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Electromagnetic scattering by periodic structures with sign-changing coefficients

Diffraction électromagnétique par un réseau périodique avec des coefficients qui changent de signe

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A B S T R A C T

We analyze the well-posedness of a scattering problem of time-harmonic electromagnetic waves by periodic structures with sign-changing coefficients. Transmission problems for Maxwell's equations with sign-changing coefficients in bounded domains have been recently studied by Bonnet-Ben Dhia and co-workers in the so-called T-coercivity framework. In this article, we generalize such a framework for periodic scattering problems relying on an integral equation approach. The periodic scattering problem is formulated by a hypersingular integral equation of Lipmann–Schwinger type. We prove that the integral equation satisfies a Gårding-type estimate, which allows us to establish the well-posedness of the problem in the sense of Fredholm.

R É S U M É

Nous analysons le caractère bien posé du problème de diffraction d'ondes électromagnétiques par des structures périodiques dont les coefficients diélectriques changent de signe. Le problème de diffraction pour les équations de Maxwell avec des coefficients qui changent de signe a été récemment étudié par Bonnet-Ben Dhia et al. en utilisant le concept de la T-coercivité. Dans cette note, nous étendons cette étude à la diffraction par un réseau périodique en se basant sur une formulation intégrale volumique du problème. Le problème de diffraction est d'abord écrit sous la forme d'une équation de type Lippmann–Schwinger avec un noyau hyper-singulier. Nous montrons ensuite que la solution de cette équation satisfait une estimation a priori de type Gårding, ce qui nous permet de conclure sur le caractère bien posé du problème au sens de Fredholm.

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1. Introduction

Electromagnetic scattering theory in periodic structures is well known as a topic of great interest in applications, e.g., for the construction and optimization of optical filters, lenses, and beam-splitters in optics (see [10]). There is a large body of applied math literature on forward and inverse scattering problems related to periodic dielectric materials, see for example [3,8] and references therein. Recently, periodic metamaterials with applications in photonics and optics have received intense research interest in the engineering and applied physics community (see for instance [1,9]). However, to the knowledge of the authors, there have been only a few works on periodic metamaterials in the applied mathematics literature. The paper [6] studies the Fredholm property of a scattering problem for periodic metamaterials known as gratings in two dimensions.

We consider in this work an electromagnetic scattering problem for biperiodic gratings consisting of dielectric materials and metamaterials. By biperiodic, we mean that the grating is periodic in the, say, x_1- and x_2-direction, while it is bounded in the x_3-direction. This is modeled by a periodic scattering problem for 3D Maxwell's equations with sign-changing coefficients. Transmission Maxwell problems with sign-changing coefficients in bounded domains have been recently studied in [4] by the so-called T-coercivity framework. In this article, we generalize such a framework for periodic scattering problems relying on an integral equation approach. It is known in [5,8] that scattering problems for Mawell's equations can be formulated as a hypersingular integral equation of Lipmann–Schwinger type. We aim to prove a Gårding-type estimate for the integral equation with sign-changing coefficients. The idea is to investigate such estimates for the integral equation in a truncation of the unit cell. This, roughly speaking, enables the use of variational formulation of the integral operators, which allows us to incorporate the idea of T-coercivity. It also turns out that one needs similar assumptions as in [4] when applying the T-coercivity framework.

The integral equation approach in this work has some advantages in the sense that the Fredholm property obtained is valid at Rayleigh frequencies, which is typically excluded in the variational approach. It further avoids technical complication that one might have when treating boundary terms, with Calderón maps, perturbed by the abstract operator T in the T-coercivity framework.

Notation: Let $\mathcal{O}$ be a bounded domain (connected and open) with Lipschitz boundary $\partial\mathcal{O}$. We indistinctly denote by $\langle\cdot,\cdot\rangle$ the inner products of $L^2(\mathcal{O})$ and $(L^2(\mathcal{O}))^3$ and by $\|\cdot\|$ the associated norms. We set $L^\infty_{\mathrm{p}}(\mathbb{R}^3) = \{v \in L^\infty(\mathbb{R}^3) : v$ is 2π-periodic in x_1 and $x_2\}$, $H_\alpha(\mathrm{curl},\mathcal{O})$ is the closure with respect to the norm $\|\cdot\| + \|\mathrm{curl}\cdot\|$ of space of smooth functions that is α-quasi-periodic in x_1 and x_2.

2. Periodic electromagnetic scattering

We consider the scattering of time-harmonic electromagnetic waves from a diffraction grating consisting of dielectric materials and metamaterials. The electric field E and the magnetic field H are governed by the time-harmonic Maxwell's equations at frequency $\omega > 0$ in $\mathbb{R}^3$

$$\mathrm{curl}\, H + \mathrm{i}\omega\varepsilon E = 0, \qquad \mathrm{curl}\, E - \mathrm{i}\omega\mu H = 0 \quad \text{in } \mathbb{R}^3, \tag{1}$$

where the electric permittivity ε, the magnetic permeability μ are real-valued functions in $L^\infty_{\mathrm{p}}(\mathbb{R}^3)$. We assume that there are positive constants ε_0 and μ_0 such that $\varepsilon = \varepsilon_0$, $\mu = \mu_0$ outside the grating. We define the wave number $k = \omega(\varepsilon_0\mu_0)^{1/2}$.

The grating is illuminated by an electromagnetic plane wave with wave vector $d = (d_1, d_2, d_3) \in \mathbb{R}^3$, $d_3 \neq 0$ such that $d \cdot d = k^2$. The polarizations $p, s \in \mathbb{R}^3$ of the incident wave satisfy $p \cdot d = 0$ and $s = 1/(\omega\varepsilon_0)(p \times d)$. With these definitions, the incident plane waves E^i and H^i are given by

$$E^i = s\, \mathrm{e}^{\mathrm{i}d\cdot x}, \qquad H^i = p\, \mathrm{e}^{\mathrm{i}d\cdot x}, \quad x \in \mathbb{R}^3. \tag{2}$$

For $d = (d_1, d_2, d_3) \in \mathbb{R}^3$ defined in (2), we set $\alpha = (\alpha_1, \alpha_2, 0) = (d_1, d_2, 0)$. Then a function $u : \mathbb{R}^3 \to \mathbb{C}^3$ is then called α-quasi-periodic if, for all $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$, $n = (n_1, n_2, 0)^\top \in \mathbb{Z}^3$,

$$u(x_1 + 2\pi n_1, x_2 + 2\pi n_2, x_3) = \mathrm{e}^{2\pi \mathrm{i}\alpha\cdot n} u(x_1, x_2, x_3).$$

Note that the incident fields E^i, H^i defined in (2) are α-quasi-periodic functions. The relative material parameters are defined by $\varepsilon_{\mathrm{r}} = \varepsilon/\varepsilon_0$, $\mu_{\mathrm{r}} = \mu/\mu_0$. We wish to reformulate (1) in terms of the scattered field u, defined by $u := H - H^i$. Since, by construction, $\mathrm{curl}\,\mathrm{curl}\, H^i - k^2 H^i = 0$, eliminating the electric field E from (1), and subtracting the latter equation implies that

$$\mathrm{curl}\left(\varepsilon_{\mathrm{r}}^{-1}\, \mathrm{curl}\, u\right) - k^2\mu_{\mathrm{r}} u = \mathrm{curl}\left(q\, \mathrm{curl}\, H^i\right) + k^2 p H^i \quad \text{in } \mathbb{R}^3, \tag{3}$$

where the contrasts q, p are defined by

$$q := 1 - \varepsilon_{\mathrm{r}}^{-1}, \qquad p := \mu_{\mathrm{r}} - 1.$$

It is required that u also be α-quasi-periodic in x_1 and x_2 and that it admits a Rayleigh expansion radiation condition of the form (see, e.g., [3,8])

$$u(x) = \sum_{n\in\mathbb{Z}^2} c_n^{\pm}\, \mathrm{e}^{\mathrm{i}(\alpha_n\cdot x \pm \beta_n(x_3 \mp h))} \quad \text{for } x_3 \gtrless \pm h, \tag{4}$$

where $c_n^{\pm} \in \mathbb{C}^3$, $\alpha_n = (\alpha_1 + n_1, \alpha_2 + n_2, 0)$, $\beta_n = \sqrt{k^2 - |\alpha_n|^2}$ with the branch of the square root chosen such that $\mathrm{Re}\,(\beta_n) \geq 0$ and $\mathrm{Im}\,(\beta_n) \geq 0$, and $h > \sup\{|x_3| : (x_1, x_2, x_3)^\top \in \mathrm{supp}(q) \cup \mathrm{supp}(p)\}$. Now we define

$$\Omega := (-\pi, \pi)^2 \times \mathbb{R}, \qquad \overline{D} := [\mathrm{supp}(q) \cup \mathrm{supp}(p)] \cap \Omega.$$

Recall that $\varepsilon_\mathrm{r}, \mu_r$ belong to $L^\infty_\mathrm{p}(\mathbb{R}^3)$. We further make the following assumption on ε_r and μ_r and D for our analysis.

Assumption 2.1. *We assume that the support D is a bounded and simply connected domain in $\mathbb{R}^3$ such that its Lipschitz boundary ∂D is connected, and that $\varepsilon_\mathrm{r}^{-1}, \mu_r^{-1}$ belong to $L^\infty_\mathrm{p}(\mathbb{R}^3)$.*

The problem (3)–(4) can be reduced to one period Ω due to its periodicity. We now consider a more general problem as follows: given $f, g \in (L^2(D))^3$, find $u : \Omega \to \mathbb{C}^3$ such that

$$\mathrm{curl}\left(\varepsilon_\mathrm{r}^{-1}\,\mathrm{curl}\, u\right) - k^2 \mu_\mathrm{r} u = \mathrm{curl}\, f + k^2 g \quad \text{in } \Omega, \tag{5}$$

and the radiation condition (4). To study the well-posedness of the latter problem, we reformulate it as an integral equation of Lippmann–Schwinger type. This has been done in [5] for scattering problems in bounded inhomogeneous media. The following lines follow from [8].

We denote by G_k the α-quasi-periodic Green's function to the Helmholtz equation in $\mathbb{R}^3$. From [3], we know that $G_k(x) = \exp(\mathrm{i}k|x|)/(4\pi|x|) + \Psi_k(x)$ for $x \neq 0$, where Ψ_k is an analytic function solving the Helmholtz equation $\Delta\Psi_k + k^2\Psi_k = 0$ in $(-2\pi, 2\pi)^2 \times \mathbb{R}$. Now, for any $R > 0$, the truncation Ω_R of the unit cell Ω is defined by

$$\Omega_R = (-\pi, \pi)^2 \times (-R, R), \qquad \Gamma_{\pm R} = (-\pi, \pi)^2 \times \{\pm R\}.$$

We define the volume potential V_k by

$$(V_k f)(x) = \int_D G_k(x - y) f(y)\,\mathrm{d}y, \quad x \in \Omega, \tag{6}$$

for $f \in L^2(D)$. The following lemma is the main ingredient for the integral equation formulation, which is also necessary for our analysis later on. Its proof can be found in chapter 3 of [8].

Lemma 2.2. *The volume potential V_k defined in (6) is bounded from $L^2(D)$ into $H^2_\alpha(\Omega_R)$ for all $R > 0$. The potentials $A_k = \mathrm{curl}\, V_k$ and $B_k = (k^2 + \nabla\mathrm{div})V_k$ are bounded from $(L^2(D))^3$ into $H_\alpha(\mathrm{curl}, \Omega_R)$ for all $R > 0$. Further, for $g \in (L^2(D))^3$, $A_k g$ and $B_k g$ are the unique solution to*

$$\int_\Omega (\mathrm{curl}\, A_k g \cdot \mathrm{curl}\,\overline{\psi} - k^2 A_k g \cdot \overline{\psi})\,\mathrm{d}x = \int_D g \cdot \mathrm{curl}\,\overline{\psi}\,\mathrm{d}x,$$

$$\int_\Omega (\mathrm{curl}\, B_k g \cdot \mathrm{curl}\,\overline{\psi} - k^2 B_k g \cdot \overline{\psi})\,\mathrm{d}x = k^2 \int_D g \cdot \overline{\psi}\,\mathrm{d}x,$$

for all $\psi \in H_\alpha(\mathrm{curl}, \Omega)$ with compact support, and additionally the radiation condition (4).

The scattering problem (4)–(5) is equivalent to the integral equation (see [8] for more details)

$$u - A_k(q\,\mathrm{curl}\, u + f) - B_k(pu + g) = 0 \quad \text{in } \Omega_h. \tag{7}$$

3. *T*-Coercivity framework

In this section, we study the framework of T-coercivity in quasi-periodic function spaces. For $\xi \in L^\infty_\mathrm{p}(\mathbb{R}^3)$, we define:

$$\mathcal{S}_\alpha(\Omega_h) = \left\{ v \in H^1_\alpha(\Omega_h) : \int_{\partial\Omega_h} v = 0 \right\},$$

$$\mathcal{V}_\alpha(\xi, \Omega_h) = \{ w \in H_\alpha(\mathrm{curl}, \Omega_h) : \langle \xi w, \nabla\psi \rangle = 0 \text{ for all } \psi \in \mathcal{S}_\alpha \},$$

$$\mathcal{X}_\alpha(\xi, \Omega_h) = \{ u \in H_\alpha(\mathrm{curl}, \Omega_h) : \mathrm{div}(\xi u) = 0 \text{ in } \Omega_h, n \times u = 0 \text{ on } \partial\Omega_h \}.$$

It is well-known that $\langle \nabla\cdot, \nabla\cdot \rangle$ defines an inner product on $\mathcal{S}_\alpha(\Omega_h)$ with an equivalent norm given by $\|u\|_{\mathcal{S}_\alpha(\Omega_h)} = \|\nabla u\|$. Let X be $\mathcal{V}_\alpha(1, \Omega_h)$ or $\mathcal{X}_\alpha(1, \Omega_h)$. The proof of the following lemmas can be done similarly as in the free space case, see [2,4].

Lemma 3.1. *The embedding of X in $(L^2(\Omega_h))^3$ is compact. Further, there exists a positive constant C such that $\|w\| \le C\|\operatorname{curl} w\|$ for all $w \in X$. Thus, $\langle \operatorname{curl}\cdot, \operatorname{curl}\cdot\rangle$ defines an inner product on X with an equivalent norm given by $\|w\|_X = \|\operatorname{curl} w\|$.*

Lemma 3.2. *A function $u \in H_\alpha(\operatorname{div}, \Omega_h)$ satisfies*

$$\operatorname{div} u = 0 \quad \text{in } \Omega_h, \quad \text{and} \quad \int_{\partial\Omega_h} (n\cdot u)\overline{\phi}\,\mathrm{d}s = 0 \quad \text{for all } \phi \in \mathcal{S}_\alpha(\Omega_h) \tag{8}$$

if and only if there exists a vector potential ψ in $\mathcal{X}_\alpha(1,\Omega_h)$ such that $u = \operatorname{curl}\psi$. Further, this function ψ is unique.

As in [4], the following assumption is important for the T-coercivity framework. We also refer to the cited paper for a detailed investigation on geometric configurations related to this assumption.

Assumption 3.3. (H^{ε_r})*: There exists an isomorphism T^{ε_r} in $H_0^1(\Omega_h)$ and a positive constant C such that*

$$|\langle \varepsilon_r \nabla u, \nabla T^{\varepsilon_r} u\rangle| \ge C\|\nabla u\|^2, \quad \text{for all } u \in H_0^1(\Omega_h).$$

(H^{μ_r})*: There exists an isomorphism T^{μ_r} in $\mathcal{S}_\alpha(\Omega_h)$ and a positive constant C such that*

$$|\langle \mu_r \nabla u, \nabla T^{\mu_r} u\rangle| \ge C\|\nabla u\|^2, \quad \text{for all } u \in \mathcal{S}_\alpha(\Omega_h).$$

Lemma 3.4. *Suppose the assumptions (H^{ε_r}) and (H^{μ_r}) hold true, there exist isomorphisms T in $\mathcal{V}_\alpha(\mu_r,\Omega_h)$ and $\widetilde{T}$ in $\mathcal{X}_\alpha(1,\Omega_h)$ such that*

$$\langle \varepsilon_r^{-1}\operatorname{curl} u, \operatorname{curl} T v\rangle = \langle \varepsilon_r^{-1}\operatorname{curl} T u, \operatorname{curl} v\rangle = \langle \operatorname{curl} u, \operatorname{curl} v\rangle, \quad \text{for all } u, v \in \mathcal{V}_\alpha(\mu_r,\Omega_h),$$
$$\langle \mu_r^{-1}\operatorname{curl} u, \operatorname{curl} \widetilde{T} v\rangle = \langle \mu_r^{-1}\operatorname{curl} \widetilde{T} u, \operatorname{curl} v\rangle = \langle \operatorname{curl} u, \operatorname{curl} v\rangle, \quad \text{for all } u, v \in \mathcal{X}_\alpha(1,\Omega_h).$$

We refer to [4] for the proof of Lemma 3.4.

Lemma 3.5. *Suppose the assumption (H^{μ_r}) holds true. $\mathcal{V}_\alpha(\mu_r,\Omega_h)$ is compactly embedded in $(L^2(\Omega_h))^3$.*

Proof. Assume that (u_n) is a bounded sequence in $\mathcal{V}_\alpha(\mu_r,\Omega_h)$. Since $\mu_r u_n$ satisfies (8), Lemma 3.2 implies that there exists $w_n \in \mathcal{X}_\alpha(1,\Omega_h)$ such that $u_n = \mu_r^{-1}\operatorname{curl} w_n$. Recall that $\mu_r^{-1} \in L^\infty(\Omega_h)$, it is sufficient now to show that $(\operatorname{curl} w_n)$ has a subsequence that converges in $(L^2(\Omega_h))^3$.

Since (u_n) is a bounded sequence in $\mathcal{V}_\alpha(\mu_r,\Omega_h)$, the equation $u_n = \mu_r^{-1}\operatorname{curl} w_n$ implies that (w_n) is also bounded in $\mathcal{X}_\alpha(1,\Omega_h)$. The compact embedding $\mathcal{X}_\alpha(1,\Omega_h) \subset (L^2(\Omega_h))^3$ deduces that (w_n) has a subsequence, still denoted (w_n), that converges in $(L^2(\Omega_h))^3$. Therefore, (w_n) is a Cauchy sequence in $(L^2(\Omega_h))^3$. Now, for $\phi \in \mathcal{X}_\alpha(1,\Omega_h)$, we have $\langle \operatorname{curl}(\mu_r^{-1}\operatorname{curl} w_{nm}), \phi\rangle = \langle \mu_r^{-1}\operatorname{curl} w_{nm}, \operatorname{curl}\phi\rangle$, where $w_{nm} = w_n - w_m$. Taking $\phi = \widetilde{T} w_{nm}$, where $\widetilde{T}$ is the isomorphism from Lemma 3.4, we find that

$$\langle \operatorname{curl} u_{nm}, \widetilde{T} w_{nm}\rangle = \langle \mu_r^{-1}\operatorname{curl} w_{nm}, \operatorname{curl}\widetilde{T} w_{nm}\rangle = \|\operatorname{curl} w_{nm}\|^2. \tag{9}$$

Furthermore, we have that $(\widetilde{T} w_n)$ is also a Cauchy sequence in $(L^2(\Omega_h))^3$ and $(\operatorname{curl} u_n)$ is bounded in $(L^2(\Omega_h))^3$. Therefore, we obtain from (9) that $(\operatorname{curl} w_n)$ is a Cauchy sequence in $(L^2(\Omega_h))^3$. □

To prove the Fredholm property for the integral equation, we need the following Hodge decomposition. We notice that for the case that μ_r does not change sign this result is classical, see [7].

Lemma 3.6. *Suppose the assumption (H^{μ_r}) holds true, we have*

$$H_\alpha(\operatorname{curl},\Omega_h) = \mathcal{V}_\alpha(\mu_r,\Omega_h) \oplus \nabla\mathcal{S}_\alpha(\Omega_h).$$

Furthermore, for all $u = u_0 + \nabla p$,

$$\|u\|^2_{H_\alpha(\operatorname{curl},\Omega_h)} = \|\operatorname{curl} u_0\|^2 + \|\nabla p\|^2.$$

Proof. For $u \in H_\alpha(\operatorname{curl},\Omega_h)$, we know from the assumption (H^{μ_r}) that there exists a unique $p \in \mathcal{S}_\alpha(\Omega_h)$ such that

$$\int_{\Omega_h} \mu_r \nabla p\cdot\nabla\overline{\varphi}\,\mathrm{d}x = \int_{\Omega_h} \mu_r u\cdot\nabla\overline{\varphi}\,\mathrm{d}x, \quad \text{for all } \varphi \in \mathcal{S}_\alpha(\Omega_h).$$

This means that $\int_{\Omega_h} \mu_r(u - \nabla p) \cdot \nabla\overline{\varphi}\,\mathrm{d}x = 0$, for all $\varphi \in \mathcal{S}_\alpha(\Omega_h)$, or $u_0 := u - \nabla p$ belongs to $\mathcal{V}_\alpha(\mu_r, \Omega_h)$. Now it remains to prove that

$$\mathcal{V}_\alpha(\mu_r, \Omega_h) \cap \nabla\mathcal{S}_\alpha(\Omega_h) = \{0\}.$$

Let $w \in \mathcal{V}_\alpha(\mu_r, \Omega_h) \cap \nabla\mathcal{S}_\alpha(\Omega_h)$, then there exists $\psi \in \mathcal{S}_\alpha(\Omega_h)$ such that $w = \nabla\psi$, and $\langle \mu_r w, \nabla\varphi\rangle = 0$ for all $\varphi \in \mathcal{S}_\alpha(\Omega_h)$. Choosing $\varphi = T^{\mu_r}\psi$ implies that

$$\langle \mu_r \nabla\psi, \nabla T^{\mu_r}\psi\rangle = 0.$$

Again, from the assumption (H^{μ_r}), we have $0 = |\langle \mu_r \nabla\psi, \nabla T^{\mu_r}\psi\rangle| \geq C\|\nabla\psi\|^2$, which allows us to deduce that $\nabla\psi = 0$ or $w = 0$. $\square$

4. Fredholm's alternative

Theorem 4.1. *Suppose that the assumptions (H^{ε_r}) and (H^{μ_r}) hold true. Then there exist an isomorphism $\mathbb{T}$, a compact operator K in $H_\alpha(\mathrm{curl}, \Omega_h)$ and a positive constant C such that, for all $u \in H_\alpha(\mathrm{curl}, \Omega_h)$,*

$$\mathrm{Re}\,\langle u - A_k(q\,\mathrm{curl}\,u) - B_k(pu), \mathbb{T}u\rangle_{H_\alpha(\mathrm{curl},\Omega_h)} \geq C\|u\|^2_{H_\alpha(\mathrm{curl},\Omega_h)} + \mathrm{Re}\,\langle Ku, u\rangle_{H_\alpha(\mathrm{curl},\Omega_h)}.$$

Proof. Let $u \in H_\alpha(\mathrm{curl}, \Omega_h)$. The Hodge decomposition implies that $u = u_0 + \nabla p$, where $u_0 \in \mathcal{V}_\alpha(\mu_\mathrm{r}, \Omega_h)$, and $p \in \mathcal{S}_\alpha(\Omega_h)$. Let us consider the operator $\mathbb{T} : H_\alpha(\mathrm{curl}, \Omega_h) \to H_\alpha(\mathrm{curl}, \Omega_h)$ defined by

$$u = (u_0 + \nabla p) \mapsto (Tu_0 + \nabla T^{\mu_\mathrm{r}} p),$$

where T and T^{μ_r} are defined in Lemma 3.4 and Assumption 3.3, respectively. It is easy to check that $\mathbb{T}$ is an isomorphism. Now we define w by

$$\begin{aligned} w &= A_i(q\,\mathrm{curl}\,u) + B_i(pu) \\ &= \mathrm{curl} \int_D G_i(\cdot - y) q(y)\,\mathrm{curl}\,u(y)\,\mathrm{d}y + (-1 + \nabla\mathrm{div}) \int_D G_i(\cdot - y) p(y) u(y)\,\mathrm{d}y. \end{aligned}$$

From Lemma 2.2 we have, for all $v \in H_\alpha(\mathrm{curl}, \Omega_h)$,

$$\langle w, v\rangle_{H_\alpha(\mathrm{curl},\Omega_h)} + \int_{\Gamma_{\pm h}} (n \times \mathrm{curl}\,w) \cdot (n \times \overline{v}) \times n\,\mathrm{d}s = \int_D (q\,\mathrm{curl}\,u \cdot \mathrm{curl}\,\overline{v}\,\mathrm{d}x - pu \cdot \overline{v})\,\mathrm{d}x$$

which implies that

$$\langle u - w, v\rangle_{H_\alpha(\mathrm{curl},\Omega_h)} = \int_{\Omega_h} (\varepsilon_\mathrm{r}^{-1}\,\mathrm{curl}\,u \cdot \mathrm{curl}\,\overline{v} + \mu_\mathrm{r} u \cdot \overline{v})\,\mathrm{d}x + \int_{\Gamma_{\pm h}} (n \times \mathrm{curl}\,w) \cdot (n \times \overline{v}) \times n\,\mathrm{d}s.$$

Choosing $v = \mathbb{T}u$, we have:

$$\begin{aligned} \langle u - w, \mathbb{T}u\rangle_{H_\alpha(\mathrm{curl},\Omega_h)} &= \int_{\Omega_h} (\varepsilon_\mathrm{r}^{-1}\,\mathrm{curl}\,u_0 \cdot \mathrm{curl}\,\overline{Tu_0} + \mu_\mathrm{r} u_0 \cdot \overline{Tu_0} + \mu_\mathrm{r}\nabla p \cdot \overline{\nabla T^{\mu_\mathrm{r}} p})\,\mathrm{d}x \\ &\quad + \int_{\Gamma_{\pm h}} (n \times \mathrm{curl}\,w) \cdot (n \times \overline{\mathbb{T}u}) \times n\,\mathrm{d}s \geq C\|u\|^2_{H_\alpha(\mathrm{curl},\Omega_h)} + \langle K_1 u, u\rangle_{H_\alpha(\mathrm{curl},\Omega_h)}. \end{aligned} \tag{10}$$

Here, due to Lemma 3.5 and the smoothness of w on $\Gamma_{\pm h}$, K_1 defined by

$$\langle K_1 u, u\rangle_{H_\alpha(\mathrm{curl},\Omega_h)} = \int_{\Omega_h} \mu_r u_0 \cdot \overline{Tu_0}\,\mathrm{d}x + \int_{\Gamma_{\pm h}} (n \times \mathrm{curl}\,w) \cdot (n \times \overline{\mathbb{T}u}) \times n\,\mathrm{d}s$$

is a compact operator in $H_\alpha(\mathrm{curl}, \Omega_h)$. From (10) we have:

$$\begin{aligned} \langle u - A_k(q\,\mathrm{curl}\,u) - B_k(pu), \mathbb{T}u\rangle_{H_\alpha(\mathrm{curl},\Omega_h)} &\geq C\|u\|^2_{H_\alpha(\mathrm{curl},\Omega_h)} + \langle K_1 u, u\rangle_{H_\alpha(\mathrm{curl},\Omega_h)} \\ &\quad - \langle (A_k - A_i)(q\,\mathrm{curl}\,u) + (B_k - B_i)(pu), \mathbb{T}u\rangle_{H_\alpha(\mathrm{curl},\Omega_h)}. \end{aligned} \tag{11}$$

Recall that the Green function $G_k(x, y) = \Phi_k(x, y) + \Psi_k(x, y)$, where $\Phi_k(x, y) = \exp(ik|x - y|)/(4\pi|x - y|)$ and $\Psi_k(x, y)$ is an analytic function. Hence thanks to the smoothness of $\Phi_k(x, y) - \Phi_i(x, y)$, the last term in (11) can be written as $\langle K_2 u, u\rangle_{H_\alpha(\mathrm{curl},\Omega_h)}$ with a compact operator K_2 in $H_\alpha(\mathrm{curl}, \Omega_h)$ (see [5,8]). The latter argument completes the proof. $\square$

Corollary 4.2. *For $f, g \in (L^2(D))^3$, the problem of finding $u \in H_\alpha(\mathrm{curl}, \Omega_h)$ such that $u - A_k(q\,\mathrm{curl}\,u) - B_k(pu) = A_k f + B_k g$ in $H_\alpha(\mathrm{curl}, \Omega_h)$, satisfies the Fredholm alternative, i.e., the uniqueness of the solution implies the existence of the solution.*

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References

[1] A. Alù, First-principles homogenization theory for periodic metamaterials, Phys. Rev. B 84 (2011) 075135.
[2] C. Amrouche, C. Bernardi, M. Dauge, V. Girault, Vector potentials in three-dimensional non-smooth domains, Math. Methods Appl. Sci. 21 (1998) 823–864.
[3] T. Arens, Scattering by biperiodic layered media: the integral equation approach, Habilitation thesis, Universität Karlsruhe, Karlsruhe, Germany, 2010.
[4] A.-S. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet Jr., T-coercivity for the Maxwell problem with sign-changing coefficients, Commun. Partial Differ. Equ. 39 (2014) 1007–1031.
[5] A. Kirsch, An integral equation approach and the interior transmission problem for Maxwell's equations, Inverse Probl. Imaging 1 (2007) 159–180.
[6] A. Lechleiter, D.-L. Nguyen, Volume integral equations for scattering from anisotropic diffraction gratings, Math. Methods Appl. Sci. 36 (2013) 262–274.
[7] P. Monk, Finite Element Methods for Maxwell's Equations, Clarendon Press, Oxford, UK, 2003.
[8] D.-L. Nguyen, Spectral methods for direct and inverse scattering from periodic structures, Ph.D. thesis, École Polytechnique, Palaiseau, France, 2012.
[9] J. Pendry, Photonics: metamaterials in the sunshine, Nat. Mater. 5 (2006) 599–600.
[10] R. Petit (Ed.), Electromagnetic Theory of Gratings, Springer, 1980.