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Complex analysis

A continuous link between the disk and half-plane cases of Grace's theorem

Un lien continu entre les cas du disque et du demi-plan dans le théorème de Grace

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ABSTRACT

We obtain a continuous link between the disk and half-plane cases of Grace's theorem and new, non-circular zero domains that stay invariant under the Schur–Szegő convolution.

RÉSUMÉ

On obtient un lien continu entre les cas du disque et du demi-plan dans le théorème de Grace, ainsi que de nouveaux domaines de zéros non cerclés, qui sont invariants par la convolution de Schur–Szegő.

1. Introduction

1.1. Main results

Let Ω be a connected set in $\mathbb{C}$. Depending on whether Ω is bounded or unbounded, we denote by $\pi_n(\Omega)$ the set of all polynomials of degree n or $\leq n$ with zeros only in Ω. A polynomial $g(z) = \sum_{k=0}^{n} b_k z^k$ of degree n is called a *multiplier* of $\pi_n(\Omega)$ if the *convolution*

$$(f * g)(z) := \sum_{k=0}^{n} a_k b_k z^k$$

of g with every $f(z) = \sum_{k=0}^{n} a_k z^k$ in $\pi_n(\Omega)$ again belongs to $\pi_n(\Omega)$. We denote the set of multipliers of $\pi_n(\Omega)$ by $\mathcal{M}_n(\Omega)$. The *pre-coefficient class* $\pi_n^*(\Omega)$ of a connected set $\Omega \subset \mathbb{C}$ is the set of all polynomials $f(z) = \sum_{k=0}^{n} b_k z^k$ for which

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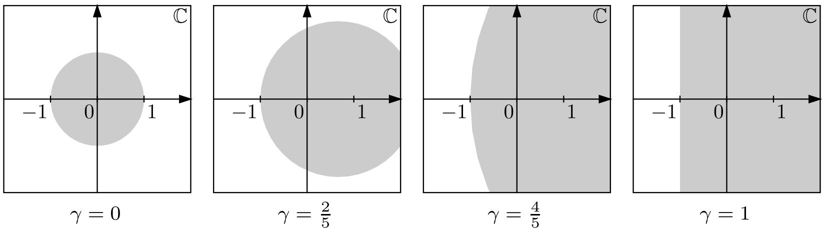

Fig. 1. The sets $\Omega_{-(1+\gamma),\gamma}$ (grey area) for certain values of γ.

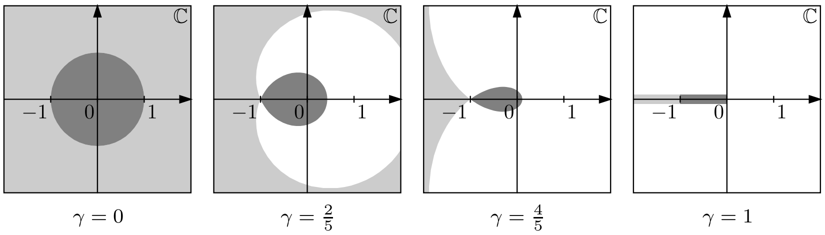

Fig. 2. The sets $\bar{I}_\gamma$ (dark grey) and $\overline{O}_\gamma$ (light grey) for certain values of γ.

$$f(z) * (1+z)^n = \sum_{k=0}^{n} \binom{n}{k} b_k z^k \in \pi_n(\Omega).$$

In this note we show that for every open or closed disk $\Omega \subset \mathbb{C}$ that contains the origin in its interior there is an associated set $\Omega^* \subset \mathbb{C}$ such that $\mathcal{M}_n(\Omega) = \pi_n^*(\Omega^*)$.

In order to give an explicit description of the sets Ω^*, note that, as explained in [5], for every open disk or half-plane Ω that contains the origin, there are two unique parameters $\tau \in \mathbb{C} \setminus \{0\}$ and $\gamma \in [0, 1]$ such that Ω is the image of the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ under a Möbius transformation of the form

$$w_{\tau,\gamma}(z) := \frac{\tau z}{1 + \gamma z}.$$

We write $\Omega_{\tau,\gamma}$ for such a domain and note that, for all $\tau \in \mathbb{C} \setminus \{0\}$ (cf. also Fig. 1),

$$\Omega_{\tau,0} = \big\{z \in \mathbb{C} : |z| < |\tau|\big\} \quad \text{and} \quad \Omega_{\tau,1} = \left\{z \in \mathbb{C} : \Re\big(\tau^{-1} z\big) < \frac{1}{2}\right\}. \tag{1}$$

For $\gamma \in [0, 1)$ we also define

$$I_\gamma := \big\{z \in \mathbb{C} : |z| + \gamma|1 + z| < 1\big\} \quad \text{and} \quad O_\gamma := \big\{z \in \mathbb{C} : |z| - \gamma|1 + z| > 1\big\}.$$

$\bar{I}_\gamma$ and $\overline{O}_\gamma$ are families of sets that, when γ increases from 0 to 1, decrease from $\bar{I}_0 = \overline{\mathbb{D}}$ and $\overline{O}_0 = \mathbb{C} \setminus \mathbb{D}$ to

$$\bar{I}_1 := \bigcap_{\gamma \in [0,1)} \bar{I}_\gamma = [-1, 0] \quad \text{and} \quad \overline{O}_1 := \bigcap_{\gamma \in [0,1)} \overline{O}_\gamma = (-\infty, -1], \tag{2}$$

respectively. For $\gamma \in (0, 1)$, I_γ is the interior of the inner loop of the limaçon of Pascal, and O_γ is the open exterior of the limaçon of Pascal (cf. Fig. 2).

Our main result can now be stated as follows.

Theorem 1.1. *Let* $\tau \in \mathbb{C} \setminus \{0\}$ *and* $\gamma \in [0, 1]$. *Then*

(i) $\mathcal{M}_n(\overline{\Omega}_{\tau,\gamma}) = \mathcal{M}_n(\Omega_{\tau,\gamma}) = \pi_n^*(\overline{I_\gamma})$, *and*
(ii) $\mathcal{M}_n(\mathbb{C} \setminus \Omega_{\tau,\gamma}) = \mathcal{M}_n(\mathbb{C} \setminus \overline{\Omega}_{\tau,\gamma}) = \pi_n^*(\overline{O}_\gamma)$.

By the definition of multiplier classes, it is clear that $f, g \in \mathcal{M}_n(\Omega)$ implies $f * g \in \mathcal{M}_n(\Omega)$. Theorem 1.1 thus leads to the following description of $\mathcal{M}_n(\Omega)$ for the domains $\Omega = I_\gamma$ and $\Omega = O_\gamma$.

Corollary 1.2. *Let* $\gamma \in [0, 1)$. *Then*

$$\mathcal{M}_n(I_\gamma) = \mathcal{M}_n(\bar{I}_\gamma) = \pi_n^*(\bar{I}_\gamma) \quad \textit{and} \quad \mathcal{M}_n(O_\gamma) = \mathcal{M}_n(\overline{O}_\gamma) = \pi_n^*(\overline{O}_\gamma).$$

1.2. Connection to the Schur–Szegő convolution

A *circular domain* in $\mathbb{C}$ is the image of the open or closed unit disk under a Möbius transformation. As we will show, Theorem 1.1 is a (surprisingly yet undiscovered) special case of the following classical result, which is a reformulation due to Szegő [7] of a theorem of Grace [2] regarding apolar polynomials. In the following, we will refer to it simply as Grace's theorem.

Theorem 1.3 *(Grace's theorem). Let*

$$F(z) = \sum_{k=0}^{n} A_k z^k \quad \text{and} \quad G(z) = \sum_{k=0}^{n} \binom{n}{k} b_k z^k$$

be polynomials of degree $n \in \mathbb{N}$ and suppose that $\Omega \subset \mathbb{C}$ is a circular domain, but not the exterior of a disk, that contains all zeros of F. Then each zero γ of the Schur–Szegő convolution *of F and G,*

$$F *_S G(z) := \sum_{k=0}^{n} A_k b_k z^k,$$

is of the form $\gamma = -\alpha\beta$ with $\alpha \in \Omega$ and $G(\beta) = 0$. If $G(0) \neq 0$, this also holds when Ω is the exterior of a disk.

This theorem almost immediately leads to a description of the multiplier classes of all disks centered at the origin, the exteriors of such disks, all half-planes, and the boundaries of all those domains. In particular, it implies the following (cf. for instance [3, Sect. 5.5]).

Corollary 1.4. *Let D be an open or closed disk centered at the origin, and H be an open or closed half-plane. Then*

(i) $\mathcal{M}_n(D) = \pi_n^*(\overline{\mathbb{D}})$ *and* $\mathcal{M}_n(\mathbb{C} \setminus D) = \pi_n^*(\mathbb{C} \setminus \mathbb{D})$,
(ii) $\mathcal{M}_n(H) = \pi_n^*([-1, 0])$, *if the interior of H contains the origin, and* $\mathcal{M}_n(H) = \pi_n^*((-\infty, -1])$, *if the closure of H does not contain the origin,*
(iii) $\mathcal{M}_n([-1, 0]) = \pi_n^*([-1, 0])$ *and* $\mathcal{M}_n((-\infty, -1]) = \pi_n^*((-\infty, -1])$.

To the best of our knowledge, the question of how the 'disk statements' (i) and the 'half-plane statements' (ii) of Corollary 1.4 are connected to each other has not been considered until now. The answer to this question is given here by Theorem 1.1. Note also that Corollary 1.2 gives a continuous link between the statements (i) and (iii) of Corollary 1.4, and that Corollary 1.2 seems to be the first result in which the multiplier classes for domains that are non-circular are determined.

In [1], Borcea and Brändén used Grace's theorem to obtain characterizations of all linear operators on the space of complex polynomials that preserve the sets $\pi_n(\Omega)$ and $\pi_n(\partial\Omega)$ for disks or half-planes Ω, and it is possible to deduce Theorem 1.1 from these very general results. In the following section, however, we will present a short proof of Theorem 1.1 that makes use only of Grace's theorem.

Finally, we would like to mention that our interest in the question considered in this paper was strongly motivated by a recent paper [5] by Ruscheweyh and Salinas (cf. also [4,6]), in which the sets I_γ and O_γ first appeared in connection with the set $\Omega_{\tau,\gamma}$ (observe that with Ω_γ^* and L_γ as defined in [5] we have $\Omega_\gamma^* = -\mathbb{C} \setminus \overline{O}_\gamma$ and $L_\gamma = -I_\gamma$).

2. Proofs

2.1. An auxiliary lemma

Lemma 2.1. *Let $\tau \in \mathbb{C} \setminus \{0\}$ and $\gamma \in [0, 1)$.*

(i) *Suppose G is of degree n. Then $F *_S G \in \pi_n(\Omega_{\tau,\gamma})$ for all $F \in \pi_n(\overline{\Omega}_{\tau,\gamma})$ if, and only if, $G \in \pi_n(I_\gamma)$.*
(ii) *Suppose G is of degree $\leq n$. Then $F *_S G \in \pi_n(\mathbb{C} \setminus \overline{\Omega}_{\tau,\gamma})$ for all $F \in \pi_n(\mathbb{C} \setminus \Omega_{\tau,\gamma})$ if, and only if, $G \in \pi_n(O_\gamma)$.*

Proof. We begin by proving (i) and suppose $G \in \pi_n(I_\gamma)$. Then $\beta \in I_\gamma$ for every zero β of G, which means:

$$\gamma|1 + \beta| < 1 - |\beta|. \tag{3}$$

This holds if, and only if,

$$|\beta| < \left|1 + \gamma z(1 + \beta)\right| \quad \text{for all } z \in \partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\},$$

and hence, by the maximum principle (note that $1/(\gamma|1+\beta|) > 1/(1-|\beta|) > 1$ by (3)), if, and only if,

$$\omega(z) = \frac{-\beta z}{1+\gamma z(1+\beta)}$$

maps $\overline{\mathbb{D}}$ into $\mathbb{D}$. Since

$$-\beta\, w_{\tau,\gamma}(z) = \frac{-\beta\tau z}{1+\gamma z} = \frac{\tau\omega(z)}{1+\gamma\omega(z)} = w_{\tau,\gamma}\big(\omega(z)\big),$$

this shows

$$-\beta\, \overline{\Omega}_{\tau,\gamma} = -\beta\, w_{\tau,\gamma}(\overline{\mathbb{D}}) \subseteq w_{\tau,\gamma}(\mathbb{D}) = \Omega_{\tau,\gamma}$$

for every zero β of G. This implies $F *_S G \in \pi_n(\Omega_{\tau,\gamma})$ by Grace's theorem.

Our argumentation also shows that if G of degree n has a zero $\beta \notin I_\gamma$, then there is an $\alpha \in \overline{\Omega}_{\tau,\gamma}$ such that $-\alpha\beta \notin \Omega_{\tau,\gamma}$. For such an α, the polynomial

$$F(z) := (1 - z/\alpha)^n = \sum_{k=0}^{n} \binom{n}{k} (-\alpha)^{-k} z^k$$

is of degree n with all zeros in $\overline{\Omega}_{\tau,\gamma}$ and we have:

$$(F *_S G)(z) = G(-z/\alpha).$$

Hence, in this case $F *_S G$ has a zero at $-\alpha\beta$ that is not in $\Omega_{\tau,\gamma}$. The proof of (i) is thus complete.

In order to prove (ii), recall that if $F(z) = \sum_{k=0}^{m} A_k z^k$ is a polynomial of degree $m \le n$ with $F(0) \ne 0$, then the *n-inverse*

$$F^{*n}(z) := z^n \overline{F\left(\bar{z}^{-1}\right)} = \sum_{k=n-m}^{n} \overline{A}_{n-k} z^k$$

of F is of degree n, and the zeros of F^{*n} are those of F reflected around the unit circle. In particular, we have that

$$F \mapsto F^{*n} \quad \text{is a bijection between } \pi_n(\mathbb{C} \setminus \Omega_{\tau,\gamma}) \text{ and } \pi_n(\overline{\Omega}_{(\gamma^2-1)/\bar{\tau},\gamma}).$$

Hence, G of degree $\le n$ is such that $F *_S G \in \pi_n(\mathbb{C} \setminus \overline{\Omega}_{\tau,\gamma})$ for all $F \in \pi_n(\mathbb{C} \setminus \Omega_{\tau,\gamma})$, if, and only if,

$$R *_S G^{*n} = (F *_S G)^{*n} \in \pi_n(\Omega_{(\gamma^2-1)/\bar{\tau},\gamma})$$

for all $F^{*n} =: R \in \pi_n(\overline{\Omega}_{(\gamma^2-1)/\bar{\tau},\gamma})$. Because of (i), this is equivalent to $G^{*n} \in \pi_n(I_\gamma)$. Since $G^{*n} \mapsto G$ is a bijection between $\pi_n(I_\gamma)$ and $\pi_n(O_\gamma)$, we have verified (ii). □

2.2. Proof of Theorem 1.1

Every polynomial in $\pi_n(\bar{I}_\gamma)$ or $\pi_n(\overline{O}_\gamma)$ can be approximated by polynomials in $\pi_n(I_\gamma)$ or $\pi_n(O_\gamma)$, respectively. The relations $\mathcal{M}_n(\overline{\Omega}_{\tau,\gamma}) \supseteq \pi_n^*(\bar{I}_\gamma)$ and $\mathcal{M}_n(\mathbb{C} \setminus \Omega_{\tau,\gamma}) \supseteq \pi_n^*(\overline{O}_\gamma)$ thus follow directly from Lemma 2.1. On the other hand, if g of degree n is such that $F * g \in \pi_n(\overline{\Omega}_{\tau,\gamma})$ for all $F \in \pi_n(\overline{\Omega}_{\tau,\gamma})$, then $(F * g)(xz) = F(z) * g(xz) \in \pi_n(\Omega_{\tau,\gamma})$ for all $x > 1$. By Lemma 2.1, this implies $g(xz) \in \pi_n^*(I_\gamma)$ for all $x > 1$, and thus $g \in \pi_n^*(\bar{I}_\gamma)$. This shows that $\mathcal{M}_n(\overline{\Omega}_{\tau,\gamma}) \subseteq \pi_n^*(\bar{I}_\gamma)$, and hence that $\mathcal{M}_n(\overline{\Omega}_{\tau,\gamma}) = \pi_n^*(\bar{I}_\gamma)$.

In a similar way, we can prove that $\mathcal{M}_n(\mathbb{C} \setminus \Omega_{\tau,\gamma}) = \pi_n^*(\overline{O}_\gamma)$. We will omit the proofs of the remaining two relations $\mathcal{M}_n(\Omega_{\tau,\gamma}) = \pi_n^*(\bar{I}_\gamma)$ and $\mathcal{M}_n(\mathbb{C} \setminus \overline{\Omega}_{\tau,\gamma}) = \pi_n^*(\overline{O}_\gamma)$, since they are very similar to the proofs of the two relations we have just shown. We have thus verified Theorem 1.1.

2.3. Proof of Corollary 1.2

If $F \in \pi_n(\bar{I}_\gamma)$ and $G \in \pi_n(I_\gamma)$, then by Lemma 2.1(i) we have $H *_S G \in \pi_n(\Omega_{\tau,\gamma})$ for all $H \in \pi_n(\overline{\Omega}_{\tau,\gamma})$, and consequently, by Theorem 1.1(i), $H *_S G *_S F \in \pi_n(\Omega_{\tau,\gamma})$ for all such H. Another application of Lemma 2.1(i) shows that $F *_S G \in \pi_n(I_\gamma)$. On the other hand, if G of degree n is such that $F *_S G \in \pi_n(I_\gamma)$ for all $F \in \pi_n(\bar{I}_\gamma)$, then in particular

$$G(z) = (1+z)^n *_S G(z) \in \pi_n(I_\gamma),$$

since $-1 \in \bar{I}_\gamma$. This proves that for a polynomial G of degree n we have $F *_S G \in \pi_n(I_\gamma)$ for all $F \in \pi_n(\bar{I}_\gamma)$ if, and only if, $G \in \pi_n(I_\gamma)$. In a similar way, one can show that for a polynomial G of degree $\le n$, we have $F *_S G \in \pi_n(O_\gamma)$ for all $F \in \pi_n(\overline{O}_\gamma)$ if, and only if, $G \in \pi_n(O_\gamma)$. Corollary 1.2 now follows from these two relations in the same way as Theorem 1.1 follows from Lemma 2.1.

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