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Complex analysis

Extremal cases for the log canonical threshold

Cas extrêmes pour le seuil log-canonique

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ABSTRACT

We show that a recent result of Demailly and Pham Hoang Hiep [12] implies a description of plurisubharmonic functions with given Monge–Ampère mass and smallest possible log canonical threshold. We also study an equality case for the inequality from [12].

RÉSUMÉ

Nous montrons qu'un résultat récent de Demailly et Pham Hoang Hiep [12] implique une description des fonctions plurisousharmoniques avec une masse de Monge–Ampère donnée et le seuil log-canonique le plus petit possible. Nous étudions aussi le cas d'égalité dans l'inégalité de [12].

1. Introduction and statement of results

Let PSH_0 denote the collection of germs of all plurisubharmonic functions at the origin of $\mathbb{C}^n$. A basic characteristic of singularity of $u \in \mathrm{PSH}_0$ is its Lelong number $\nu_u = \nu_u(0) = \liminf u(z)/\log|z|$ as $z \to 0$. One more characteristic, introduced in various contexts by several authors (first, probably, in [22]), and which attracted recently considerable attention (e.g., [2,3,11–13,15–17]), is the *integrability index* (at 0) $\lambda_u = \inf\{\lambda > 0 : e^{-u/\lambda} \in L^2_{loc}(0)\}$. For an ideal $\mathcal{I} = \mathcal{I}(f_1, \dots, f_m) \subset \mathcal{O}_0$ generated by analytic germs $f_1, \dots, f_m$, the value $c(\mathcal{I}) = \lambda^{-1}_{\log|f|}$ is the log-canonical threshold of $\mathcal{I}$. Accordingly, $c_u = \lambda_u^{-1}$ is called the *log canonical threshold* of u.

A classical result due to Skoda [22] states that

$$\nu_u^{-1} \le c_u \le n\,\nu_u^{-1}, \tag{1}$$

the extremal situations (equalities) being realized, for example, for $u = \log|z_1|$ (for the first inequality) and $u = \log|z|$ (for the second one). A description of all functions u with $c_u = n\nu_u^{-1}$ was given in [21]. The other extremal relation seems to be more involved. The only known-to-us result in this direction concerns the case $n = 2$, where the functions satisfying $c_u = \nu_u^{-1}$ are proved in [15] to be of the form $u = c\log|f| + v$, where f is an analytic function regular at 0, and $v \in \mathrm{PSH}_0$ has zero Lelong number at 0.

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In this note, we concentrate on lower bounds for the log canonical threshold, with the main focus when the inequalities become equalities.

In [9] and [18], the log canonical threshold of a zero dimensional ideal $\mathcal{I} \subset \mathcal{O}_0$ was related to its Samuel multiplicity $e(\mathcal{I})$:

$$c(\mathcal{I}) \ge n\, e(\mathcal{I})^{-1/n}, \tag{2}$$

with an equality if and only if the integral closure of $\mathcal{I}$ is a power of the maximal ideal $\mathfrak{m}_0 \subset \mathcal{O}_0$. It was used by Demailly [11] for a corresponding bound for plurisubharmonic functions u with isolated singularity at 0, and extended then by Zeriahi [24] to all u with $(dd^c u)^n$ well defined (more precisely, for all u from the Cegrell class $\mathcal{E}$ [6]),

$$c_u \ge n\, e_n(u)^{-1/n}. \tag{3}$$

Here $e_k(u)$ are the Lelong numbers of the currents $(dd^c u)^k$ at 0:

$$e_k(u) = \left(dd^c u\right)^k \wedge \left(dd^c \log|z|\right)^{n-k}(0), \quad 1 \le k \le n,$$

and $d = \partial + \bar\partial$, $d^c = (\partial - \bar\partial)/2\pi i$. The Cegrell class $\mathcal{F}(D)$ is formed by limits of decreasing sequences of bounded plurisubharmonic functions u_j in D such that $u_j = 0$ on ∂D and $\sup_j \int_D (dd^c u_j)^n < \infty$, and $u \in \mathcal{E}(D)$ if, for any $K \Subset D$, one can find $v \in \mathcal{F}(D)$ such that $u = v$ on K, see [6]. In particular, all negative plurisubharmonic functions that are bounded outside a compact subset of D belong to $\mathcal{E}(D)$.

Note that $e_1(u) = \nu_u$. When $\mathcal{I} = \mathcal{I}(f_1, \ldots, f_p) \subset \mathcal{O}_0$ is a zero-dimensional ideal, then $e_n(\log|f|) = e(\mathcal{I})$, see [11]. If $\operatorname{codim} V(\mathcal{I}) = k$, the values $e_k(\log|f|)$ are mixed Rees multiplicities $e_k(\mathcal{I})$ of $\mathcal{I}$ and the maximal ideal $\mathfrak{m}_0$ that were considered, e.g., in [4].

A direct proof of Demailly's inequality (3) without using (2) was obtained in [2]. In [11], the question of equality in (3) has been raised, and it was conjectured that, similarly to the analytic case $u = \log|f|$, the extremal functions should be plurisubharmonic functions with logarithmic singularity at 0.

In [21], Demailly's inequality was used to get the 'intermediate' bounds

$$c_u \ge k\, e_k(u)^{-1/k}, \quad 1 \le k \le l, \tag{4}$$

where l is the codimension of an analytic set A such that $u^{-1}(-\infty) \subset A$. None of the bounds for different values of k can be deduced from the others.

In a recent paper [12], an optimal bound for the integrability index in terms of the Lelong numbers was obtained: if $u \in \mathcal{E}$ near 0 and $e_1(u) > 0$, then

$$c_u \ge E_n(u) := \sum_{1 \le k \le n} \frac{e_{k-1}(u)}{e_k(u)}, \tag{5}$$

where $e_0(u) = 1$. It is easy to see that this bound implies all the relations (4) for the case when $l = n$ (that is, for u with isolated singularity). Here we will show that it also gives an answer to the aforementioned question on equality in (3).

To state it, we need the following notion from [20]. Let D be a bounded, hyperconvex neighborhood of 0. Given a function $u \in \mathrm{PSH}^-(D)$ (negative and plurisubharmonic in D), its *greenification* g_u at 0 is the regularized upper envelope of all functions $v \in \mathrm{PSH}^-(D)$ such that $v \le u + O(1)$ near 0.

The greenification of $\log|z|$ is the standard pluricomplex Green function with pole at 0. For u satisfying $(dd^c u)^n = 0$ on a punctured neighborhood of the origin, g_u is the Green function in the sense of Zahariuta [23]. The greenification of a *multi-circled singularity* $u(z) = u(|z_1|, \ldots, |z_n|) + O(1)$ in the unit polydisk $\mathbb{D}^n$ is the so-called *indicator*: a multi-circled function satisfying $g_u(|z_1|^c, \ldots, |z_n|^c) = c g_u(z)$ $\forall c > 0$ [21].

One has always $(dd^c g_u)^n = 0$ on $\{g_u > -\infty\}$. Evidently, $g_u \ge u$, while the relation $g_u = u + O(1)$ needs not be true. Nevertheless, the greenification keeps the considered characteristics of singularity:

Lemma 1.1. *Let $u \in \mathrm{PSH}_0$ and let g_u be its greenification on a bounded hyperconvex neighborhood D of 0. Then $\lambda_{g_u} = \lambda_u$. If, in addition, $u \in \mathcal{E}$ on a neighborhood of 0, then $g_u \in \mathcal{F}(D)$, $(dd^c g_u)^n = 0$ on $D \setminus \{0\}$, and $e_k(g_u) = e_k(u)$ for all k.*

Therefore, the only information on the asymptotic behavior of u that one can expect from the values of c_u and e_k is the one on its greenifications g_u.

Theorem 1.2. *For any $u \in \mathcal{E}$ near 0, the relation $c_u = n\, e_n(u)^{-1/n}$ holds if and only if its greenification for some (and then for any) bounded hyperconvex domain D satisfies $g_u = e_1(u)\log|z| + O(1)$ as $z \to 0$.*

Corollary 1.3. *Let $u \in \mathcal{F}(D)$, $e_1(u) = 1$, and $\int_D (dd^c u)^n = (n\lambda_u)^n$. Then u is the pluricomplex Green function for D with logarithmic singularity at 0.*

In the case of analytic singularities, $u = \log|f|$, statement (i) of Theorem 1.2, recovers the aforementioned result from [9] on equality in the bound for log canonical thresholds.

The next question is when equalities in (4) and (5) occur. Moreover, the latter bound can be extended to the case of functions not from $\mathcal{E}$, which rises a question on the equality cases.

Theorem 1.4. *If $u \in \mathrm{PSH}_0$ is locally bounded outside an analytic set of codimension $l > 1$, then*

$$c_u \ge E_l(u) := \sum_{1\le k\le l} \frac{e_{k-1}(u)}{e_k(u)}. \tag{6}$$

(Note that relation (6) for $l = 1$ is the lower bound in Skoda's inequalities (1) and it does not require any assumption on u.)

For multi-circled singularities $\varphi(z) = \varphi(|z_1|, \dots, |z_n|) + O(1)$ and any l, it was proved in [21] that the relation $c_\varphi = l\, e_l(\varphi)^{-1/l}$ holds if and only if its greenification g_φ in $\mathbb{D}^n$ equals $e_1(z) \max_{j\in J} \log|z_j|$ for an l-tuple $J \subset \{1, \dots, n\}$.

Theorem 1.5. *If a multi-circled plurisubharmonic singularity φ satisfies $c_\varphi = E_l(\varphi)$, then*

$$g_\varphi(z) = \max_{j\in J} \frac{\log|z_j|}{a_j} \tag{7}$$

for some l-tuple $J = (j_1, \dots, j_l) \subset \{1, \dots, n\}$ and $a_j > 0$.

A characterization of functions of the form (7) is that they generate *monomial valuations* $\mathfrak{v}_\varphi$ on plurisubharmonic singularities u by $\mathfrak{v}_\varphi(u) = \liminf u(z)/\varphi(z)$ as $z \to 0$. One could ask if the statement of Theorem 1.5 remains true for φ generating *quasi-monomial valuations*, i.e., monomial ones on birational models [5]. As the following example shows, the answer is no.

Example 1. As follows from [14], the function $\varphi = \log(|z_1^4| + |z_1^3 - z_2^2|)$ generates a quasi-monomial valuation. Since $u = \log|z_1^3 - z_2^2| \le \varphi \le v = \log(|z_1^4| + |z_1^3| + |z_2^2|)$ and $c_u = c_v = 5/6$, we have $c_\varphi = 5/6 > E_2(\varphi) = 3/4$.

2. Proofs

1. *Proof of Lemma 1.1.* Evidently, $c_{g_u} \ge c_u$. By the Choquet lemma, there exists a sequence u_j increasing a.e. to g_u and such that $u_j \le u + O(1)$ and so, $c_{u_j} \le c_u$. Semicontinuity theorem [13] shows then $c_{g_u} \le c_u$.

Let $u \in \mathcal{E}(\omega)$, $0 \in \omega \subset D$. Then there exists $v \in \mathcal{F}(\omega)$ such that $v = u$ near 0. Furthermore, there exists $w \in \mathcal{F}(D)$ such that $w \le v$ on ω [8]. Since $w \le g_u$, the function g_u belongs to $\mathcal{F}(D)$. The relation $(dd^c g_u)^n = 0$ outside 0 follows by standard arguments, because the maximality of $v \in \mathcal{E}$ on an open set U is equivalent to $(dd^c v)^n(U) = 0$.

To prove $e_k(g_u) = e_k(u)$, we take again a sequence u_j increasing a.e. to g_u; u_j can be chosen to be from the class $\mathcal{F}(D)$; otherwise we replace them by $\max\{u_j, w\}$. Therefore, the currents $(dd^c u_j)^k$ converge to $(dd^c g_u)^k$ [7] (the result is stated there only on the convergence of $(dd^c u_j)^n$, while the proof uses induction in the degree k). By the semicontinuity theorem for the Lelong numbers [10], this implies $\limsup_{j\to\infty} e_k(u_j) \le e_k(g_u)$. On the other hand, the relations $u_j \le u + O(1) \le g_u$ give us, by the comparison theorem for the Lelong numbers [10], $e_k(u) \le \limsup e_k(u_j)$ and $e_k(g_u) \le e_k(u)$. □

2. Further proofs are based essentially on estimate (5) and the following uniqueness result.

Lemma 2.1. *(See [1, Thm. 3.7]; for greenifications of isolated singularities, [20, Lem. 6.3].) If $u, v \in \mathcal{F}(D)$ are such that $u \le v$ and $(dd^c u)^n = (dd^c v)^n$, then $u = v$. As a consequence, if $u, v \in \mathcal{E}$, $u \le v + O(1)$ near 0, and $e_n(u) = e_n(v)$, then $g_u = g_v$.*

3. *Proof of Theorem 1.2.* By Lemma 1.1, we can assume $u = g_u$. Relation (5) gives us $E_n(u) = n\, e_n(u)^{-1/n}$, and by the arithmetic–geometric mean theorem, we get then

$$\frac{e_{k-1}(u)}{e_k(u)} = \frac{e_{l-1}(u)}{e_l(u)}$$

for any $k, l \le n$, which implies $e_n(u) = [e_1(u)]^n$. Let $v = e_1(u)G$, where G denotes the pluricomplex Green function for D with logarithmic pole at 0. Since $u \in \mathrm{PSH}^-(D)$ satisfies $u \le e_1(u)\log|z| + O(1)$ as $z \to 0$, we have $u \le v$ on D, while $e_n(u) = e_n(v)$. By Lemma 2.1, we conclude then $u = v$. □

4. *Proof of Theorem 1.4.* The restriction u_L of u to a generic l-dimensional subspace $L \in G(l, n)$ has isolated singularity at 0 and, by Siu's theorem, $e_k(u_L) = e_k(u)$. By [13, Prop. 2.2], we have also $c_u \ge c_{u_L}$. Therefore, we can apply (5) to u_L and get the bound (6). □

5. *Proof of Proposition 1.5.* By considering again the restriction to a generic l-dimensional coordinate plane, we can assume $l = n$ and φ to coincide with its greenification g_φ in $\mathbb{D}^n$.

As was proved in [12], the bound (5) for multi-circled functions follows from the inequality

$$\varphi \le \Phi(z) := \left|\varphi(z^*)\right| \max_j \frac{\log|z_j|}{a_j}$$

with $a_j = |\log|z_j^*||$ (e.g., [19, Prop. 3]), where $z^* \in \Pi = \{z : |z_1 \cdot \ldots \cdot z_n| = 1/e\}$ is chosen such that $|\varphi(z^*)| = |\min\{\varphi(z) : z \in \Pi\}| = \lambda_\varphi$ [17, Thm. 5.8]. Namely, $E_k(\varphi) \le E_k(\Phi)$ for all k and $c_\varphi = c_\Phi = E_n(\Phi)$. Therefore, $c_\varphi = E_n(\varphi)$ implies $E_n(\varphi) = E_n(\Phi)$.

Following [12], we set $t_0 = 1$ and consider the function $f(t) = \sum_1^n t_{j-1}/t_j$ on the convex set $\{t \in \mathbb{R}_+^n : t_j^2 \le t_{j-1}t_{j+1}\}$. The function is decreasing in each variable t_j and strictly decreasing in t_n. Note that $E_n(v) = f_n(e_1(v), \ldots, e_n(v))$ for any $v \in \mathrm{PSH}_0$ with isolated singularity. If $e_n(\varphi) > e_n(\Phi)$, then we would have $E_n(\varphi) < E_n(\Phi)$, which is not true. Therefore, $e_n(\varphi) = e_n(\Phi)$ and, since $\varphi \le \Phi$, we get $\varphi = \Phi$ by Lemma 2.1.

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