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Partial differential equations

Long-time existence for semilinear Klein–Gordon equations on compact manifolds for a generic mass

Stabilité en grands temps pour des équations semi-linéaires de Klein–Gordon sur des variétés compactes avec une masse générique

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A R T I C L E I N F O

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A B S T R A C T

The purpose of this note is to recap the results of long-time existence of small solutions for the semilinear Klein–Gordon equations on a boundaryless compact Riemannian manifold. Using a result by Zhang on the harmonic oscillator and Delort–Szeftel's estimates, we will explain how we can easily obtain a result that seems to be new: we improve the local existence time on compact manifolds whose eigenvalues are integers (like finite product of spheres).

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R É S U M É

L'objet de cette note est de résumer les résultats de stabilité en temps grand pour les petites solutions de l'équation semi-linéaire de Klein–Gordon sur une variété riemannienne compacte sans bord. Nous expliquerons aussi comment obtenir facilement un résultat qui semble nouveau en utilisant un résultat de Zhang sur l'oscillateur harmonique et des estimées de Delort et Szeftel : nous améliorons le temps d'existence sur des variétés compactes dont les valeurs propres sont des entiers (comme des produits finis de sphères).

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Version française abrégée

Soit X une variété lisse riemannienne compacte sans bord. On s'intéresse à l'équation de Klein–Gordon

$$\partial_t^2 w - \Delta w + m^2 w = w^{n+1}, \quad (t,x) \in \mathbb{R} \times X, \tag{1}$$

où $m > 0$ est la masse et n un entier supérieur ou égal à 2. On cherche des solutions dans l'espace

$$\mathcal{C}^0((-T,+T), H^{s+1}(X)) \cap \mathcal{C}^1((-T,+T), H^s(X)),$$

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avec $T > 0$, $s > \frac{1}{2}\dim(X)$ et où $H^s(X)$ désigne l'usuel espace de Sobolev. La théorie locale assure que, si la condition initiale $(w(0), \dot{w}(0)) \in H^{s+1}(X) \times H^s(X)$ est d'ordre $\varepsilon > 0$ (avec ε suffisamment petit), alors l'équation (1) admet une unique solution telle que, pour tout $t \in [-C\varepsilon^{-n}, C\varepsilon^{-n}]$, on a $||w(t)||_{H^{s+1}(X)} + ||\dot{w}(t)||_{H^s(X)} \leq K\varepsilon$, où $C > 0$ et $K > 0$ sont indépendants de $\varepsilon > 0$. On s'intéresse à la possibilité de prolonger le temps d'existence locale ε^{-n} tout en conservant le contrôle de la solution par $K\varepsilon$. Le premier résultat dans cette direction est celui de Bourgain, qui montre l'*existence presque globale* pour $X = \mathbb{T}$: pour tout réel $A > 1$ et pour presque toute masse $m > 0$, le temps d'existence locale ε^{-n} peut être amélioré en ε^{-An}. Actuellement, le meilleur résultat est celui où X est une variété de Zoll (c'est-à-dire dont le flot géodésique est périodique) et a été traité par Bambusi, Delort, Grébert et Szeftel [3].

Pour les tores $\mathbb{T}^d$, Delort, Fang et Zhang ont développé des arguments qui montrent que l'on peut remplacer ε^{-n} par ε^{-An} pour certains réels explicites $A > 1$ (sans pour autant atteindre $A \to +\infty$) [6,9]. Dans [13], Zhang traite le cas de l'équation de Klein–Gordon avec oscillateur harmonique :

$$\partial_t^2 w - \Delta w + x^2 w = w^{n+1}, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d.$$

L'analyse du papier précédent donne l'espoir de se passer de l'analyse harmonique inhérente à l'opérateur de Laplace–Beltrami d'une variété compacte. En combinant des estimées multilinéaires prouvées par Delort et Szeftel et le résultat de Zhang, on montre que l'on peut traiter le cas des variétés X dont l'opérateur de Laplace–Beltrami Δ a toutes ses valeurs propres entières. Le théorème suivant, par exemple, est applicable à $X = \mathbb{S}^2 \times \mathbb{S}^3 \times \mathbb{S}^4$.

Théorème 0.1. *Supposons qu'il existe $\rho > 0$ tel que toute valeur propre de Δ appartient à $\rho\mathbb{Z}$ et considérons un nombre réel $A \in]1, \frac{4}{3}[$. Il existe un sous-ensemble $\mathcal{E} \subset]0, +\infty[$ de mesure de Lebesgue pleine tel que pour tout $m \in \mathcal{E}$, il existe $s_0 = s_0(X, n, A, m) > 0$, de sorte que pour tout $s \geq s_0$, on peut trouver $\varepsilon_0(X, n, A, m, s) \in (0, 1)$ tel que pour tout couple $(w_0, w_1) \in H^{s+1}(X) \times H^s(X)$ de fonctions réelles, avec $||w_0||_{H^{s+1}(X)} + ||w_1||_{H^s(X)} \leq 1$, l'équation (2) admet une unique solution dans l'espace*

$$\mathcal{C}^0((-C\varepsilon^{-An}, +C\varepsilon^{-An}), H^{s+1}(X)) \cap \mathcal{C}^1((-C\varepsilon^{-An}, +C\varepsilon^{-An}), H^s(X)), \quad C = C(X, n, A, m, s) > 0,$$

avec condition initiale $(w(0, \cdot), \dot{w}(0, \cdot)) = (\varepsilon w_0, \varepsilon w_1)$ et $\varepsilon \in (0, \varepsilon_0)$. En outre, il existe $K = K(X, n, A, m, s) > 0$ tel que

$$\forall t \in (-C\varepsilon^{-An}, +C\varepsilon^{-An}) \qquad ||w(t)||_{H^{s+1}(X)} + ||\dot{w}(t)||_{H^s(X)} \leq K\varepsilon.$$

1. The semilinear Klein–Gordon equation

Let us consider a smooth compact Riemannian manifold X without boundary and denote by Δ the negative Laplace–Beltrami operator. We are interested in studying the solutions to the semilinear Klein–Gordon equation

$$\partial_t^2 w - \Delta w + m^2 w = w^{n+1}, \quad (t, x) \in \mathbb{R} \times X, \tag{2}$$

where $m > 0$ is the mass and n an integer that is larger than or equal to 2. In all our purpose, the non-linearity w^{n+1} can be replaced by $F(w)$, where $F : \mathbb{R} \to \mathbb{R}$ is a smooth function that vanishes at least with order $n + 1$ at 0. We seek solutions in the space $\mathcal{C}^0((-T, +T), H^{s+1}(X)) \cap \mathcal{C}^1((-T, +T), H^s(X))$, where $(-T, +T)$ is the time interval, $s > 0$ will be a large number and $H^s(X)$ is the usual Sobolev space of the functions $f : X \to \mathbb{C}$ such that $(I - \Delta)^{\frac{s}{2}} f$ belongs to $L^2(X)$. Let us recall the following easy result.

Proposition 1.1. *Assume that one has $s > \frac{1}{2}\dim(X)$. For any couple $(w_0, w_1) \in H^{s+1}(X) \times H^s(X)$ of real-valued functions such that $||w_0||_{H^{s+1}(X)} + ||w_1||_{H^s(X)} \leq 1$, there are $C = C(X, m, s, n) > 0$ and $\varepsilon_0 = \varepsilon_0(X, m, s, n) \in (0, 1)$ such that Eq. (2) admits a unique solution in the space*

$$\mathcal{C}^0((-C\varepsilon^{-n}, +C\varepsilon^{-n}), H^{s+1}(X)) \cap \mathcal{C}^1((-C\varepsilon^{-n}, +C\varepsilon^{-n}), H^s(X)),$$

with initial data $(w(0, \cdot), \dot{w}(0, \cdot)) = (\varepsilon w_0, \varepsilon w_1)$, with $\varepsilon \in (0, \varepsilon_0)$. Furthermore, there is a constant $K = K(X, m, s) > 0$ such that

$$\forall t \in (-C\varepsilon^{-n}, +C\varepsilon^{-n}) \qquad ||w(t)||_{H^{s+1}(X)} + ||\dot{w}(t)||_{H^s(X)} \leq K\varepsilon.$$

Proof. We use the following reformulation of the equation

$$w(t) = \cos\left(t\sqrt{-\Delta + m^2}\right) w_0 + \frac{\sin\left(t\sqrt{-\Delta + m^2}\right)}{\sqrt{-\Delta + m^2}} w_1 + \int_0^t \frac{\sin\left((t - \tau)\sqrt{-\Delta + m^2}\right) w(\tau)^{n+1}}{\sqrt{-\Delta + m^2}} \mathrm{d}\tau.$$

Introducing $u = -\mathrm{i}\partial_t w + (-\Delta + m^2)^{\frac{1}{2}} w$ and $u_0 = -\mathrm{i}\varepsilon w_1 + (-\Delta + m^2)^{\frac{1}{2}} \varepsilon w_0$, the former equation becomes

$$u(t) = \mathrm{e}^{\mathrm{i}t\sqrt{-\Delta + m^2}} u_0 - \mathrm{i} \int_0^t \mathrm{e}^{\mathrm{i}(t-\tau)\sqrt{-\Delta + m^2}} \left((-\Delta + m^2)^{-\frac{1}{2}} \left(\frac{u(\tau) + \overline{u(\tau)}}{2} \right) \right)^{n+1} \mathrm{d}\tau.$$

Since $H^s(X)$ is an algebra, we get the a priori estimate

$$||u(t)||_{H^s(X)} \leq ||u_0||_{H^s(X)} + C(X,m,s,n)|t| \sup_{0\leq|\tau|\leq|t|} ||u(\tau)||^{n+1}_{H^s(X)}.$$

We easily conclude by a classical fix-point argument in the Banach space $\mathcal{C}^0\left([-C'\varepsilon^{-n}, +C'\varepsilon^{-n}], \overline{B_{H^s}(0, K'\varepsilon)}\right)$ for some universal constant $K' > 0$ and a constant $C' > 0$ that depends on C. We can conclude noticing that $||u(t)||_{H^s}$ and $||w(t)||_{H^{s+1}(X)} + ||\dot{w}(t)||_{H^s(X)}$ are similar up to a multiplicative constant, which depends on (X, m, s). □

Therefore, a question one can ask is: can we improve the local existence time ε^{-n} by a larger one if ε tends to 0^+? We will see that it is possible for s large enough and if the mass m is chosen generically. To compare several results, it is convenient to set the following definition.

Definition 1.2. Let us consider a real number $A > 1$, we say that the local existence time ε^{-n} of Eq. (2) can be improved to ε^{-An} generically in m if there is a full Lebesgue measure subset $\mathcal{E} \subset (0, +\infty)$ such that the following holds. For any $m \in \mathcal{E}$, there is $s_0 = s_0(X, n, A, m) > 0$ such that for any $s \geq s_0$, one can find $\varepsilon_0(X, n, A, m, s) \in (0, 1)$ such that for any couple $(w_0, w_1) \in H^{s+1}(X) \times H^s(X)$ of real-valued functions, with $||w_0||_{H^{s+1}(X)} + ||w_1||_{H^s(X)} \leq 1$, Eq. (2) admits a unique solution in the space:

$$\mathcal{C}^0((-C\varepsilon^{-An}, +C\varepsilon^{-An}), H^{s+1}(X)) \cap \mathcal{C}^1((-C\varepsilon^{-An}, +C\varepsilon^{-An}), H^s(X)), \quad C = C(X, n, A, m, s) > 0,$$

with initial data $(w(0,\cdot), \dot{w}(0,\cdot)) = (\varepsilon w_0, \varepsilon w_1)$ and $\varepsilon \in (0, \varepsilon_0)$. Furthermore, there is $K = K(X, n, A, m, s) > 0$ such that

$$\forall t \in (-C\varepsilon^{-An}, +C\varepsilon^{-An}) \qquad ||w(t)||_{H^{s+1}(X)} + ||\dot{w}(t)||_{H^s(X)} \leq K\varepsilon.$$

The first result in this direction is the one-dimensional torus $X = \mathbb{S}^1$ (see [5,1]): one can increase the local existence time ε^{-n} to ε^{-An} generically in m for any $A > 1$. Such a conclusion is called the *almost global existence*. Several articles have made significant progresses to extend the family of manifolds for which the almost global existence holds (see [10,2,4,7,8,3]). Let us recall that X is said to be a Zoll manifold if each geodesic is closed with the same length (say 2π for simplicity). It turns out that those manifolds fulfill the following spectral property

$$\exists b \in \mathbb{R} \quad \exists c > 0 \quad \mathrm{Sp}\left(\sqrt{-\Delta}\right) \subset \bigcup_{n\geq 0}\left[n + b - \frac{c}{n+1}, n + b + \frac{c}{n+1}\right]. \tag{3}$$

The following theorem is the most general result of almost global existence [3].

Theorem 1.3. *Assume that X is a Zoll manifold and consider a real number $A > 1$. Then, the local existence time ε^{-n} of Eq. (2) can be improved to ε^{-An} generically in m.*

We will use below a very interesting element of the proof of Theorem 1.3 that does not require X to be a Zoll manifold. Indeed, the eigenfunctions on an arbitrary compact boundaryless Riemannian manifold satisfy universal multilinear estimates (see [8, Proposition 1.2.1]).

Proposition 1.4. *Consider a sequence of compact intervals $I_k := [\lambda_k - \alpha_k, \lambda_k + \alpha_k]$ with $\sup_{k\in\mathbb{N}} \alpha_k < +\infty$ and $\lambda_k > 0$ for each $k \in \mathbb{N}$. Denote by $\Pi_k : L^2(X) \to L^2(X)$ the spectral projector $\mathbf{1}_{I_k}(\sqrt{-\Delta})$ on the subspace of functions whose spectrum lies in I_k. For any positive integer $n \geq 2$, there is $\nu = \nu(n, \dim X) > 0$ such that for any $(k_1, \dots, k_{n+2}) \in (\mathbb{N}\backslash\{0\})^{n+2}$ with $\lambda_{k_{n+2}} \leq \dots \leq \lambda_{k_2} \leq \lambda_{k_1}$, any $(u_1, \dots, u_{n+2}) \in \mathcal{C}(X)^{n+2}$ and any integer $N \in \mathbb{N}^\star$, one has*

$$\left|\int_X \Pi_{k_1}(u_1)\dots\Pi_{k_{n+2}}(u_{n+2})\mathrm{d}x\right| \leq C(X, n, N)\lambda_{k_3}^{\nu}\left(1 + \frac{\lambda_{k_1} - \lambda_{k_2}}{\lambda_{k_3}}\right)^{-N} \prod_{j=1}^{n+2} ||u_j||_{L^2(X)}. \tag{4}$$

We have written the previous inequalities for the product of $n+2$ functions because they are relevant for the analysis of the nonlinearity w^{n+1}. Let us recall that the famous Lagrange's four-square theorem ensures that the spectrum of $\sqrt{-\Delta}$ on $\mathbb{T}^d$, for any $d \geq 4$, is nothing else than

$$\left\{\sqrt{n_1^2 + \dots + n_d^2}, \quad (n_1, \dots, n_d) \in \mathbb{N}^d\right\} = \{\sqrt{k}, k \in \mathbb{N}\}.$$

Clearly, the previous spectrum is badly separated by comparison with (3). With a new method, Delort proved in [6] that the local existence time for the semilinear Klein–Gordon equation (2) on the torus $\mathbb{T}^d$, with $d \geq 2$, can be improved to ε^{-An}

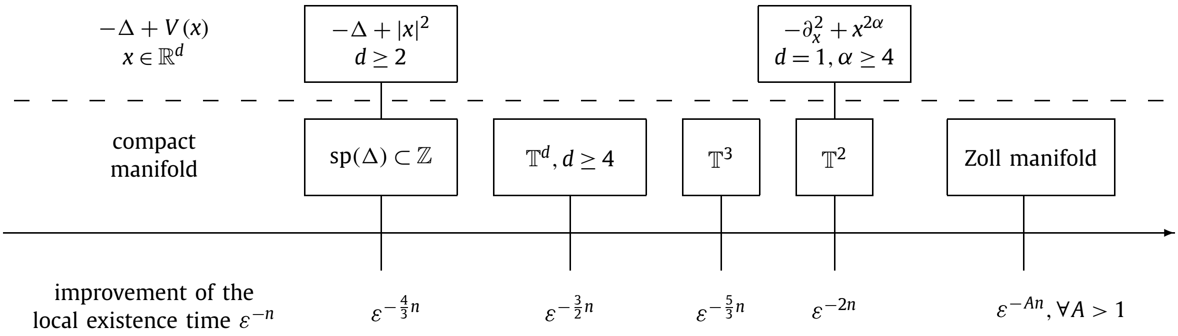
(up to a logarithmic term) generically in m for any $A \in (1, 1+\frac{2}{d})$. Combining the previous proof and considering packets of eigenfunctions on $\mathbb{T}^d$ that are associated with the same eigenvalues, Fang and Zang obtained a time of order $\varepsilon^{-\frac{3}{2}n}$ (see [9]), which is independent of the dimension of the torus. Such an analysis has also been used by Zhang in [13] for the following Klein–Gordon equation with the harmonic oscillator:

$$\partial_t^2 w - \Delta w + |x|^2 w + m^2 w = w^{n+1}, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \quad d \geq 2. \tag{5}$$

The time $\varepsilon^{-\frac{4}{3}n}$ is reached generically in m for the previous equation. Such an analysis has also been extended to the Klein–Gordon equation with a superquadratic oscillator $-\partial_x^2 + x^{2\alpha}$, and we get the longer time ε^{-2n} for any integer $\alpha \geq 4$ (see more explanations in [12] about how the separation of the spectrum of $-\partial_x^2 + x^{2\alpha}$ ensures a longer time). We will explain below that the combination of the proof of [13] and Proposition 1.4 lead to the following new theorem (which applies for instance to $X = \mathbb{S}^2 \times \mathbb{S}^3 \times \mathbb{S}^4$).

Theorem 1.5. *Assume there is $\rho > 0$ such that any eigenvalue of Δ belongs to $\rho\mathbb{Z}$. For any real number $A \in (1, \frac{4}{3})$, the local existence time ε^{-n} of Eq. (2) can be improved to ε^{-An} generically in m.*

However, the time $\varepsilon^{-\frac{4}{3}n}$ is less than the one Delort, Fang and Zhang get for tori. That means that it is probably possible to improve the time of Theorem 1.5 if we make use of the harmonic analysis on specific manifolds as in [6] or [9]. The following diagram recaps the previous discussion:



2. Explanation of the proof of Theorem 1.5

The natural Sobolev spaces of $-\Delta + |x|^2$ on $L^2(\mathbb{R}^d)$ are defined by

$$\forall s \geq 0 \quad \mathcal{H}^s(\mathbb{R}^d) := \mathrm{Dom}((-\Delta + |x|^2)^{\frac{s}{2}}) = \left\{ f \in H^s(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} |x|^{2s} |f(x)|^2 \mathrm{d}x < +\infty \right\}.$$

Let us also recall that the spectrum of $-\Delta + |x|^2$ on $L^2(\mathbb{R}^d)$ is $d + 2\mathbb{N}$. Let us repeat a third time the type of result that Zhang obtained in [13], but for that new category of Sobolev spaces.

Theorem 2.1. *Assume $d \geq 2$ and consider a real number $A \in (1, \frac{4}{3})$. There is a full Lebesgue measure subset $\mathcal{E} \subset (0, +\infty)$ such that the following holds. For any $m \in \mathcal{E}$, there is $s_0 = s_0(d, n, A, m) > 0$ such that for any $s \geq s_0$, one can find $\varepsilon_0(d, n, A, m, s) \in (0, 1)$ such that for any couple $(w_0, w_1) \in \mathcal{H}^{s+1}(\mathbb{R}^d) \times \mathcal{H}^s(\mathbb{R}^d)$ of real-valued functions, with $||w_0||_{\mathcal{H}^{s+1}(\mathbb{R}^d)} + ||w_1||_{\mathcal{H}^s(\mathbb{R}^d)} \leq 1$, Eq. (5) admits a unique solution in the space:*

$$\mathcal{C}^0((-C\varepsilon^{-An}, +C\varepsilon^{-An}), \mathcal{H}^{s+1}(\mathbb{R}^d)) \cap \mathcal{C}^1((-C\varepsilon^{-An}, +C\varepsilon^{-An}), \mathcal{H}^s(\mathbb{R}^d)), \quad C = C(d, n, A, m, s) > 0,$$

with initial data $(w(0,\cdot), \dot{w}(0,\cdot)) = (\varepsilon w_0, \varepsilon w_1)$ and $\varepsilon \in (0, \varepsilon_0)$. Furthermore, there is $K = K(d, n, A, m, s) > 0$ such that

$$\forall t \in (-C\varepsilon^{-An}, +C\varepsilon^{-An}) \qquad ||w(t)||_{\mathcal{H}^{s+1}(\mathbb{R}^d)} + ||\dot{w}(t)||_{\mathcal{H}^s(\mathbb{R}^d)} \leq K\varepsilon.$$

A careful reading shows that [13] uses a normal form procedure that only needs the two following facts:

Fact 1: there is a positive number ρ such that the spectrum of $-\Delta + |x|^2$ on $L^2(\mathbb{R}^d)$ is included in $\rho\mathbb{N}$. That point is of interest in view to get adequate estimates of small divisors (see [13, Part 2.3]).

Fact 2: the eigenfunctions of the harmonic oscillator $-\Delta + |x|^2$ fulfill the estimates given by the following result.

Proposition 2.2. *For any integer $k \in \mathbb{N}$, let us denote by $\Pi_k : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ the spectral projector on $\ker(-\Delta + |x|^2 - \lambda_k^2)$ with $\lambda_k = \sqrt{2k+d}$. For any positive integer $n \geq 2$, there is $\nu = \nu(d,n) > 0$ such that for any $(k_1, \dots, k_{n+2}) \in (\mathbb{N}\backslash\{0\})^{n+2}$ with $k_{n+2} \leq \dots \leq k_1$, any $(u_1, \dots, u_{n+2}) \in \mathcal{C}(X)^{n+2}$ and any integer $N \in \mathbb{N}^\star$, one has*

$$\left| \int_X \Pi_{k_1}(u_1) \dots \Pi_{k_{n+2}}(u_{n+2}) \mathrm{d}x \right| \leq C(d,n,N) \lambda_{k_3}^{\nu} \left(\frac{\lambda_{k_2}\lambda_{k_3}}{\lambda_{k_2}\lambda_{k_3} + \lambda_{k_1}^2 - \lambda_{k_2}^2} \right)^N \prod_{j=1}^{n+2} ||u_j||_{L^2}. \tag{6}$$

Proof. It suffices to follow the proof of [11, Proposition 3.6]. □

Since we do not need a specific harmonic analysis of the eigenfunctions of $-\Delta + |x|^2$, this gives a hope to handle compact manifolds X whose eigenvalues are integers but for which we have no knowledge of eigenfunctions. The estimates (4) and (6) play the same role to decompose the non-linearity w^{n+1} in (2) and (5). Their proofs are different, but rely on the same idea of computing commutators. However, it turns out that (4) and (6) are equivalent because of the following inequalities that hold for any nonnegative integers $k_3 \leq k_2 \leq k_1$:

$$\left(\frac{\lambda_{k_3}}{\lambda_{k_3} + \lambda_{k_1} - \lambda_{k_2}} \right)^2 \leq \frac{\lambda_{k_2}\lambda_{k_3}}{\lambda_{k_2}\lambda_{k_3} + \lambda_{k_1}^2 - \lambda_{k_2}^2} \leq \frac{\lambda_{k_3}}{\lambda_{k_3} + \lambda_{k_1} - \lambda_{k_2}}. \tag{7}$$

The right-hand side of (7) is obvious thanks to the inequality $\lambda_{k_1}^2 \geq \lambda_{k_1}\lambda_{k_2}$. The left-hand side of (7) is a consequence of the following straightforward computation:

$$\lambda_{k_3}\lambda_{k_2}(\lambda_{k_3} + \lambda_{k_1} - \lambda_{k_2})^2 - \lambda_{k_3}^2(\lambda_{k_2}\lambda_{k_3} + \lambda_{k_1}^2 - \lambda_{k_2}^2) = \lambda_{k_3}(\lambda_{k_1} - \lambda_{k_2})^2(\lambda_{k_2} - \lambda_{k_3}) \geq 0.$$

In conclusion, one could use the same strategy of that of Theorem 2.1 to prove Theorem 1.5: we order the spectrum of $\sqrt{-\Delta}$ on the compact Riemannian manifold X as an increasing sequence $(\lambda_k)_{k\geq 0}$ and we make use of the Delort–Szeftel estimates (4) to handle the non-linearity w^{n+1} with a normal form.

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