle $\varrho\,\mathrm{e}^{\mathrm{i}\psi}$, $\psi \in [\theta, 2\pi - \theta]$. The above facts confirm that

$$f(\Delta) \subset \tilde{D}_{\varrho,\theta} ,$$

or equivalently

$$f(\Delta) \subset \tilde{f}_{\varrho,\theta}(\Delta) . \tag{13}$$

The form of the sets $\tilde{D}_{\varrho,\theta}$ and $D_\theta = H_\theta(\Delta)$ makes

$$D_\theta = \frac{R(\theta)}{\varrho}\tilde{D}_{\varrho,\theta} ,$$

where $R(\theta)$ is given by (11). Hence

$$\tilde{f}_{\varrho,\theta}(z) = \frac{\varrho}{R(\theta)} H_\theta(z) . \tag{14}$$

Since H_θ is univalent, from (13) and (14) we conclude

$$f \prec \frac{\varrho}{R(\theta)} H_\theta .$$

For this reason

$$1 = f'(0) \leq \frac{\varrho}{R(\theta)} H'_\theta(0) = \frac{\varrho}{R(\theta)} ,$$

which gives $\varrho \geq R(\theta)$. It means that for $\theta \in (0, \pi)$ the extremal functions are H_θ. □

Observe that

$$\lim_{\theta\to 0} R(\theta) = 1 \quad \text{and} \quad \lim_{\theta\to\pi} R(\theta) = 1/2\,.$$

3. Koebe set for $Y \cap S^*$

For any $\varrho > 0$ and $\theta \in [0,\pi]$, we denote by $\tilde{E}_{\varrho,\theta}$ the set of the form

$$\tilde{E}_{\varrho,\theta} = \Delta_\varrho \cup \{w : |\arg w| < \theta\}\,.$$

From this definition, one infers

$$\tilde{E}_{\varrho,0} = \Delta_\varrho \quad \text{and} \quad \tilde{E}_{\varrho,\pi} = \mathbb{C} \setminus \{x \in \mathbb{R} : x \le -\varrho\}\,.$$

For $\theta \in (0,\pi)$, the boundary of $\tilde{E}_{\varrho,\theta}$ consists of an arc of the circle centered on the origin with radius ϱ and two rays emanating from $\varrho\,\mathrm{e}^{\mathrm{i}\theta}$ and $\varrho\,\mathrm{e}^{-\mathrm{i}\theta}$; the prolongations of these rays contain the origin. The slope angles between the rays and the positive real half-axis are equal to θ and $-\theta$.

According to the Riemann theorem, there exists an univalent function $\tilde{g}_{\varrho,\theta}$, such that $\tilde{g}_{\varrho,\theta}(\Delta) = \tilde{E}_{\varrho,\theta}$, with $\tilde{g}_{\varrho,\theta}(0) = 0$ and $\tilde{g}'_{\varrho,\theta}(0) > 0$. Additionally, we define $g_\theta = \tilde{g}_{\varrho,\theta}/\tilde{g}'_{\varrho,\theta}(0)$ and $E_\theta = g_\theta(\Delta)$.

From the definition of $\tilde{E}_{\varrho,\theta}$, we conclude that $\tilde{g}_{\varrho,\theta}$ is circularly symmetric and starlike. Furthermore, $g_\theta \in Y \cap S^*$.

Lemma 4. *Let $\theta \in [0,\pi]$ be fixed and let $g_\theta \in Y \cap S^*$ map Δ onto E_θ. Then*

$$\frac{z g'_\theta(z)}{g_\theta(z)} = \sqrt{1 + b^2 \frac{z}{(1-z)^2}} \tag{15}$$

for some $b \in [0,2]$.

Proof. The equality

$$\left.\frac{z f'(z)}{f(z)}\right|_{z=r\,\mathrm{e}^{\mathrm{i}\varphi}} = \frac{\partial}{\partial\varphi}\left(\arg f(r\,\mathrm{e}^{\mathrm{i}\varphi})\right) - i\frac{\partial}{\partial\varphi}\left(\log\left|f(r\,\mathrm{e}^{\mathrm{i}\varphi})\right|\right)$$

results in the following relations for a function g_θ and some $\varphi_0 \in (0,\pi)$:

$$\mathrm{Re}\left.\frac{z g'_\theta(z)}{g_\theta(z)}\right|_{z=\mathrm{e}^{\mathrm{i}\varphi}} = 0 \quad \text{for } \varphi \in (0,\varphi_0] \tag{16}$$

$$\mathrm{Im}\left.\frac{z g'_\theta(z)}{g_\theta(z)}\right|_{z=\mathrm{e}^{\mathrm{i}\varphi}} = 0 \quad \text{for } \varphi \in [\varphi_0, 2\pi - \varphi_0] \tag{17}$$

$$\mathrm{Re}\left.\frac{z g'_\theta(z)}{g_\theta(z)}\right|_{z=\mathrm{e}^{\mathrm{i}\varphi}} = 0 \quad \text{for } \varphi \in [2\pi - \varphi_0, 2\pi)\,. \tag{18}$$

From (1) we know that $\frac{z g'_\theta(z)}{g_\theta(z)} \in \tilde{T} \cap P$. Given the above, a function $p(z) = \frac{z g'_\theta(z)}{g_\theta(z)}$ maps Δ onto the right half-plane with some segment excluded; the segment lies on the real axis and has one endpoint in the origin. For this reason, we can take

$$p(z) = \sqrt{1 + b^2 \frac{z}{(1-z)^2}}\,. \tag{19}$$

If a positive number b in (19) is such that $1 - b^2/4 \ge 0$, then the image of the unit disk under a function $1 + b^2\frac{z}{(1-z)^2}$ is $\mathbb{C} \setminus \{x \in \mathbb{R} : x \le 1 - b^2/4\}$. Hence, for $b \in [0,2]$:

$$p(\Delta) = \left\{ w : \mathrm{Re}\, w > 0,\, w \notin \left(0, \sqrt{1 - b^2/4}\right]\right\}\,.$$

Moreover, for $\varphi \in (0, 2\pi)$, there is

$$p(\mathrm{e}^{\mathrm{i}\varphi}) = \frac{\sqrt{4\sin^2\frac{\varphi}{2} - b^2}}{2\sin\frac{\varphi}{2}}\,.$$

This results in

$$\mathrm{Re}\, p(\mathrm{e}^{\mathrm{i}\varphi}) = 0 \quad \text{for} \quad \varphi \in (0,\varphi_0] \cup [2\pi - \varphi_0, 2\pi)$$

and

$$\operatorname{Im} p(\mathrm{e}^{\mathrm{i}\varphi}) = 0 \quad \text{for} \quad \varphi \in [\varphi_0, 2\pi - \varphi_0]\,,$$

where

$$\varphi_0 = 2\arcsin(b/2)\,. \qquad \square \tag{20}$$

For $b = 0$ directly from (15), we obtain $g_\theta(z) = z$. Combining it with the description of E_θ, one can see that in this case $\theta = 0$. Similarly, for $b = 2$ there is $g_\theta(z) = \frac{z}{(1-z)^2}$. In this case, θ is equal to π. The general correspondence between b and θ is given in the next lemma.

Lemma 5. *Let $\theta \in [0, \pi]$ be fixed and let g_θ be defined by (15). Then*

1. *for $b \in (0, 2]$ a function g_θ is of the form*

$$g_\theta(z) = \frac{4z}{(1 + z + q(z))^2} \cdot \left(\frac{b - 1 + z + q(z)}{b + 1 - z - q(z)}\right)^b\,, \tag{21}$$

where

$$q(z) = \sqrt{1 + (b^2 - 2)z + z^2}\,, \tag{22}$$

2. *$g_\theta(\Delta) = \tilde{E}_{\varrho,\theta}$, where*

$$\varrho = \varrho(\theta) = \begin{cases} \left(1 - \left(\frac{\theta}{\pi}\right)^2\right)^{-1} \left(\frac{\pi-\theta}{\pi+\theta}\right)^{\frac{\theta}{\pi}}, & \theta \in [0, \pi) \\ 1/4, & \theta = \pi\,. \end{cases} \tag{23}$$

Proof.
ad 1. Consider the functions g_θ of the form (21), where $b \in (0, 2]$. From the logarithmic derivative of g_θ, we obtain

$$\frac{z g_\theta'(z)}{g_\theta(z)} = 1 + 2z\left(1 + q'(z)\right)\left[\frac{b^2}{b^2 - (1 - z - q(z))^2} - \frac{1}{q(z) + 1 + z}\right].$$

But (22) leads to

$$b^2 z = q(z)^2 - (1 - z)^2\,,$$

$$1 + q'(z) = 1 + \frac{2z + b^2 - 2}{2q(z)} = \frac{(q(z) + z + 1)(q(z) + z - 1)}{2zq(z)}\,,$$

and

$$\frac{b^2}{b^2 - (1 - z - q(z))^2} = \frac{q(z)^2 - (1 - z)^2}{q(z)^2 - (1 - z)^2 - z(1 - z - q(z))^2} = \frac{q(z) + 1 - z}{(1 - z)(q(z) + 1 + z)}\,.$$

The two above relations and the correspondence $q(z) = (1 - z)p(z)$ that connects p and q defined by (19) and (22) respectively yield that

$$\frac{z g_\theta'(z)}{g_\theta(z)} = 1 + \frac{q(z) + z - 1}{q(z)}\left(\frac{q(z) + 1 - z}{1 - z} - 1\right) = \frac{q(z)}{1 - z} = p(z)\,,$$

which assures us that the functions g_θ satisfy (15) for $b \in (0, 2]$.

ad 2. By Lemma 4 $g_\theta(\Delta) = E_\theta$. We shall prove that $E_\theta = \tilde{E}_{\varrho,\theta}$, where $\varrho = \varrho(\theta)$ is given by (23). In other words, the boundary of E_θ contains an arc of the circle with radius $\varrho(\theta)$.

For $z_0 = \mathrm{e}^{\mathrm{i}\varphi_0}$, where φ_0 is of the form (20), $\cos\varphi_0 = 1 - b^2/2$, and hence

$$q(z_0) = \sqrt{z_0(2\cos\varphi_0 + b^2 - 2)} = 0\,.$$

Consequently,

$$g_\theta(z_0) = \frac{4z_0}{(1 + z_0)^2} \cdot \left(\frac{b - 1 + z_0}{b + 1 - z_0}\right)^b \quad \text{for} \quad b \in (0, 2)$$

and

$$g_0(z_0) = 1 \quad \text{and} \quad g_\pi(z_0) = -\frac{1}{4}\,,$$

respectively for $b = 0$ and $b = 2$.

If $b \in (0, 2)$, then

$$g_\theta(z_0) = \frac{4}{4-b^2}\left(\frac{b-1+\cos\varphi_0 + \mathrm{i}\sin\varphi_0}{b+1-\cos\varphi_0 - \mathrm{i}\sin\varphi_0}\right)^b = \frac{4}{4-b^2}\left(\frac{1-b/2+\mathrm{i}\sqrt{1-b^2/4}}{1+b/2-\mathrm{i}\sqrt{1-b^2/4}}\right)^b$$

$$= \frac{4}{4-b^2}\left(\frac{1-b/2}{1+b/2}\right)^{b/2}\left(\frac{\sqrt{1-b/2}+\mathrm{i}\sqrt{1+b/2}}{\sqrt{1+b/2}-\mathrm{i}\sqrt{1-b/2}}\right)^b = \frac{4}{4-b^2}\left(\frac{2-b}{2+b}\right)^{b/2} \mathrm{e}^{\mathrm{i}b\frac{\pi}{2}}\,.$$

For this reason, the parameters of the set $\tilde{E}_{\varrho,\theta}$ are given by the following parametric formulae

$$\varrho = \frac{4}{4-b^2}\left(\frac{2-b}{2+b}\right)^{b/2} \tag{24}$$

and

$$\theta = b\frac{\pi}{2}\,, \tag{25}$$

which proves (23). □

The main theorem of this section is as follows.

Theorem 2. *Let $\varrho = \varrho(\theta)$ be defined by (23). The Koebe set $K_{Y\cap S^*}$ is a bounded domain, symmetric with respect to the real axis. Its boundary is given by the polar equation*

$$w = \varrho(|\theta|)\,\mathrm{e}^{\mathrm{i}\theta} \quad \theta \in (-\pi, \pi]\,. \tag{26}$$

Proof. Let K denote the desired Koebe set for $Y \cap S^*$. Because of the real coefficients of the functions in this class, the set K is symmetric with respect to the real axis.

It is known that the Koebe set for the class S of all univalent functions is the one-quarter disk. Hence, $\Delta_{1/4} \subset K$, and in particular, $(-1/4, 1/4) \subset K \cap \mathbb{R}$. But $g(z) = \frac{z}{(1-z)^2} \in Y \cap S^*$, so $-1/4$ cannot be improved. An argument similar to the one given in the proof of Theorem 1 leads to $K \cap \mathbb{R} = (-1/4, 1)$.

Let $w = \varrho\,\mathrm{e}^{\mathrm{i}\theta}$ be a boundary point of K and let $\theta \in (0, \pi)$, $\varrho > 0$. There exists $g \in Y \cap S^*$ such that $w \in \partial g(\Delta)$.

The starlikeness of g provides that for $t \geq 1$

$$g(z) \neq tw \quad \text{and} \quad g(z) \neq t\overline{w}\,.$$

Furthermore, f is circularly symmetric. Consequently, $f(\Delta)$ is disjoint from the arc of the circle $\varrho\,\mathrm{e}^{\mathrm{i}\psi}$, $\psi \in [\theta, 2\pi - \theta]$. Hence,

$$g(\Delta) \subset \tilde{E}_{\varrho,\theta}\,,$$

or equivalently

$$g(\Delta) \subset \tilde{g}_{\varrho,\theta}(\Delta)\,. \tag{27}$$

Due to the form of $\tilde{E}_{\varrho,\theta}$ and $E_\theta = g_\theta(\Delta)$, we can write

$$E_\theta = \frac{\varrho(\theta)}{\varrho}\tilde{E}_{\varrho,\theta}\,,$$

so

$$\tilde{g}_{\varrho,\theta}(z) = \frac{\varrho}{\varrho(\theta)}g_\theta(z)\,. \tag{28}$$

But g_θ is univalent. From (27) and (28)

$$g \prec \frac{\varrho}{\varrho(\theta)}g_\theta\,.$$

By this subordination

$$1 = g'(0) \leq \frac{\varrho}{\varrho(\theta)}g'_\theta(0) = \frac{\varrho}{\varrho(\theta)}\,,$$

which means that $\varrho \geq \varrho(\theta)$ for $\theta \in (0, \pi)$. One can check that $\lim_{\theta\to\pi^-}\varrho(\theta) = \frac{1}{4}$. □

4. Concluding remarks

Summing up, it is worth repeating that the above technique for the determination of Koebe sets does not require the knowledge of class representation formulae. A similar situation can also be observed for the subclass of Y consisting of functions that are starlike and convex in the direction of the imaginary axis. Despite the fact that we do not know a representation formula for $Y \cap S^* \cap K(i)$, it is possible to select extremal sets and to determine the Koebe set in an analogous way as was done in Section 2.

Theorem 3. *The Koebe set $K_{Y\cap S^*\cap K(i)}$ is a bounded domain, symmetric with respect to the real axis. Its boundary is given by the polar equation $w = \varrho(|\theta|)\,\mathrm{e}^{\mathrm{i}\theta} \quad \theta \in (-\pi, \pi]$,*

$$\varrho(\theta) = \begin{cases} \left(1 - \left(\frac{\theta}{\pi}\right)^2\right)^{-1} \left(\frac{\pi-\theta}{\pi+\theta}\right)^{\frac{\theta}{\pi}}, & \theta \in [0, \pi/2] \\ \frac{\pi \sin\theta}{(2\pi-\theta)\sin\frac{\pi\theta}{2\pi-\theta}}, & \theta \in [\pi/2, \pi) \\ 1/2, & \theta = \pi \ . \end{cases} \tag{29}$$

Proof. Let K denote the Koebe set for $Y \cap S^* \cap K(i)$; it is symmetric with respect to the real axis.

From the inclusions $Y \cap S^* \cap K(i) \subset Y \cap K(i)$ and $Y \cap S^* \cap K(i) \subset Y \cap S^*$ it follows that $K_{Y\cap K(i)} \subset K$ and $K_{Y\cap S^*} \subset K$. This results in

$$K_{Y\cap K(i)} \cup K_{Y\cap S^*} \subset K \tag{30}$$

In particular, $(-1/2, 1) \subset K \cap \mathbb{R}$. This interval cannot be enlarged because the functions $f(z) = \frac{z}{1-z}$ and $f(z) = z$ belong to $Y \cap S^* \cap K(i)$ (see the proofs of Theorem 1 and Theorem 2). Hence,

$$K \cap \mathbb{R} = (-1/2, 1) \ .$$

Let $w = \varrho\,\mathrm{e}^{\mathrm{i}\theta} \in \partial K$, $\varrho > 0$. It means that there exists a function $h \in Y \cap S^* \cap K(i)$ such that $w \in \partial h(\Delta)$.

Assume that $\theta \in (0, \pi/2]$. From the starlikeness of h, we can see that for $t \ge 1$

$$h(z) \ne tw \quad \text{and} \quad h(z) \ne t\overline{w} \ . \tag{31}$$

Since $h \in K(i)$

$$h(z) \ne \varrho\cos\theta + \mathrm{i}(\varrho\sin\theta + t) \quad \text{and} \quad h(z) \ne \varrho\cos\theta - \mathrm{i}(\varrho\sin\theta + t) \ . \tag{32}$$

Moreover, h is circularly symmetric. For this reason, $h(\Delta)$ is disjoint with the arc of the circle $\varrho\,\mathrm{e}^{\mathrm{i}\psi}$, $\psi \in [\theta, 2\pi - \theta]$.

Taking into consideration the above facts, we can see that if $\theta \in (0, \pi/2]$, then

$$h(\Delta) \subset \tilde{E}_{\varrho,\theta} \ .$$

Suppose now $\theta \in [\pi/2, \pi)$. Combining three properties of h, we obtain

$$h(\Delta) \subset \tilde{D}_{\varrho,\theta} \ .$$

It is enough to apply the same argument as in the final parts of the proofs of Theorems 1 and 2. □

The result of Theorem 3 can be rewritten in another way,

$$K_{Y\cap K(i)} \cup K_{Y\cap S^*} = K_{Y\cap S^*\cap K(i)} \ .$$

We have obtained an interesting example of two different classes A, B, such that the first one is not contained in the other one, for which $K_A \cup K_B = K_{A\cap B}$.

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