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Extension formulas and deformation invariance of Hodge numbers



Formules d'extension et invariance par déformation des nombres de Hodge

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ABSTRACT

We introduce a canonical isomorphism from the space of pure-type complex differential forms on a compact complex manifold to the one on its infinitesimal deformations. By use of this map, we generalize an extension formula in a recent work of K. Liu, X. Yang and the second author. As a direct corollary of the extension formulas, we prove several deformation invariance theorems for Hodge numbers on some certain classes of complex manifolds, without using the Frölicher inequality or the topological invariance of the Betti numbers.

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RÉSUMÉ

Nous introduisons un isomorphisme canonique entre l'espace des formes différentielles complexes de type pur sur une variété complexe, compacte, et celui de ses déformations infinitésimales, et nous l'utilisons pour généraliser la formule d'extension récemment obtenue par K. Liu, X. Yang et le second auteur. Comme corollaire direct des formules d'extension, nous établissons plusieurs théorèmes d'invariance par déformation des nombres de Hodge des variétés complexes, sans avoir recours à l'inégalité de Frölicher ou à l'invariance topologique des nombres de Betti.

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1. Introduction and main results

This paper aims at studying the deformation invariance of Hodge numbers using an iteration method to construct an explicit extension of Dolbeault cohomology classes.

Let $\pi: \mathfrak{X} \to \triangle$ be a holomorphic family of n-dimensional compact complex manifolds with the central fiber $\pi^{-1}(0) = X_0$ and its infinitesimal deformations $\pi^{-1}(t) = X_t$, where \triangle is a small disk in \mathbb{C} for simplicity. Then there exists a transversely

holomorphic trivialization $F_{\sigma}: \mathfrak{X} \xrightarrow{(\sigma,\pi)} X_0 \times \triangle$ (cf. [20, Proposition 9.5] and [3, Appendix A]), which gives us the Kuranishi data $\varphi(t)$ (or φ), depending holomorphically on t, with the integrability

$$\overline{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]. \tag{1.1}$$

Fix an open coordinate covering $\{\mathfrak{U}: (w_j^{\alpha},t)\in U^{\alpha}, U^{\alpha}\in \mathfrak{U}\}$ of \mathfrak{X} , with a restricted covering $\{\mathfrak{U}_0: z_j^{\alpha}\in U_0^{\alpha}:=U^{\alpha}\cap X_0, U^{\alpha}\cap X_0\in \mathfrak{U}_0\}$ of X_0 . As we focus on one coordinate chart, the superscript α is suppressed. As in [3,10,9], the operator $e^{i\varphi}$ is defined by

$$e^{i\varphi} = \sum_{k=0}^{\infty} \frac{1}{k!} i_{\varphi}^{k},$$

where i_{φ}^k denotes k times of the contraction operator $i_{\varphi} = \varphi_{\perp}$ and $e^{i_{\overline{\varphi}}}$ is similarly defined. It is known that $\{e^{i_{\varphi}}(\mathrm{d}z^i)\}_{i=1}^n$ and $\{e^{i_{\varphi}}(\mathrm{d}z^i)\}_{i=1}^n$ are the local bases of $T_{X_t}^{*(1,0)}$ and $T_{X_t}^{*(0,1)}$, respectively. Inspired by these, we introduce:

Definition 1.1. A canonical map between $A^{p,q}(X_0)$ and $A^{p,q}(X_t)$ is defined as:

$$e^{i_{\varphi}|i_{\overline{\varphi}}}: A^{p,q}(X_0) \rightarrow A^{p,q}(X_t)$$

 $\omega \mapsto e^{i_{\varphi}|i_{\overline{\varphi}}}(\omega),$

where

$$e^{i_{\varphi}|i_{\overline{\varphi}}}(\omega) = \sum_{\substack{i_1,\dots,i_p\\i_1,\dots,i_q\\i_2,\dots,i_q}} \frac{1}{p!q!} \omega_{i_1,\dots,i_p;j_1,\dots,j_q}(z) \left(e^{i_{\varphi}} \left(dz^{i_1} \wedge \dots \wedge dz^{i_p} \right) \right) \wedge \left(e^{i_{\overline{\varphi}}} \left(d\overline{z}^{j_1} \dots \wedge d\overline{z}^{j_q} \right) \right)$$

and ω is locally written as $\sum_{\substack{i_1,\cdots,i_p\\j_1,\cdots,j_q}}\frac{1}{p!q!}\omega_{i_1,\cdots,i_p;j_1,\cdots,j_q}(z)\mathrm{d}z^{i_1}\wedge\cdots\wedge\mathrm{d}z^{i_p}\wedge\mathrm{d}z^{j_1}\wedge\cdots\mathrm{d}z^{j_q}$.

It is easy to check that $e^{i_{\varphi}|i_{\overline{\varphi}}}$ is independent of the choice of the local coordinates and is actually a real isomorphism. From the explicit formula of φ (cf. [11, pp. 150]), a careful calculation yields:

Lemma 1.2.

$$\begin{cases} \mathrm{d} w^{\alpha} &= \frac{\partial w^{\alpha}}{\partial z^{i}} \left(e^{i\varphi} \left(\mathrm{d} z^{i} \right) \right) \\ \frac{\partial}{\partial w^{\alpha}} &= \left((\mathbb{1} - \varphi \overline{\varphi})^{-1} \left(\frac{\partial w}{\partial z} \right)^{-1} \right)_{\alpha}^{j} \frac{\partial}{\partial z^{j}} - \left((\mathbb{1} - \overline{\varphi} \varphi)^{-1} \overline{\varphi} \left(\frac{\partial w}{\partial z} \right)^{-1} \right)_{\alpha}^{j} \frac{\partial}{\partial \overline{z}^{j}}, \end{cases}$$

where $\overline{\varphi}\varphi := \varphi \lrcorner \overline{\varphi}$ and $\varphi \overline{\varphi}$ is similarly defined.

$$\textbf{Corollary 1.3.} \ \tfrac{\partial w^\alpha}{\partial z^l} \, \tfrac{\partial}{\partial w^\alpha} = \left((\mathbb{1} - \varphi \overline{\varphi})^{-1} \right)_i^j \, \tfrac{\partial}{\partial z^j} - \left((\mathbb{1} - \overline{\varphi} \varphi)^{-1} \, \overline{\varphi} \right)_i^j \, \tfrac{\partial}{\partial \overline{z}^j}.$$

Then we get the following useful local formula:

Lemma 1.4.

$$d\left(e^{i\varphi}\left(dz^{i}\right)\right) = \left((\mathbb{1} - \overline{\varphi}\varphi)^{-1}\overline{\varphi}\right)_{\bar{k}}^{\bar{l}}\frac{\partial\varphi_{\bar{l}}^{l}}{\partial z^{j}}\left(e^{i\varphi}\lrcorner dz^{k}\right) \wedge \left(e^{i\varphi}\lrcorner dz^{j}\right) \\ - \left((\mathbb{1} - \overline{\varphi}\varphi)^{-1}\right)_{\bar{k}}^{\bar{l}}\frac{\partial\varphi_{\bar{l}}^{l}}{\partial z^{j}}\left(\overline{e^{i\varphi}\lrcorner dz^{k}}\right) \wedge \left(e^{i\varphi}\lrcorner dz^{j}\right),$$

which describes the d-operator under the local frames $\{e^{i_{\varphi}}\left(dz^{i}\right),\overline{e^{i_{\varphi}}\left(dz^{i}\right)}\}_{i=1}^{n}$

Using these, one has:

Proposition 1.5. Let f be a smooth function on X_0 . Then

$$\mathrm{d} f = e^{i\varphi|i\overline{\varphi}} \Big((\mathbb{1} - \varphi\overline{\varphi})^{-1} \mathsf{J} (\partial - \overline{\varphi} \mathsf{J}\overline{\partial}) f + (\mathbb{1} - \overline{\varphi}\varphi)^{-1} \mathsf{J} (\overline{\partial} - \varphi \mathsf{J}\partial) f \Big).$$

Since $\mathrm{d} f$ can be decomposed into $\partial_t f + \overline{\partial}_t f$ on X_t , $\overline{\partial}_t f = e^{i_{\varphi}|i_{\overline{\varphi}}} \Big((\mathbb{1} - \overline{\varphi}\varphi)^{-1} \mathbb{1}(\overline{\partial} - \varphi \mathbb{1}\partial) f \Big)$. Thus f is holomorphic with respect to the complex structure of X_t if and only if

$$(\overline{\partial} - \varphi \lrcorner \partial) f = 0,$$

by the invertibility of $(1 - \overline{\varphi}\varphi)^{-1}$. Hence, we reprove this important criterion (cf. [13] and also [11, pp. 151–152]) in the deformation theory.

Then we get two extension formulas on (p, 0) and (0, q)-forms.

Proposition 1.6. For $\omega \in A^{p,0}(X_0)$,

$$\begin{split} \mathbf{d}(e^{i_{\varphi}|i_{\overline{\varphi}}}(\omega)) &= e^{i_{\varphi}|i_{\overline{\varphi}}} \Big((\mathbb{1} - \overline{\varphi}\varphi)^{-1} \lrcorner (\overline{\partial}\omega + \partial(\varphi \lrcorner \omega) - \varphi \lrcorner \partial\omega) \\ &+ (\mathbb{1} - \varphi\overline{\varphi})^{-1} \lrcorner \partial\omega - p\partial\omega - ((\mathbb{1} - \varphi\overline{\varphi})^{-1} \lrcorner \overline{\varphi}) \lrcorner (\overline{\partial}\omega + \partial(\varphi \lrcorner \omega)) \Big). \end{split}$$

Corollary 1.7. For $\omega \in A^{0,q}(X_0)$,

$$\begin{split} \mathbf{d}(e^{i_{\varphi}|i_{\overline{\varphi}}}(\omega)) &= e^{i_{\varphi}|i_{\overline{\varphi}}} \Big((\mathbb{1} - \varphi \overline{\varphi})^{-1} \mathbb{1}(\partial \omega + \overline{\partial}(\overline{\varphi} \mathbb{1}\omega) - \overline{\varphi} \mathbb{1}\overline{\partial}\omega) \\ &+ (\mathbb{1} - \overline{\varphi}\varphi)^{-1} \mathbb{1}\overline{\partial}\omega - q\overline{\partial}\omega - ((\mathbb{1} - \overline{\varphi}\varphi)^{-1} \mathbb{1}\varphi) \mathbb{1}(\partial \omega + \overline{\partial}(\overline{\varphi} \mathbb{1}\omega)) \Big). \end{split}$$

Based on these two, we use the iteration method, initiated by [10] and developed in [16,17,9,22], to achieve two theorems on deformation invariance of Hodge numbers, by constructing explicit extension, without use of Frölicher inequality or the topological invariance of Betti numbers (cf. [6, Section 5.1] and [20, Section 9.3.2]). We need:

Definition 1.8. Define a complex manifold $X \in \mathcal{E}^{p,q}$, $\mathfrak{D}^{p,q}$ and $\mathfrak{B}^{p,q}$, if for any $\overline{\partial}$ -closed $\partial g \in A^{p,q}(X)$, the equation

$$\overline{\partial} x = \partial g$$

has a solution, \overline{a} ∂ -closed solution and a ∂ -exact solution, respectively. It is obvious that $\mathfrak{B}^{p,q} \subset \mathfrak{D}^{p,q} \subset \mathcal{E}^{p,q}$ and that X, satisfying the $\partial \overline{\partial}$ -lemma, lies in $\mathfrak{B}^{p,q}$.

Set
$$h_t^{p,q} = dim_{\mathbb{C}} H^{p,q}(X_t, \mathbb{C})$$
. Then:

Theorem 1.9. For $1 \le p \le n$ and $X_0 \in \mathfrak{D}^{p,1} \cap \mathcal{E}^{p+1,0}$, $h_t^{p,0}$ are independent of t.

Theorem 1.10. For $1 \le q \le n$ and $X_0 \in \mathfrak{B}^{1,q'} \cap \mathcal{E}^{q',0} \cap \mathfrak{D}^{q',1}$ with all $1 \le q' \le q$, $h_t^{0,q}$ are independent of t.

By Theorem 1.9 and the standard Hodge theory on compact complex surfaces (such as in Section IV.2 of [2]), we obtain:

Corollary 1.11. All the Hodge numbers of a compact complex surface are infinitesimal deformation invariant.

For the jumping phenomenon of Hodge numbers, we refer to [12,21]. More generally than Proposition 1.6 and Corollary 1.7, we achieve:

Proposition 1.12. For $\omega \in A^{\star,\star}(X_0)$,

$$\begin{split} \mathrm{d} \left(e^{i_{\varphi} \mid i_{\overline{\varphi}}} (\omega) \right) &= e^{i_{\varphi} \mid i_{\overline{\varphi}}} \left(\partial \omega + \left(\varphi \overline{\varphi} (\mathbb{1} - \varphi \overline{\varphi})^{-1} \right) \square \partial \omega - \partial \left(\left(\varphi \overline{\varphi} (\mathbb{1} - \varphi \overline{\varphi})^{-1} \right) \square \omega \right) + \left(\partial \left(\overline{\varphi} (\mathbb{1} - \varphi \overline{\varphi})^{-1} \right) \square \varphi \right) \square \omega \right) \\ &- \left(\overline{\varphi} (\mathbb{1} - \varphi \overline{\varphi})^{-1} \right) \square \overline{\partial} \omega + \overline{\partial} \left(\left(\overline{\varphi} (\mathbb{1} - \varphi \overline{\varphi})^{-1} \right) \square \omega \right) - \left(\overline{\partial} (\mathbb{1} - \varphi \overline{\varphi})^{-1} \square \overline{\varphi} \right) \square \omega \right) \\ &+ \overline{\partial} \omega + \left(\overline{\varphi} \varphi (\mathbb{1} - \overline{\varphi} \varphi)^{-1} \right) \square \overline{\partial} \omega - \overline{\partial} \left(\left(\overline{\varphi} \varphi (\mathbb{1} - \overline{\varphi} \varphi)^{-1} \right) \square \omega \right) + \left(\overline{\partial} \left(\varphi (\mathbb{1} - \overline{\varphi} \varphi)^{-1} \right) \square \overline{\varphi} \right) \square \omega \right) \\ &- \left(\varphi (\mathbb{1} - \overline{\varphi} \varphi)^{-1} \right) \square \partial \omega + \partial \left(\left(\varphi (\mathbb{1} - \overline{\varphi} \varphi)^{-1} \right) \square \omega \right) - \left(\partial (\mathbb{1} - \overline{\varphi} \varphi)^{-1} \square \varphi \right) \square \omega \right). \end{split}$$

More details and applications (especially for Proposition 1.12) will appear in [15].

2. The ideas of proofs

We shall describe the main ideas in the proofs of Theorems 1.9 and 1.10 in this section. Throughout this section, X_t is assumed to be determined by the integrable Kuranishi data $\varphi(t) = \sum_{k=1}^{\infty} t^k \varphi_k$ with (1.1). Theorem 1.9 is obtained by Kodaira-Spencer's upper semi-continuity theorem and the following iteration procedure.

Proposition 2.1. Let $X_0 \in \mathfrak{D}^{p,1} \cap \mathcal{E}^{p+1,0}$. Then for any holomorphic (p,0)-form σ_0 on X_0 , there exits a power series

$$\sigma_t = \sigma_0 + \sum_{k=1}^{\infty} t^k \sigma_k \in A^{p,0}(X_0),$$

such that $e^{i_{\varphi(t)}}(\sigma_t) \in A^{p,0}(X_t)$ is holomorphic with respect to the complex structure on X_t .

Sketch of proof. By Grauert's formal function theorem [5], we only need to construct σ_t order by order. Proposition 1.6 yields that the holomorphicity of $e^{i\varphi(t)}$ (σ_t) is equivalent to the resolution of the equation

$$\overline{\partial}\sigma_t = -\partial(\varphi(t) \lrcorner \sigma_t) + \varphi(t) \lrcorner \partial \sigma_t$$

by the invertibility of the operators $e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}$ and $\left(\mathbb{1}-\overline{\varphi(t)}\varphi(t)\right)^{-1}$. By comparing the coefficients of t^k , it suffices to resolve the system of equations

$$\begin{cases} \overline{\partial}\sigma_0 = 0, \\ \overline{\partial}\sigma_k = -\partial(\sum_{i=1}^k \varphi_i \lrcorner \sigma_{k-i}), & \text{for each } k \ge 1, \\ \partial\sigma_k = 0, & \text{for each } k \ge 0. \end{cases}$$
 (2.1)

By $X_0 \in \mathcal{E}^{p+1,0}$, the equation $\overline{\partial} x = \partial \sigma_0$ has solutions, which implies $\partial \sigma_0 = 0$ by type consideration. Let's resolve (2.1) inductively. Since $X_0 \in \mathfrak{D}^{p,1}$, our task is to verify

$$\overline{\partial}\partial(\sum_{i=1}^k \varphi_i \lrcorner \sigma_{k-i}) = 0$$

for $k \ge 1$. Set $\eta_k = -\partial (\sum_{i=1}^k \varphi_i \lrcorner \sigma_{k-i})$ for simplicity. For k=1, one has

$$\overline{\partial}\eta_1 = -\overline{\partial}\partial(\varphi_1 \lrcorner \sigma_0) = \partial(\overline{\partial}\varphi_1 \lrcorner \sigma_0 + \varphi_1 \lrcorner \overline{\partial}\sigma_0) = 0,$$

since $\overline{\partial}\varphi_1=0$ by (1.1) and $\overline{\partial}\sigma_0=0$. Thus σ_1 is got by $X_0\in \mathfrak{D}^{p,1}$. By induction, we assume that (2.1) is solved for all $k\leq l$ and thus we have $\partial\sigma_k=0$ for $0\leq k\leq l$. By $X_0\in \mathfrak{D}^{p,1}$, we only need to show $\overline{\partial}\eta_{l+1}=0$. We resort to a useful commutative formula (cf. [18,19,1,4,3,7-9]) on a complex manifold X. For $\phi,\psi\in A^{0,1}(X,T_X^{1,0})$ and $\alpha\in A^{*,*}(X)$,

$$[\phi,\psi] \lrcorner \alpha = -\partial (\psi \lrcorner (\phi \lrcorner \alpha)) - \psi \lrcorner (\phi \lrcorner \partial \alpha) + \phi \lrcorner \partial (\psi \lrcorner \alpha) + \psi \lrcorner \partial (\phi \lrcorner \alpha).$$

Hence, by this formula and (1.1), one has

$$\begin{split} \overline{\partial}\eta_{l+1} &= \partial \left(\sum_{i=2}^{l+1} \overline{\partial} \varphi_i \lrcorner \sigma_{l+1-i} + \sum_{i=1}^{l} \varphi_i \lrcorner \overline{\partial} \sigma_{l+1-i} \right) \\ &= \partial \left(\frac{1}{2} \sum_{i=2}^{l+1} \sum_{j=1}^{i-1} [\varphi_j, \varphi_{i-j}] \lrcorner \sigma_{l+1-i} - \sum_{i=1}^{l} \varphi_i \lrcorner \partial \left(\sum_{j=1}^{l+1-i} \varphi_j \lrcorner \sigma_{l+1-i-j} \right) \right) \\ &= \partial \left(\frac{1}{2} \sum_{i=2}^{l+1} \sum_{j=1}^{i-1} \left(-\partial \left(\varphi_j \lrcorner \left(\varphi_{i-j} \lrcorner \sigma_{l+1-i} \right) \right) - \varphi_j \lrcorner \varphi_{i-j} \lrcorner \partial \sigma_{l+1-i} \right) + \varphi_j \lrcorner \partial \left(\varphi_{i-j} \lrcorner \sigma_{l+1-i} \right) + \varphi_{i-j} \lrcorner \partial \left(\varphi_j \lrcorner \sigma_{l+1-i} \right) \right) \\ &- \sum_{i=1}^{l} \varphi_i \lrcorner \partial \left(\sum_{j=1}^{l+1-i} \varphi_j \lrcorner \partial_{l+1-i-j} \right) \right) \\ &= \partial \left(\sum_{1 \leq j < i \leq l+1} \varphi_j \lrcorner \partial \left(\varphi_{i-j} \lrcorner \sigma_{l+1-i} \right) - \sum_{i=1}^{l} \sum_{j=1}^{l+1-i} \varphi_i \lrcorner \partial \left(\varphi_j \lrcorner \sigma_{l+1-i-j} \right) \right) \\ &= 0. \quad \Box \end{split}$$

The proof of Theorem 1.10 is a bit different from that of Theorem 1.9 and we need:

Lemma 2.2. (See [14], Lemma 3.1.) Each Dolbeault class $[\alpha]$ of type (p,q) on a complex manifold $X \in \mathfrak{B}^{p+1,q}$ can be represented by a d-closed (p,q)-form γ_{α} .

Lemma 2.3. Let γ_{α_1} and γ_{α_2} be two d-closed representatives of the same Dolbeault class $[\alpha_1] = [\alpha_2]$ as in the above lemma on $X \in \mathcal{E}^{q,0} \cap \mathfrak{B}^{1,q}$. Then $\gamma_{\alpha_1} = \gamma_{\alpha_2}$.

Proof. From $\gamma_{\alpha_i} = \alpha_i + \overline{\partial} \beta_{\alpha_i}$, i = 1, 2, there exists some $\beta \in A^{0,q-1}(X)$ such that

$$\gamma_{\alpha_2} - \gamma_{\alpha_1} = \overline{\partial} \beta.$$

Since γ_{α_1} , γ_{α_2} are *d*-closed, we have $\partial \overline{\partial} \beta = 0$. Hence, by $X \in \mathcal{E}^{q,0}$, the equation

$$\overline{\partial} x = \partial \bar{\beta}$$

has solutions. From type consideration, $\partial \bar{\beta} = 0$, which implies $\gamma_{\alpha_1} = \gamma_{\alpha_2}$. \square

We shall construct a correspondence from $H^{0,q}(X_0)$ to $H^{0,q}(X_t)$ by sending $[\alpha] \in H^{0,q}(X_0)$ to $[e^{i\overline{\psi}}(\gamma_{\alpha}(t))] \in H^{0,q}(X_t)$, where

$$\gamma_{\alpha}(t) = \gamma_{\alpha} + \sum_{k=1}^{\infty} \gamma_{\alpha}^{k} \bar{t}^{k} \in A^{0,q}(X_{0}).$$

Here γ_{α} is uniquely determined by the Dolbeault class $[\alpha]$ from the above two lemmas. To guarantee that this correspondence cannot send a nonzero class in $H^{0,q}(X_0)$ to a zero class in $H^{0,q}(X_t)$, one needs $h_t^{0,q-1} = h_0^{0,q-1}$. Therefore, for each $1 \le q \le n$, we use induction to reduce Theorem 1.10 to the following proposition with all $1 \le q' \le q$.

Proposition 2.4. Let $X_0 \in \mathfrak{B}^{1,q'} \cap \mathcal{E}^{q',0} \cap \mathfrak{D}^{q',1}$. Then for any d-closed (0,q')-form σ_0 on X_0 , there exits a power series on X_0

$$\sigma_t = \sigma_0 + \sum_{k=1}^{\infty} \bar{t}^k \sigma_k \in A^{0,q'}(X_0)$$

such that $e^{i\overline{\varphi(t)}}(\sigma_t) \in A^{0,q'}(X_t)$ is $\overline{\partial}_t$ -closed with respect to the complex structure on X_t .

Sketch of Proof. By Corollary 1.7, the invertibility of the operators $e^{i_{\varphi(t)}|i_{\overline{\varphi(t)}}}$ yields that the desired $\overline{\partial}_t$ -closed condition is equivalent to the resolution of the equation

$$\left(\mathbb{1} - \overline{\varphi(t)}\varphi(t)\right)^{-1} \lrcorner \overline{\partial}\sigma_t - q'\overline{\partial}\sigma_t - \left(\left(\mathbb{1} - \overline{\varphi(t)}\varphi(t)\right)^{-1} \lrcorner \varphi(t)\right) \lrcorner (\partial\sigma_t + \overline{\partial}(\overline{\varphi(t)} \lrcorner \sigma_t)) = 0.$$

By comparing the coefficients of \bar{t}^k , it suffices to resolve the system of equations

$$\begin{cases} \overline{\partial}\sigma_t = 0, \\ \partial\sigma_t + \overline{\partial}(\overline{\varphi(t)} \lrcorner \sigma_t) = 0, \end{cases}$$

or equivalently, by conjugation,

$$\begin{cases} d\sigma_0 = 0, \\ \overline{\partial} \overline{\sigma_k} = -\partial(\sum_{i=1}^k \varphi_i \lrcorner \overline{\sigma_{k-i}}), & \text{for each } k \ge 1, \\ \partial \overline{\sigma_k} = 0, & \text{for each } k \ge 1. \end{cases}$$
 (2.2)

Hence, analogously to the proof of Proposition 2.1, we are able to resolve (2.2) inductively by the assumption on X_0 and Lemmata 2.2, 2.3. \Box

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