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Mathematical analysis

Profile decomposition and phase control for circle-valued maps in one dimension

Décomposition en profils et contrôle des phases des applications unimodulaires en dimension un

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A B S T R A C T

When $1 < p < \infty$, maps f in $W^{1/p,p}((0,1);\mathbb{S}^1)$ have $W^{1/p,p}$ phases φ, but the $W^{1/p,p}$-seminorm of φ is not controlled by the one of f. Lack of control is illustrated by "the kink": $f = e^{\imath\varphi}$, where the phase φ moves quickly from 0 to 2π. A similar situation occurs for maps $f : \mathbb{S}^1 \to \mathbb{S}^1$, with Moebius maps playing the role of kinks. We prove that this is the only loss of control mechanism: each map $f : \mathbb{S}^1 \to \mathbb{S}^1$ satisfying $|f|^p_{W^{1/p,p}} \le M$ can be written as $f = e^{\imath\psi} \prod_{j=1}^{K} (M_{a_j})^{\pm 1}$, where M_{a_j} is a Moebius map vanishing at $a_j \in \mathbb{D}$, while the integer $K = K(f)$ and the phase ψ are controlled by M. In particular, we have $K \le c_p M$ for some c_p. When $p = 2$, we obtain the sharp value of c_2, which is $c_2 = 1/(4\pi^2)$. As an application, we obtain the existence of minimal maps of degree one in $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ with $p \in (2-\varepsilon, 2)$.

R É S U M É

Si $1 < p < \infty$, les applications f appartenant à $W^{1/p,p}((0,1);\mathbb{S}^1)$ ont des phases φ dans $W^{1/p,p}$, mais la seminorme $W^{1/p,p}$ de φ n'est pas contrôlée par celle de f. L'absence de contrôle est illustrée par « le pli » : $f = e^{\imath\varphi}$, où la phase φ augmente rapidement de 0 à 2π. Pour des applications $f : \mathbb{S}^1 \to \mathbb{S}^1$, le même phénomène apparaît, avec les transformations de Moebius jouant le rôle des plis. Nous prouvons que cet exemple est essentiellement le seul : toute application $f : \mathbb{S}^1 \to \mathbb{S}^1$ telle que $|f|^p_{W^{1/p,p}} \le M$ s'écrit $f = e^{\imath\psi} \prod_{j=1}^{K} (M_{a_j})^{\pm 1}$, où M_{a_j} est une transformation de Moebius s'annulant en $a_j \in \mathbb{D}$, tandis que l'entier $K = K(f)$ et la phase ψ sont contrôlés par M. En particulier, nous avons $K \le c_p M$ pour une constante c_p. Pour $p = 2$, nous obtenons la valeur optimale de c_2, qui est $c_2 = 1/(4\pi^2)$.

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Comme application, nous obtenons l'existence d'une application minimale de degré un dans $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ avec $p \in]2-\varepsilon, 2[$.

1. Introduction

Let $0 < s < 1$, $1 \le p < \infty$ and let $f : (0,1) \to \mathbb{S}^1$ belong to the space $W^{s,p}$. Then f can be written as $f = e^{\imath\varphi}$, where $\varphi \in W^{s,p}$ [3]. Once the existence of φ is known, a natural question is whether we can control $|\varphi|_{W^{s,p}}$ in terms of $|f|_{W^{s,p}}$. For most of s, p, the answer is positive. The exceptional cases are provided precisely by the spaces $W^{1/p,p}((0,1);\mathbb{S}^1)$, with $1 < p < \infty$ [3]. In these spaces, lack of control is established via the following explicit example. For $n \ge 1$, we define φ_n as follows:

$$\varphi_n(x) := \begin{cases} 0, & \text{for } 0 < x < 1/2 \\ 2\pi n(x - 1/2), & \text{for } 1/2 < x < 1/2 + 1/n \\ 2\pi, & \text{for } 1/2 + 1/n < x < 1 \end{cases}.$$

Then $|\varphi_n|_{W^{1/p,p}} \to \infty$ (since $\varphi_n \to \varphi = 2\pi\, \chi_{(1/2,1)}$ a.e., and φ does not belong to $W^{1/p,p}$). On the other hand, if we extend $u_n := e^{\imath\varphi_n}$ with the value 1 outside $(0,1)$ and still denote the extension u_n then, by scaling,

$$|u_n|_{W^{1/p,p}((0,1))} \le |u_n|_{W^{1/p,p}(\mathbb{R})} = |u_1|_{W^{1/p,p}(\mathbb{R})} < \infty.$$

Thus $|u_n|_{W^{1/p,p}((0,1))} \lesssim 1$ and $|\varphi_n|_{W^{1/p,p}((0,1))} \to \infty$. Finally, we invoke the fact that $W^{1/p,p}$ phases are unique mod 2π [3].

If one considers instead maps $f : \mathbb{S}^1 \to \mathbb{S}^1$, always in the critical case $f \in W^{1/p,p}$, $1 < p < \infty$, then a new phenomenon occurs: f has a degree $\deg f$, and does not have a $W^{1/p,p}$ phase at all when $\deg f \ne 0$ [11, Remark 10]. However, even if $\deg f = 0$ (and thus f has a $W^{1/p,p}$ phase φ), we have a loss-of-control phenomenon similar to the one on $(0,1)$. Indeed, let $M_a(z) := \dfrac{a - z}{1 - \overline{a} z}$, $a \in \mathbb{D}$, $z \in \overline{\mathbb{D}}$, be a Moebius transform (that we identify with its restriction to $\mathbb{S}^1$, $M_a : \mathbb{S}^1 \to \mathbb{S}^1$). Let $f_a(z) := \overline{z} M_a(z)$, so that f_a is smooth and $\deg f_a = 0$. One may prove (see below) that $|M_a|_{W^{1/p,p}} = |\mathrm{Id}|_{W^{1/p,p}}$, and thus f_a is bounded in $W^{1/p,p}$. However, if $a \to \alpha = e^{\imath\xi} \in \mathbb{S}^1$, then the smooth phase φ_a of f_a converges a.e. to $\varphi(e^{\imath\theta}) := \begin{cases} \xi - \theta, & \text{if } \xi - \pi < \theta < \xi \\ 2\pi + \xi - \theta, & \text{if } \xi < \theta < \xi + \pi \end{cases}$, which does not belong to $W^{1/p,p}$. (Here, uniqueness of the phases and convergence hold mod 2π.) Thus φ_a is not bounded as $a \to \alpha \in \mathbb{S}^1$. On the other hand, the plot of φ_a shows that φ_a has a "kink shape", and thus we have here the analog of the example on $(0,1)$.

There are evidences that this loss of control mechanism is the only possible one. For example, the phase of the kink is not bounded in $W^{1/p,p}$, but clearly is in $W^{1,1}$ (same for f_a). Bourgain and Brézis [4] proved that for every $f \in W^{1/2,2}((0,1);\mathbb{S}^1)$, we may split $f = e^{\imath\psi} v$, with ψ and $v = e^{\imath\eta}$ satisfying

$$|\psi|_{W^{1/2,2}} \lesssim |f|_{W^{1/2,2}} \text{ and } |\eta|_{W^{1,1}} = |v|_{W^{1,1}} \lesssim |f|^2_{W^{1/2,2}}. \tag{1}$$

Intuitively, one should think at v as at "the kink part of f". The above result was extended by Nguyen [18] to $1 < p < \infty$: for every $1 < p < \infty$ and every $f \in W^{1/p,p}((0,1);\mathbb{S}^1)$, we may split $f = e^{\imath\psi} v$, with ψ and $v = e^{\imath\eta}$ satisfying

$$|\psi|_{W^{1/p,p}} \le C_p\, |f|_{W^{1/p,p}} \text{ and } |\eta|_{W^{1,1}} = |v|_{W^{1,1}} \le C_p\, |f|^p_{W^{1/p,p}}. \tag{2}$$

Here we present another result in this direction, written for simplicity on the unit circle.

Theorem 1. *Let $1 < p < \infty$ and $M > 0$. Then there exist constants c_p and $F(M)$ such that: every map $f \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ satisfying $|f|^p_{W^{1/p,p}} \le M$ can be written as $f = e^{\imath\psi} \prod_{j=1}^{K} (M_{a_j})^{\varepsilon_j}$, with $\varepsilon_j \in \{-1, 1\}$,*

$$K \le c_p\, M, \tag{3}$$

and

$$|\psi|^p_{W^{1/p,p}} \le F(M). \tag{4}$$

When $p = 2$, we may take $c_2 = 1/(4\pi^2)$, and this constant is optimal.

Corollary 1. *Let $1 < p < \infty$ and let $f_n, f \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ be such that $f_n \rightharpoonup f$ in $W^{1/p,p}$. Then, up to a subsequence, there exist $K \in \mathbb{N}$, $\varepsilon_j \in \{-1, 1\}$, $a_{j_n} \in \mathbb{D}$, $\alpha_j \in \mathbb{S}^1$, $j = 1, \dots, K$, $\psi_n \in W^{1/p,p}(\mathbb{S}^1;\mathbb{R})$, and a constant C, such that:*

i) $f_n = \mathrm{e}^{\imath\psi_n}\prod_{j=1}^{K}(M_{a_{j_n}})^{\varepsilon_j} f$;
ii) $a_{j_n} \to \alpha_j$ *as* $n \to \infty$;
iii) $\psi_n \rightharpoonup C$ *in* $W^{1/p,p}$ *as* $n \to \infty$.

The theorem and the corollary are reminiscent of profile decompositions obtained in different, often geometrical, contexts. We mention, e.g., the work of Sacks and Uhlenbeck [19] on minimal 2-spheres, the analysis of Brézis and Coron [6–8] of constant mean curvature surfaces, or the one of Struwe [20] of equations involving the critical Sobolev exponent. There are also abstract approaches to bubbling as in the work of Lions [16] about concentration-compactness or the characterization of the lack of compactness of critical embeddings in Gérard [12], Jaffard [15] or Bahouri, Cohen and Koch [1].

Let us comment on the connection between (2) and our theorem. First, (2) has the following version for maps on $\mathbb{S}^1$: we may split $f = \mathrm{e}^{\imath\psi} v$, with $|\psi|_{W^{1/p,p}} \le C_p|f|_{W^{1/p,p}}$ and $|v|_{W^{1,1}} \le C_p|f|_{W^{1/p,p}}$. Next, a Moebius map satisfies $|M_a|_{W^{1,1}} = 2\pi$, and thus

$$\left|\prod_{j=1}^{K}(M_{a_j})^{\varepsilon_j}\right|_{W^{1,1}} \le 2\pi\, K \le 2\pi\, c_p\, M. \tag{5}$$

Estimate (5) shows that (3) is a refinement of the second part of (2). On the other hand, (4) is weaker than the first part of (2), since $F(M)$ need not have a linear growth (and actually we do not have any control on F). This suggests the following conjecture.

Conjecture. *Let $1 < p < \infty$. Then there exist constants c_p, d_p such that every $f \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ satisfying $|f|^p_{W^{1/p,p}} \le M$ can be decomposed as $f = \mathrm{e}^{\imath\psi}\prod_{j=1}^{K}(M_{a_j})^{\varepsilon_j}$, with $\varepsilon_j \in \{-1, 1\}$,*

$$K \le c_p\, M, \tag{6}$$

and

$$|\psi|^p_{W^{1/p,p}} \le d_p\, M. \tag{7}$$

In addition, when $p = 2$, we may take $c_2 = 1/(4\pi^2)$.

2. Proofs

We start by recalling or establishing few auxiliary results. Given $1 \le p < \infty$, f, f_n will denote maps in $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$. When $1 < p < \infty$, "$\rightharpoonup$" refers to weak convergence in $W^{1/p,p}$.

1. Recall that, up to a multiplicative factor $\alpha \in \mathbb{S}^1$, the Moebius transforms give all the conformal representations $u : \mathbb{D} \to \mathbb{D}$. In particular, $M_a : \mathbb{S}^1 \to \mathbb{S}^1$ is a smooth orientation preserving diffeomorphism, and thus $\deg M_a = 1$. Consequence: if $g : \mathbb{S}^1 \to \mathbb{S}^1$ is continuous, then $\deg[g \circ M_a] = \deg g$.

2. If $1 \le p < \infty$ and $a \in \mathbb{D}$, then $|f \circ M_a|_{W^{1/p,p}} = |f|_{W^{1/p,p}}$. (Here, we let $|f|_{W^{1,1}} := \int_{\mathbb{S}^1}|\dot f| = \int_0^{2\pi}|\mathrm{d}[f(\mathrm{e}^{\imath\theta})]/\mathrm{d}\theta|\,\mathrm{d}\theta$ and, for $1 < p < \infty$, $|f|^p_{W^{1/p,p}} := \int_{\mathbb{S}^1}\int_{\mathbb{S}^1}|f(x) - f(y)|^p/|x-y|^2\,\mathrm{d}x\,\mathrm{d}y$.) In order to prove the desired equality when $p = 1$, we write $M_a(\mathrm{e}^{\imath\theta}) = \mathrm{e}^{\imath\varphi(\theta)}$, $0 \le \theta \le 2\pi$, with φ smooth and increasing. Then

$$|f \circ M_a|_{W^{1,1}} = \int_0^{2\pi}\left|\frac{\mathrm{d}}{\mathrm{d}\theta}[f(\mathrm{e}^{\imath\varphi(\theta)})]\right|\mathrm{d}\theta = \int_{\varphi^{-1}(0)}^{\varphi^{-1}(2\pi)}\left|\frac{\mathrm{d}}{\mathrm{d}\theta}[f(\mathrm{e}^{\imath\theta})]\right|\mathrm{d}\theta = \int_0^{2\pi}\left|\frac{\mathrm{d}}{\mathrm{d}\theta}[f(\mathrm{e}^{\imath\theta})]\right|\mathrm{d}\theta = |f|_{W^{1,1}}.$$

When $1 < p < \infty$, we rely on the following identity, valid for measurable functions $F : \mathbb{S}^1 \times \mathbb{S}^1 \to [0, \infty]$:

$$\int_{\mathbb{S}^1}\int_{\mathbb{S}^1}\frac{F(M_a(x), M_a(y))}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}x = \int_{\mathbb{S}^1}\int_{\mathbb{S}^1}\frac{F(x,y)}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}x. \tag{8}$$

Proof of (8): We have $[M_a]^{-1} = M_a$ and thus, after change of variables, (8) amounts to

$$|x-y|^2\,|\dot M_a(x)|\,|\dot M_a(y)| = |M_a(x) - M_a(y)|^2,\ \forall x, y \in \mathbb{S}^1. \tag{9}$$

In turn, (9) follows immediately from the straightforward equality $|\dot M_a(x)| = \dfrac{1 - |a|^2}{|1 - \bar a x|^2}$.

3. If $1 \le p < \infty$ and $a \in \mathbb{D}$, then $\deg[f \circ M_a] = \deg f$. Indeed, to start with, such f has a degree, since $W^{1/p,p} \hookrightarrow \mathrm{VMO}$ and VMO maps gave a degree stable with respect to BMO convergence [11]. By item **1**, the desired equality holds true for smooth f. The general case follows by density of $C^\infty(\mathbb{S}^1;\mathbb{S}^1)$ into $W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$ [11, Lemmas A.11 and A.12] and by stability of the VMO degree.

4. If $1 \le p < \infty$ and the degree of f is d, then we may write $f(z) = e^{\imath\psi(z)} z^d$, with $\psi \in W^{1/p,p}(\mathbb{S}^1;\mathbb{R})$. This follows easily from the fact that maps $f \in W^{1/p,p}((0,1);\mathbb{S}^1)$ lift within $W^{1/p,p}$ [3].

5. Let $1 < p < \infty$. For $f \in W^{1/p,p}(\mathbb{S}^1;\mathbb{S}^1)$, let $u = u(f)$ be its harmonic extension. Set $c'_p := \inf\{|f|^p_{W^{1/p,p}};\ u(0) = 0\}$. Clearly, c'_p is achieved, and therefore $c'_p > 0$.

6. When $p = 2$, we have the following straightforward calculations: if $f = \sum_{n\in\mathbb{Z}} a_n e^{\imath n\theta}$, then $|f|^2_{W^{1/2,2}} = 4\pi^2 \sum_{n\in\mathbb{Z}} |n|\,|a_n|^2$ [10, Chapter 13], and $\deg f = \sum_{n\in\mathbb{Z}} n\,|a_n|^2$ [11, eq. (25)]. This leads to $4\pi^2\,|\deg f| \le |f|^2_{W^{1/2,2}}$, with equality e.g. when $f(z) := z^d$. On the other hand, if $u(f)(0) = 0$, then $a_0 = 0$ and thus

$$|f|^2_{W^{1/2,2}} = 4\pi^2 \sum_{n\neq 0} |n|\,|a_n|^2 \ge 4\pi^2 \sum_{n\neq 0} |a_n|^2 = 4\pi^2 \sum_{n\in\mathbb{Z}} |a_n|^2 = 2\pi\,\|f\|^2_{L^2} = 4\pi^2.$$

Thus $c'_2 \ge 4\pi^2$, and the example $f(z) := z$ shows that $c'_2 = 4\pi^2$.

7. For $1 < p < \infty$, there exists some constant c''_p such that $c''_p\,|\deg f| \le |f|^p_{W^{1/p,p}}$, $\forall\, f \in W^{1/p,p}(\mathbb{S}^1,\mathbb{S}^1)$ [5, Corollary 0.5]. We let c''_p be the best constant such that this estimate holds, and set $c^*_p := \min\{c'_p, c''_p\}$. We also set $c_p := 1/c^*_p$. By item **6**, for $p = 2$ we have $c''_2 = c'_2 = c^*_2 = 4\pi^2$, and $c_2 = 1/(4\pi^2)$.

8. Let $1 < p < \infty$. Let $\delta > 0$ and assume that $|u(f)| \ge \delta$ in $\mathbb{D}$. Then there exists some $C = C(\delta, p)$ such that

$$f = e^{\imath\psi},\ \text{with } \psi \in W^{1/p,p}(\mathbb{S}^1;\mathbb{R}) \text{ and } |\psi|_{W^{1/p,p}} \le C\,|f|_{W^{1/p,p}}. \tag{10}$$

Indeed, set $v := u/|u|$, and write $v = e^{\imath\varphi}$, with smooth φ. By standard properties of the functional calculus and of trace theory, and by the lifting estimates in [3], we have $\varphi \in W^{2/p,p}(\mathbb{D};\mathbb{R})$, and then $\psi := \operatorname{tr}\varphi \in W^{1/p,p}(\mathbb{S}^1;\mathbb{R})$ satisfies

$$|\psi|_{W^{1/p,p}} \le C(p)\,|\varphi|_{W^{2/p,p}} \le C(p)\,|v|_{W^{2/p,p}} \le C(\delta,p)\,|u|_{W^{2/p,p}} \le C(\delta,p)\,|f|_{W^{1/p,p}}.$$

9. Let $1 < p < \infty$ and $c < c'_p$. If $|f|^p_{W^{1/p,p}} \le c$, then there exists some $\delta > 0$ such that $|u(f)| \ge \delta$ in $\mathbb{D}$. Proof by contradiction: assume that $|f_n|^p_{W^{1/p,p}} \le c$, $f_n \rightharpoonup g$ and $|u(f_n)(a_n)| \le 1/n$. Since $u(g \circ M_a) = [u(g)] \circ M_a$, we may assume (by item **2**) that $a_n = 0$. We find that $u(f)(0) = 0$ and $|f|^p_{W^{1/p,p}} < c'_p$, which is impossible.

10. Let $1 < p < \infty$. Assume that $f_n \rightharpoonup f$ and $f_n \to f$ a.e. Then $|f_n|^p_{W^{1/p,p}} = |f|^p_{W^{1/p,p}} + |f_n\,\overline{f}|^p_{W^{1/p,p}} + o(1)$. Indeed, if we set $g_n := f_n\,\overline{f}$, then this follows from the Brézis–Lieb lemma [9] and the identity

$$\overline{g_n}(x)\,[f_n(x) - f_n(y)] = f(x) - f(y) + \overline{g_n}(x)\,f(y)\,[g_n(x) - g_n(y)].$$

Proof of Theorem 1. The proof is by complete induction on the integer part $L := I(c_p M) = I(M/c^*_p)$ of $c_p M$. The case where $L = 0$ follows from items **8** and **9**. Let $L > 0$ and let M be such that $I(M/c^*_p) = L$. Assume, by contradiction, that the theorem does not hold for M. We may thus find a sequence (f_n) with the following properties:

(a) $|f_n|^p_{W^{1/p,p}} \le M$;

(b) for any $K \le L$ and any choice of $a_1, \ldots, a_K \in \mathbb{D}$ and of signs $\varepsilon_j = \pm 1$ such that $\sum_{j=1}^K \varepsilon_j = \deg f_n$, if we write $f_n = e^{\imath\psi_n} \prod_{j=1}^K (M_{a_j})^{\varepsilon_j}$, then we have $|\psi_n|_{W^{1/p,p}} \to \infty$. (It is always possible to take K, a_j, ε_j and ψ_n as above: it suffices to let $K := |\deg f| \le I(M/c''_p) \le I(M/c^*_p) = L$, $\varepsilon_j := \operatorname{sgn}\deg f$, and $a_j = 0$.)

By item **8** and property **(b)**, there exist points $a_n \in \mathbb{D}$ such that $u(f_n)(a_n) \to 0$. By item **2**, we may assume in addition that $a_n = 0$. Thus, in addition to **(a)** and **(b)**, we may assume:

(c) $f_n \rightharpoonup f$ and $f_n \to f$ a.e., for some f with $u(f)(0) = 0$.

Set $g_n := f_n\,\overline{f}$. By item **10** and the definition of c'_p, we have $|f|^p_{W^{1/p,p}} \ge c'_p \ge c^*_p$, and $|g_n|^p_{W^{1/p,p}} = M - |f|^p_{W^{1/p,p}} + o(1)$. Let $N > M - |f|^p_{W^{1/p,p}}$ be such that $I(N/c^*_p) = I((M - |f|^p_{W^{1/p,p}})/c^*_p) \le L - 1$. For large n, we have $|g_n|^p_{W^{1/p,p}} \le N$. By the induction hypothesis, we may write (possibly up to a subsequence) $g_n = e^{\imath\eta_n} \prod_{j=1}^R (M_{b_{jn}})^{\varepsilon_j}$, with $|\eta_n|^p_{W^{1/p,p}} \le F(N)$ and $R \le N/c^*_p$. On the other hand, if $d := \deg f$, $b_{jn} := 0$ and $\varepsilon_j := \operatorname{sgn} d$, then we may write $f = e^{\imath\eta} \prod_{j=R+1}^{R+|d|} (M_{b_{jn}})^{\varepsilon_j}$, with $\eta \in W^{1/p,p}$ (item **4**). In addition, we have $|d| \le |f|^p_{W^{1/p,p}}/c''_p$ (item **7**). Finally, with $\psi_n := \eta_n + \eta$ and $K := R + |d| \le M/c^*_p$, we have $f_n = e^{\imath\psi_n} \prod_{j=1}^K (M_{b_{jn}})^{\varepsilon_j}$, and (ψ_n) is bounded in $W^{1/p,p}$. This contradiction completes the proof of the first part of the theorem.

Optimality of (3) when $p = 2$ follows from the fact that, by item **6**, $f(z) := z^d$, $d > 0$, satisfies $|f|^2_{W^{1/2,2}} = c_2\,d$ and requires at least d Moebius maps in its decomposition. □

Proof of Corollary 1. By replacing f_n with $f_n\,\overline{f}$, we may assume that $f_n \rightharpoonup 1$. Up to a subsequence, we may write $f_n = e^{\imath\eta_n} \prod_{j=1}^P (M_{a_{jn}})^{\varepsilon_j}$, with $a_{jn} \to \alpha_j \in \overline{\mathbb{D}}$, $j = 1, \ldots, P$, and $\eta_n \rightharpoonup \eta$. With no loss of generality, we assume that $\alpha_1, \ldots, \alpha_K \in \mathbb{S}^1$ and $\alpha_{K+1}, \ldots, \alpha_P \in \mathbb{D}$. Since (clearly) $M_{a_{jn}} \rightharpoonup \alpha_j$, $j = 1, \ldots, K$, we find that $1 = e^{\imath(\eta - C)} \prod_{j=K+1}^P (M_{\alpha_j})^{\varepsilon_j}$ for some appropriate C. Thus, with $\zeta_n := \eta_n - \eta$, we have

$$f_n = \mathrm{e}^{\imath(\zeta_n+C)} \prod_{j=1}^{K} (M_{a_{jn}})^{\varepsilon_j} \prod_{j=K+1}^{P} \left(M_{a_{jn}} M_{\alpha_j}^{-1} \right)^{\varepsilon_j} = \mathrm{e}^{\imath \psi_n} \prod_{j=1}^{K} (M_{a_{jn}})^{\varepsilon_j},$$

for some ψ_n such that $\psi_n - \zeta_n \to C$ in $W^{1/p,p}$, and thus $\psi_n \rightharpoonup C$. □

Remark. The corollary implies the "bubbling-off of circles along a sequence of graphs". More specifically, the behavior of weakly converging sequences of manifold-valued maps can be investigated within the theory of Cartesian currents of Giaquinta, Modica and Souček [13]; see also [14,17] for the specific case of $W^{1/2,2}(\mathbb{S}^1;\mathbb{S}^1)$. When $p = 2$, it is possible to define (as a current) the graph $\mathcal{G}_f$ of $f \in W^{1/2,2}(\mathbb{S}^1;\mathbb{S}^1)$. With the notation in the corollary, if $p = 2$ and $f_n \rightharpoonup f$, "bubbling-off" reads

$$\mathcal{G}_{f_n} \rightharpoonup \mathcal{G}_f + \sum_{j=1}^{K} \varepsilon_j \, \delta_{\alpha_j} \times [\mathbb{S}^1] \text{ in } \mathcal{D}_1(\mathbb{S}^1 \times \mathbb{S}^1). \tag{11}$$

This can be obtained directly from (1) [17, Proposition 3.1], but also as an immediate consequence of the corollary. Details are left to the reader.

3. Applications

We start with an immediate consequence of Theorem 1.

Corollary 2. *Let d be a non-negative integer and $\delta > 0$. Then there exists a constant $F(d,\delta)$ such that: every map $f \in W^{1/2,2}(\mathbb{S}^1;\mathbb{S}^1)$ satisfying $\deg f = d$ and $|f|^2_{W^{1/2,2}} \le 4\pi^2 (d+1) - \delta$ can be written as $f = \mathrm{e}^{\imath\psi} \prod_{j=1}^{d} M_{a_j}$, with $|\psi|^2_{W^{1/2,2}} \le F(d,\delta)$.*

Corollary 2 with $d = 1$, as well as a weak version of the corollary when $d \ge 2$ were obtained in [2, Theorem 4.4, Theorem 4.8]. As an application of Corollary 2, we obtain the following theorem.

Theorem 2. *There exists some $\varepsilon > 0$ such that, for $p \in (2-\varepsilon, 2]$,*

$$m_p := \min\{|f|^p_{W^{1/p,p}};\ \deg f = 1\}$$

is achieved.

Proof. When $p = 2$, it follows from item **6** that m_2 is achieved by multiples of Moebius maps.

When $1 < p < 2$, consider a minimizing sequence for m_p. Since $m_p \le |\mathrm{Id}|^p_{W^{1/p,p}} := I_p$, we may assume that

$$|f_n|^p_{W^{1/p,p}} \le I_p \to I_2 = 4\pi^2 \text{ as } p \to 2. \tag{12}$$

On the other hand, when $f : \mathbb{S}^1 \to \mathbb{S}^1$ we have $|f|^2_{H^{1/2}} \le 2^{2-p} |f|^p_{W^{1/p,p}}$. Thus

$$|f_n|^2_{H^{1/2}} \le J_p := 2^{2-p} I_p \to 4\pi^2 \text{ as } p \to 2. \tag{13}$$

For p sufficiently close to 2 and fixed $\delta > 0$, we have $J_p \le 8\pi^2 - \delta$. We next apply Corollary 2 to f_n and write $f_n = \mathrm{e}^{\imath\psi_n} M_{a_n}$, with $|\psi_n|_{W^{1/2,2}} \le F(1,\delta)$. Set $g_n := f_n \circ M_{a_n}$. By item **2**, (g_n) is a minimizing sequence for m_p. On the other hand, we have $g_n = \mathrm{e}^{\imath\varphi_n}\,\mathrm{Id}$, with $\varphi_n := \psi_n \circ M_{a_n}$ bounded in $W^{1/2,2}(\mathbb{S}^1;\mathbb{R})$ (by (8)). Therefore, up to a subsequence $\varphi_n \rightharpoonup \varphi$ in $W^{1/2,2}$, and thus $g_n \rightharpoonup g := \mathrm{e}^{\imath\varphi}\,\mathrm{Id}$ in $W^{1/2,2}$. We find that $\deg g = 1$. Since (g_n) is bounded in $W^{1/p,p}$, we obtain that $g_n \rightharpoonup g$ in $W^{1/p,p}$. By a standard argument, g achieves m_p. □

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