C. R. Acad. Sci. Paris, Ser. I 352 (2014) 993–998

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Mathematical analysis/Partial differential equations

Multi-marginal Monge–Kantorovich transport problems: A characterization of solutions

Problèmes de transport multi-marginal de Monge–Kantorovich : Une caractérisation des solutions

Abbas Moameni [1]

School of Mathematics and Statistics, Carleton University, Ottawa, ON, K1S 5B6, Canada

A R T I C L E I N F O

Article history:
Received 22 May 2014
Accepted after revision 8 October 2014
Available online 23 October 2014

Presented by the Editorial Board

A B S T R A C T

We shall present a measure theoretical approach that, together with the Kantorovich duality, provides an efficient tool to study the optimal transport problem. Specifically, we study the support of optimal plans where the cost function does not satisfy the classical twist condition in the two marginal problem as well as in the multi-marginal case when twistedness is limited to certain subsets.

R É S U M É

Dans cet article, nous étudions le problème de transport optimal du point de vue de la théorie de la mesure, à l'aide de la dualité de Kantorovich. En particulier, nous étudions le support des plans optimaux où la fonction coût ne satisfait pas la condition de « twist » dans le problème à deux marginales, ainsi que dans le cas multi-marginales quand la condition « twist » est limitée à des sous-ensembles précis.

1. Introduction

We consider the Monge–Kantorovich transport problem for Borel probability measures $\mu_1, \mu_2, \ldots, \mu_n$ on smooth manifolds $X_1, X_2, \ldots, X_n$. The cost function $c : X_1 \times X_2 \times \ldots \times X_n \to \mathbb{R}$ is bounded and continuous. Let $\Pi(\mu_1, \ldots, \mu_n)$ be the set of Borel probability measures on $X_1 \times X_2 \times \ldots \times X_n$ that have X_i-marginal μ_i for each $i \in \{1, 2, \ldots, n\}$. The transport cost associated with a transport plan $\pi \in \Pi(\mu_1, \ldots, \mu_n)$ is given by:

$$I_c(\pi) = \int\limits_{X_1 \times X_2 \times \ldots \times X_n} c(x_1, \ldots, x_n)\, \mathrm{d}\pi\,.$$

E-mail address: momeni@math.carleton.ca.

[1] Supported by the Natural Sciences and Engineering Research Council of Canada, Grant number 315920.

http://dx.doi.org/10.1016/j.crma.2014.10.004

We consider the Monge–Kantorovich transport problem,

$$\inf\{I_c(\pi); \pi \in \Pi(\mu_1, ..., \mu_n)\}. \tag{MK}$$

If a transport plan minimizes the cost, it will be called an optimal plan. We say that an optimal plan γ induces a Monge solution if it is concentrated on the graph $\{(x, T(x)); x \in X_1\}$ of a measurable map $T : X_1 \to X_2 \times ... \times X_n$. Contrary to the Monge problem, the Kantorovich problem always admits solutions as soon as the cost function is a non-negative lower semi-continuous function (see [17] for a proof). When $n = 2$, a general criterion for the existence and uniqueness of an optimal transport map known as the twist condition dictates the map $y \to D_1c(x, y)$ to be injective for fixed $x \in X_1$. Under the twist condition and the absolute continuity of μ_1, the optimal plan γ that solves the Monge–Kantorovich problem (MK) is supported on the graph of an optimal transport map T, i.e., $\gamma = (\mathrm{Id} \times T)_\# \mu$. For larger n, questions regarding the existence and uniqueness are not fully understood yet. By now, there are many interesting results for the multi-marginal problem in the general case as well as particular models (see for instance [3–8,10–12,15,16], the bibliography is not exhaustive). When $n > 2$, as shown in [11], the twist condition can be replaced by twistedness on c-splitting sets.

Definition 1.1. A set $S \subset X_1 \times X_2 \times ... \times X_n$ is a c-splitting set if there exists Borel functions $u_i : X_i \to \mathbb{R}$ such that for all $(x_1, x_2, ..., x_n)$,

$$\sum_{i=1}^{n} u_i(x_i) \le c(x_1, x_2, ..., x_n)$$

with equality whenever $(x_1, x_2, ..., x_n) \in S$. The n-tuple $(u_1, ..., u_n)$ is called the c-splitting tuple for S.

In [13], for the case $n = 2$, the author relaxed the twist condition by a new property, i.e.,

- the generalized-twist condition: we say that c satisfies the generalized-twist condition if for any $\bar{x}_1 \in X_1$ and $\bar{x}_2 \in X_2$ the set $L_{(\bar{x}_1,\bar{x}_2)} := \{x_2 \in X_2; D_1c(\bar{x}_1, x_2) = D_1c(\bar{x}_1, \bar{x}_2)\}$ is a finite subset of X_2. Moreover, if there exists $m \in \mathbb{N}$ such that for each $\bar{x}_1 \in X_1$ and $\bar{x}_2 \in X_2$ the cardinality of the set $L_{(\bar{x}_1,\bar{x}_2)}$ does not exceed m, then we say that c satisfies the m-twist condition.

Under the m-twist condition, it is shown that, for each optimal plan γ of (MK), there exist a sequence of non-negative measurable real functions $\{\alpha_i\}_{i=1}^m$ on X_1 with $\sum_{i=1}^m \alpha_i = 1$, and Borel measurable maps $G_1, ..., G_m : X_1 \to X_2$ such that $\gamma = \sum_{i=1}^m \alpha_i (\mathrm{Id} \times G_i)_\# \mu$.

Our aim in this work is to extend this result to the multi-marginal case. For the rest of the paper we always assume that c is non-negative, lower semi-continuous, $\bigoplus_{i=1}^n \mu_i$-a.e. differentiable with respect to the first variable and that $I_c(\gamma)$ is finite for some transport plan γ. We also assume that the Kantorovich dual problem admits a solution $(\varphi_1, ..., \varphi_n)$ such that φ_1 is differentiable μ_1-a.e., $\varphi_1(x_1) + ... + \varphi_n(x_n) \le c(x_1, ..., x_n)$ for all $(x_1, ..., x_n)$ and

$$\int c \, \mathrm{d}\gamma = \sum_{i=1}^{n} \int_{X_i} \varphi_i(x_i) \, \mathrm{d}\mu_i.$$

We denote by $D_1(c)$ the set of points at which c is differentiable with respect to the first variable. The generalized twist structure takes the following form in the multi-marginal case.

Definition 1.2. Let c be a Borel measurable function.

1. **m-Twist condition:** say that c is m-twisted on c-splitting sets if, for any c-splitting set $S \subset X_1 \times X_2 \times ... \times X_n$ and any $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) \in S \cap D_1(c)$, the cardinality of the set
$$\{(\bar{x}_1, x_2, ..., x_n) \in S \cap D_1(c); D_{x_1}c(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) = D_{x_1}c(\bar{x}_1, x_2, ..., x_n)\},$$
is at most m.
2. **Generalized-twist condition:** say that c satisfies the generalized twist condition on c-splitting sets if, for any c-splitting set $S \subset X_1 \times X_2 \times ... \times X_n$ and any $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) \in S \cap D_1(c)$, the set
$$\{(\bar{x}_1, x_2, ..., x_n) \in S \cap D_1(c); D_{x_1}c(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) = D_{x_1}c(\bar{x}_1, x_2, ..., x_n)\},$$
is a finite subset of S.

The following result provides a connection between the generalized twist condition and the local 1-twistedness.

Proposition 1.1. *Assume that c is continuously differentiable with respect to the first variable and S is a compact c-splitting set. If c is locally 1-twisted on S, then c satisfies the generalized-twist condition on S.*

We now state our main result in this paper.

Theorem 1.3. *Assume that the cost function c satisfies the m-twist condition on c-splitting sets and μ_1 is non-atomic. Then for each optimal plan γ of (MK) with $Supp(\gamma) \subset D_1(c)$, there exist $k \le m$, a sequence of non-negative measurable real functions $\{\alpha_i\}_{i=1}^m$ on X_1, and Borel measurable maps $G_1, \ldots, G_k : X_1 \to X_2 \times \ldots \times X_n$ such that*

$$\gamma = \sum_{i=1}^{k} \alpha_i (\mathrm{Id} \times G_i)_{\#}\mu, \tag{1}$$

where $\sum_{i=1}^{k} \alpha_i(x) = 1$ for μ_1-a.e. $x \in X_1$.

As shown in [13], the most interesting examples of costs satisfying the generalized-twist condition are non-degenerate costs on smooth n-dimensional manifolds X and Y. Denote by $D^2_{xy}c(x_0, y_0)$ the $n \times n$ matrix of mixed second-order partial derivatives of the function c at the point (x_0, y_0). A cost $c \in C^2(X \times Y)$ is non-degenerate provided $D^2_{xy}c(x_0, y_0)$ is non-singular, that is $\det(D^2_{xy}c(x_0, y_0)) \ne 0$. In our forthcoming work [14], following an idea in [16] together with Proposition 1.1, a differential condition similar to the non-degeneracy condition (in $n = 2$) is derived for the multi-marginal case that guaranties the general twist property on c-splitting sets and, consequently, the characterization of the support of optimal plans due to Theorem 1.3.

In the next section, we shall discuss the key ingredients for our methodology in this work. Section 3 is devoted to the proof of the main results.

2. Measurable sections and extremity

Let $(X, \mathcal{B}, \mu)$ be a finite, not necessarily complete measure space, and (Y, Σ) a measurable space. The completion of $\mathcal{B}$ with respect to μ is denoted by $\mathcal{B}_\mu$; when necessary, we identify μ with its completion on $\mathcal{B}_\mu$. The push forward of the measure μ by a map $T : (X, \mathcal{B}, \mu) \to (Y, \Sigma)$ is denoted by $T_{\#}\mu$, i.e.

$$T_{\#}\mu(A) = \mu\big(T^{-1}(A)\big), \quad \forall A \in \Sigma.$$

Definition 2.1. Let $T : X \to Y$ be $(\mathcal{B}, \Sigma)$-measurable and ν a positive measure on Σ. We call a map $F : Y \to X$ a $(\Sigma_\nu, \mathcal{B})$-measurable section of T if F is $(\Sigma_\nu, \mathcal{B})$-measurable and $T \circ F = \mathrm{Id}_Y$.

If X is a topological space, we denote by $\mathcal{B}(X)$ the set of Borel sets on X. The space of Borel probability measures on a topological space X is denoted by $\mathcal{P}(X)$. For a measurable map $T : (X, \mathcal{B}(X)) \to (Y, \Sigma, \nu)$ denote by $\mathcal{M}(T, \nu)$ the set of all measures λ on $\mathcal{B}$ so that T pushes λ forward to ν, i.e.

$$\mathcal{M}(T, \nu) = \big\{\lambda : T_{\#}\lambda = \nu\big\}.$$

Evidently $\mathcal{M}(T, \nu)$ is a convex set. A measure λ is an extreme point of $\mathcal{M}(T, \nu)$ if the identity $\lambda = \theta\lambda_1 + (1-\theta)\lambda_2$ with $\theta \in (0, 1)$ and $\lambda_1, \lambda_2 \in \mathcal{M}(T, \nu)$ implies that $\lambda_1 = \lambda_2$. The set of extreme points of $\mathcal{M}(T, \nu)$ is denoted by $ext\, \mathcal{M}(T, \nu)$.

We recall the following result from [9] in which a characterization of the set $ext\, \mathcal{M}(T, \nu)$ is given.

Theorem 2.2. *Let (Y, Σ, ν) be a probability space, $(X, \mathcal{B}(X))$ be a Hausdorff space with a Radon probability measure λ, and let $T : X \to Y$ be an $(\mathcal{B}(X), \Sigma)$-measurable mapping. Assume that T is surjective and Σ is countably separated. The following conditions are equivalent:*

(i) *λ is an extreme point of $M(T, \nu)$;*
(ii) *there exists a $(\Sigma_\nu, \mathcal{B}(X))$-measurable section $F : Y \to X$ of the mapping T with $\lambda = F_{\#}\nu$.*

By making use of the Choquet theory in the setting of noncompact sets of measures [18], each $\lambda \in M(T, \nu)$ can be represented as a Choquet type integral over $ext\, M(T, \nu)$. Denote by $\Sigma_{ext\, M(T,\nu)}$ the σ-algebra over $ext\, M(T, \nu)$ generated by the functions $\varrho \to \varrho(B)$, $B \in \mathcal{B}(X)$. We have the following result (see [13] for a proof).

Theorem 2.3. *Let X and Y be complete separable metric spaces and ν a probability measure on $\mathcal{B}(Y)$. Let $T : (X, \mathcal{B}(X)) \to (Y, \mathcal{B}(Y))$ be a surjective measurable mapping and let $\lambda \in M(T, \nu)$. Then there exists a Borel probability measure ξ on $\sum_{ext\, M(T,\nu)}$ such that for each $B \in \mathcal{B}(X)$,*

$$\lambda(B) = \int\limits_{ext\, M(T,\nu)} \varrho(B)\, \mathrm{d}\xi(\varrho), \quad \big(\varrho \to \varrho(B) \text{ is measurable}\big).$$

3. Proofs

In this section, we shall proceed with the proofs of the statements in the introduction. We first state some preliminaries required for the proofs. Let γ be a solution of (MK) such that $Supp(\gamma) \subset D_1(c)$. It is standard that $\gamma \in \Pi(\mu_1, \ldots, \mu_n)$ is non-atomic if and only if at least one μ_i is non-atomic (see for instance [1]). Set $Y = X_2 \times \ldots \times X_n$. Since μ_1 is non-atomic, it follows that the Borel measurable spaces $(X_1, \mathcal{B}(X_1), \mu_1)$ and $(X_1 \times Y, \mathcal{B}(X_1 \times Y), \gamma)$ are isomorphic. Thus, there exists an isomorphism $T = (T_1, T_2, \ldots, T_n)$ from $(X_1, \mathcal{B}(X_1), \mu_1)$ onto $(X_1 \times Y, \mathcal{B}(X_1 \times Y), \gamma)$. It can be easily deduced that $T_i : X_1 \to X_i$ are surjective maps and

$$T_i \# \mu_1 = \mu_i, \quad i = 1, 2, \ldots, n.$$

Consider the convex set:

$$\mathcal{M}(T_1, \mu_1) = \{\lambda \in \mathcal{P}(X_1);\ T_1 \# \lambda = \mu_1\},$$

and note that $\mu_1 \in \mathcal{M}(T_1, \mu_1)$. The following result is established in [13].

Lemma 3.1. *Suppose F_1, F_2 are two distinct sections of T_1. Then the set*

$$\{x \in X_1;\ F_1(x) = F_2(x)\}$$

is a null set with respect to the measure μ_1.

Proof of Theorem 1.3. Since $\mu_1 \in \mathcal{M}(T_1, \mu_1)$, it follows from Theorem 2.3 that there exists a Borel probability measure ξ on $\sum_{ext\, M(T_1,\mu_1)}$ such that for each $B \in \mathcal{B}(X_1)$,

$$\mu(B) = \int_{ext\, M(T_1,\mu_1)} \varrho(B)\, \mathrm{d}\xi(\varrho), \quad \big(\varrho \to \varrho(B) \text{ is measurable}\big). \tag{2}$$

On the other hand, there exist functions $\{\varphi_i\}_{i=1}^n$ such that $\varphi_1(x_1) + \ldots + \varphi_n(x_n) \le c(x_1, \ldots, x_n)$, φ_1 is μ_1-a.e. differentiable, and that

$$\int c\, \mathrm{d}\gamma = \sum_{i=1}^n \int_{X_i} \varphi_i(x_i)\, \mathrm{d}\mu_i.$$

Let S be the c-splitting set generated by the n-tuple $(\varphi_1, \ldots, \varphi_n)$, that is,

$$S = \left\{(x_1, \ldots, x_n);\ c(x_1, \ldots, x_n) = \sum_{i=1}^n \varphi_i(x_i)\right\}.$$

As $T = (T_1, T_2, \ldots, T_n)$ is an isomorphism from $(X_1, \mathcal{B}(X_1), \mu_1)$ onto $(X_1 \times Y, \mathcal{B}(X_1 \times Y), \gamma)$, it follows that

$$\int_{X_1} c(T_1x_1, T_2x_1, \ldots, T_nx_1)\, \mathrm{d}\mu_1 = \sum_{i=1}^n \int_{X_1} \varphi_i(T_ix_1)\, \mathrm{d}\mu_1.$$

from which, together with the fact that $\sum_{i=1}^n \varphi_i(x_i) \le c(x_1, \ldots, x_n)$, we obtain,

$$c(T_1x_1, T_2x_1, \ldots, T_nx_1) = \sum_{i=1}^n \varphi_i(T_ix_1). \quad \mu_1\text{-a.e.} \tag{3}$$

Since φ_1 is μ almost surely differentiable and $T_{1\#}\mu_1 = \mu_1$, it follows that

$$D_1c(T_1x_1, T_2x_1, \ldots, T_nx_1) = \nabla\varphi_1(T_1x_1) \quad \mu_1\text{-a.e.} \tag{4}$$

where D_1c stands for the partial derivative of c with respect to the first variable. Let $A_\gamma \in \mathcal{B}(X_1)$ be the set with $\mu(A_\gamma) = 1$ such that (3) and (4) hold for all $x_1 \in A_\gamma$, i.e.

$$c(T_1x_1, T_2x_1, \ldots, T_nx_1) = \sum_{i=1}^n \varphi_i(T_ix_1) \quad \text{and} \quad D_1c(T_1x_1, T_2x_1, \ldots, T_nx_1) = \nabla\varphi_1(T_1x_1) \quad \forall x_1 \in A_\gamma. \tag{5}$$

Since $\mu_1(X_1 \setminus A_\gamma) = 0$, it follows from (2) that

$$\int_{ext\,M(T_1,\mu_1)} \varrho(X_1 \setminus A_\gamma)\,d\xi(\varrho) = \mu(X_1 \setminus A_\gamma) = 0,$$

and therefore there exists a ξ-full measure subset K_γ of $ext\,M(T_1,\mu_1)$ such that $\varrho(X_1 \setminus A_\gamma) = 0$ for all $\varrho \in K_\gamma$.

For every $B \in \mathcal{B}(X)$ it follows from (2) that

$$\mu_1(B) = \int_{ext\,M(T_1,\mu_1)} \varrho(B)\,d\xi(\varrho) = \int_{K_\gamma} \varrho(B)\,d\xi(\varrho),$$

from which we obtain that μ is absolutely continuous with respect to $\sum_{i=1}^k \varrho_i$. It follows $d\mu/d(\sum_{i=1}^k \varrho_i) = \alpha(x)$ for some measurable non-negative function α. Assume that $F_1, ..., F_k$ are $(\mathcal{B}(X_1)_\mu, \mathcal{B}(X))$-measurable sections of the mapping T_1 with $\varrho_i = F_{i\#}\mu_1$. Setting $\alpha_i = \alpha \circ F_i$, it follows from $T_{1\#}\mu_1 = \mu_1$ that $\sum_{i=1}^k \alpha_i(x) = 1$ for μ_1-a.e. $x \in X_1$. It also follows from Corollary 6.7.6 in [2] that each F_i is μ_1-a.e. equal to a $(\mathcal{B}(X_1), \mathcal{B}(X_1))$-measurable function still denoted by F_i. For each $i \in \{1, ..., k\}$, let $G_i = (T_2 \circ F_i, ..., T_n \circ F_i)$. We now show that $\gamma = \sum_{i=1}^k \alpha_i (\mathrm{Id} \times G_i)_\# \mu$. For each bounded continuous function $f : X_1 \times Y \to \mathbb{R}$ it follows that

$$\begin{aligned}\int_{X_1\times Y} f(x,y)\,d\gamma = \int_{X_1} f(T_1x_1, T_2x_1, ..., T_nx_1)\,d\mu_1 &= \sum_{i=1}^k \int_{X_1} \alpha(x_1) f(T_1x_1, T_2x_1, ..., T_nx_1)\,d\varrho_i \\ &= \sum_{i=1}^k \int_{X_1} \alpha\big(F_i(x_1)\big) f\big(T_1 \circ F_i(x_1), T_2 \circ F_i(x_1), ..., T_n \circ F_i(x_1)\big)\,d\mu_1 \\ &= \sum_{i=1}^k \int_{X_1} \alpha_i(x) f\big(x_1, G_i(x_1)\big)\,d\mu_1.\end{aligned}$$

Therefore, $\gamma = \sum_{i=1}^k \alpha_i(\mathrm{Id} \times G_i)_\# \mu$. This completes the proof. □

We conclude this section by proving the generalized-twist property for the locally 1-twisted costs.

Proof of Proposition 1.1. Assume that $S \subset X_1 \times ... \times X_n$ is a c-splitting set. Fix $(\bar{x}_1, ..., \bar{x}_n) \in S$. We need to show that the set

$$L = \big\{(\bar{x}_1, x_2, ..., x_n) \in S;\ D_1c(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) = D_1c(\bar{x}_1, x_2, ...x_n)\big\}$$

is finite. If L is not finite, there exists an infinitely countable subset $\{(\bar{x}_1, x_2^k, ...x_n^k)\}_{k\in\mathbb{N}} \subset L$. Since S is compact, then the sequence $\{(\bar{x}_1, x_2^k, ...x_n^k)\}_{k\in\mathbb{N}}$ has an accumulation point $(\bar{x}_1, x_2^0, ...x_n^0) \in S$ and there exists a subsequence still denoted by $\{(\bar{x}_1, x_2^k, ...x_n^k)\}_{k\in\mathbb{N}}$ such that $x_i^k \to x_i^0$ as $k \to \infty$ for $i = 2, ..., n$. Since D_1c is continuous, it follows that $(\bar{x}_1, x_2^0, ...x_n^0) \in L$. Since c is locally 1-twisted on S, this leads to a contradiction as $(\bar{x}_1, x_2^0, ...x_n^0)$ is an accumulation point of the sequence $\{(\bar{x}_1, x_2^k, ...x_n^k)\}_{k\in\mathbb{N}}$ and

$$D_1c\big(\bar{x}_1, x_2^0, ...x_n^0\big) = D_1c\big(\bar{x}_1, x_2^k, ...x_n^k\big), \quad \forall k \in \mathbb{N}.$$

This completes the proof. □

References

[1] K.P.S. Bhaskara Rao, M. Bhaskara Rao, A remark on nonatomic measures, Ann. Math. Stat. 43 (1) (1972) 369–370.
[2] V.I. Bogachev, Measure Theory, vols. I, II, Springer-Verlag, Berlin, 2007.
[3] G. Buttazzo, L. De Pascale, P. Gori-Giorgi, Optimal transport formulation of electronic density-functional theory, Phys. Rev. A 85 (Jun 2012) 062502.
[4] G. Carlier, On a class of multidimensional optimal transportation problems, J. Convex Anal. 10 (2) (2003) 517–529.
[5] G. Carlier, B. Nazaret, Optimal transportation for the determinant, ESAIM Control Optim. Calc. Var. 14 (4) (2008) 678–698.
[6] P.-A. Chiappori, R.J. McCann, L.P. Nesheim, Hedonic price equilibria, stable matching, and optimal transport: equivalence, topology, and uniqueness, Econom. Theory 42 (2010) 317–354.
[7] W. Gangbo, A. Swiech, Optimal maps for the multidimensional mongekantorovich problem, Commun. Pure Appl. Math. 51 (1) (1998) 23–45.
[8] N. Ghoussoub, A. Moameni, A self-dual polar factorization for vector fields, Commun. Pure Appl. Math. 66 (6) (2013) 905–933.
[9] S. Graf, Induced σ-homomorphisms and a parametrization of measurable sections via extremal preimage measures, Math. Ann. 247 (1) (1980) 67–80.
[10] H. Heinich, Probleme de Monge pour n probabilité, C. R. Acad. Sci. Paris, Ser. I 334 (9) (2002) 793–795.
[11] Y.-H. Kim, B. Pass, A general condition for Monge solutions in the multi-marginal optimal transport problem, arXiv:1307.6293.
[12] V. Levin, Abstract cyclical monotonicity and Monge solutions for the general Monge–Kantorovich problem, Set-Valued Anal. 7 (1) (1999) 7–32.
[13] A. Moameni, A characterization for solutions of the Monge–Kantorovich mass transport problem, submitted for publication.

[14] A. Moameni, B. Pass, Solutions to multi-marginal optimal transport problems supported on several graphs, in preparation.
[15] B. Pass, Uniqueness and Monge solutions in the multimarginal optimal transportation problem, SIAM J. Math. Anal. 43 (6) (2011) 2758–2775.
[16] B. Pass, On the local structure of optimal measures in the multimarginal optimal transportation problem, Calc. Var. Partial Differ. Equ. 43 (2012) 529–536.
[17] C. Villani, Optimal Transport, Old and New, Grundlehren Math. Wiss., Springer-Verlag, Berlin, 2009.
[18] H. von Weizsäcker, G. Winkler, Integral representation in the set of solutions of a generalized moment problem, Math. Ann. 246 (1) (1979/1980) 23–32.