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Geometry

Polyhedral metrics on the boundaries of convex compact quasi-Fuchsian manifolds

Métriques polyèdrales sur les bords des variétés quasi-fuchsiennes convexes compactes

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ABSTRACT

We show the existence of a convex compact subset in a quasi-Fuchsian manifold such that the induced metric on the boundary of the subset coincides with a prescribed hyperbolic polyhedral metric.

RÉSUMÉ

On demontre l'existence d'un sous-ensemble convexe compact dans une variété quasi-fuchsienne tel que la métrique induite de bord du sous-ensemble soit une métrique polyèdrale hyperbolique prescrite.

1. Introduction

First, let us recall one classical theorem due to A.D. Alexandrov, which concerns the realization of polyhedral surfaces in the spaces of constant curvature. As in [2], R_K stands for a spherical 3-space of curvature K in the case $K > 0$; R_K stands for a hyperbolic 3-space of curvature K when $K < 0$; in the case $K = 0$, R_K denotes a Euclidean 3-space. Then the result of A.D. Alexandrov reads as follows:

Theorem 1.1. *Let h be a metric of a constant sectional curvature K with cone singularities on a sphere S^2 such that the total angle around every singular point of h does not exceed 2π. Then there exists a closed convex polyhedron (polyhedral surface) in R_K such that its induced metric coincides with h. This polyhedron is unique up to the isometries of R_K. Here we include the doubly covered convex polygons, which are plane in R_K, in the set of convex polyhedra.*

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Next, following [4], we say that a compact hyperbolic manifold M is *strictly convex* if any two points in M can be joined with a minimizing geodesic that lies inside the interior of M. This condition implies that the intrinsic curvature of ∂M is greater than -1 everywhere (the term "hyperbolic" means for us "of a constant curvature equal to -1 everywhere").

In 1992 F. Labourie [4] proved the following theorem.

Theorem 1.2. *Let M be a compact manifold with boundary (different from the solid torus) that admits the structure of a strictly convex hyperbolic manifold. Let h be a C^∞-regular metric on ∂M of a sectional curvature that is strictly greater than -1 everywhere. Then there exists a convex hyperbolic metric g on M that induces h on ∂M.*

Jean-Marc Schlenker [6] demonstrated the uniqueness of the metric g in Theorem 1.2.

Our main goal is to obtain the following extension of Theorem 1.2, which can also be considered as an analogue of Theorem 1.1 for the convex hyperbolic manifolds with polyhedral boundary.

Theorem 1.3. *Let $\mathcal{M}$ be a compact connected 3-manifold with boundary of the type $\mathcal{S} \times [-1, 1]$ where $\mathcal{S}$ is a closed connected surface of genus at least 2. Let h be a hyperbolic metric with cone singularities of angle less than 2π on $\partial\mathcal{M}$ such that every singular point of h possesses a neighborhood in $\partial\mathcal{M}$ that does not contain other singular points of h. Then there exists a hyperbolic metric g in $\mathcal{M}$ with a convex boundary $\partial\mathcal{M}$ such that the metric induced on $\partial\mathcal{M}$ is h.*

We say that the manifold M from the statement of Theorem 1.3 admits a structure of *a quasi-Fuchsian convex compact manifold*.

At last, recall that *a pleated surface* [3] in a hyperbolic 3-manifold $\mathcal{M}$ is a complete hyperbolic surface $\mathcal{S}$ together with an isometric map $f : \mathcal{S} \to \mathcal{M}$ such that every $s \in \mathcal{S}$ is in the interior of some geodesic arc that is mapped by f to a geodesic arc in $\mathcal{M}$.

A pleated surface resembles a polyhedron in the sense that it has flat faces that meet along edges. Unlike a polyhedron, a pleated surface has no vertices, but it may have infinitely many edges that form a lamination.

Remark 1. The surfaces serving as the connected components of the boundary $\partial\mathcal{M}$ of the manifold $\mathcal{M}$ from the statement of Theorem 1.3, which are equipped by assumption with hyperbolic polyhedral metrics, do not necessarily have to be polyhedra embedded in $\mathcal{M}$: these surfaces can be partially pleated.

2. Outline of the proof of Theorem 1.3

Introducing more convenient notations, we clarify that in Theorem 1.3 we want to show that there exists a quasi-Fuchsian manifold $\mathcal{M}^\circ_\infty$:

- which contains a convex compact domain $\mathcal{M}_\infty \subset \mathcal{M}^\circ_\infty$,
- such that the induced metrics of the connected components $\mathcal{S}^+_\infty$ and $\mathcal{S}^-_\infty$ of $\partial\mathcal{M}_\infty$ coincide with the prescribed hyperbolic polyhedral metrics h^+_∞ and h^-_∞.

Remark 2. $\mathcal{S}^+_\infty$ and $\mathcal{S}^-_\infty$ are topologically the same surface $\mathcal{S}$ from the statement of Theorem 1.3.

First, we construct two sequences of C^∞-metrics $\{h^+_n\}_{n\in\mathbb{N}}$ and $\{h^-_n\}_{n\in\mathbb{N}}$ on $\mathcal{S}$ of sectional curvature strictly greater than -1 everywhere, which converge to h^+_∞ and h^-_∞ as $n \to \infty$. Then, by Theorem 1.2, for any $n \in \mathbb{N}$, there is a convex compact domain $\mathcal{M}_n$ in a quasi-Fuchsian manifold $\mathcal{M}^\circ_n$ equipped with a hyperbolic metric g_n such that the induced metrics of the boundary components $\mathcal{S}^+_n$ and $\mathcal{S}^-_n$ of $\partial\mathcal{M}_n \overset{\text{def}}{=} \mathcal{S}^+_n \cup \mathcal{S}^-_n$ are exactly h^+_n and h^-_n.

Note that the universal coverings $\widetilde{\mathcal{M}}^\circ_n$ of the quasi-Fuchsian manifolds $\mathcal{M}^\circ_n$ are copies of the hyperbolic 3-space $\mathbb{H}^3$, $n \in \mathbb{N}$. Thus, we may consider the holonomy representations $\rho^{\mathcal{S}}_n : \pi_1(\mathcal{S}) \to \mathcal{I}(\widetilde{\mathcal{M}}^\circ_n)(= \mathcal{I}(\mathbb{H}^3))$ such that $\mathcal{M}^\circ_n = \widetilde{\mathcal{M}}^\circ_n/[\rho^{\mathcal{S}}_n(\pi_1(\mathcal{S}))] = \mathbb{H}^3/[\rho^{\mathcal{S}}_n(\pi_1(\mathcal{S}))]$, and moreover such that the universal coverings $\widetilde{\mathcal{M}}_n$, $\widetilde{\mathcal{S}}^+_n$, and $\widetilde{\mathcal{S}}^-_n$ of the domains $\mathcal{M}_n$ and of the surfaces $\mathcal{S}^+_n$ and $\mathcal{S}^-_n$, $n \in \mathbb{N}$, satisfy the following conditions: $\mathcal{M}_n = \widetilde{\mathcal{M}}_n/[\rho^{\mathcal{S}}_n(\pi_1(\mathcal{S}))]$, $\mathcal{S}^+_n = \widetilde{\mathcal{S}}^+_n/[\rho^{\mathcal{S}}_n(\pi_1(\mathcal{S}))]$, and $\mathcal{S}^-_n = \widetilde{\mathcal{S}}^-_n/[\rho^{\mathcal{S}}_n(\pi_1(\mathcal{S}))]$. Here the surfaces $\widetilde{\mathcal{S}}^+_n$ and $\widetilde{\mathcal{S}}^-_n$ are the connected components of the boundary of a convex domain $\widetilde{\mathcal{M}}_n$ in $\mathbb{H}^3$. Note also that the induced metrics $\tilde{g}_n$, $\tilde{h}^+_n$, and $\tilde{h}^-_n$ of $\widetilde{\mathcal{M}}_n$, $\widetilde{\mathcal{S}}^+_n$, and $\widetilde{\mathcal{S}}^-_n$ in $\mathbb{H}^3$ are the pull-backs of the metrics g_n, h^+_n, and h^-_n of $\mathcal{M}_n$, $\mathcal{S}^+_n$, and $\mathcal{S}^-_n$, respectively.

Let $\widetilde{\mathcal{S}}^+$ and $\widetilde{\mathcal{S}}^-$ be two copies of the universal covering of the surface $\mathcal{S}$. Then we can introduce the developing maps $\tilde{f}_{\widetilde{\mathcal{S}}^+_n} : \widetilde{\mathcal{S}}^+ \to \widetilde{\mathcal{S}}^+_n$ and $\tilde{f}_{\widetilde{\mathcal{S}}^-_n} : \widetilde{\mathcal{S}}^- \to \widetilde{\mathcal{S}}^-_n$. Consider some special neighborhoods of $\widehat{\Delta}^+ \subset \widetilde{\mathcal{S}}^+$ and $\widehat{\Delta}^- \subset \widetilde{\mathcal{S}}^-$ of two special fundamental domains $\Delta^+ \subset \widetilde{\mathcal{S}}^+$ and $\Delta^- \subset \widetilde{\mathcal{S}}^-$ of the surface $\mathcal{S}$. Then we can show Lemma 2.1.

Lemma 2.1. *For each $n \in \mathbb{N}$, both domains $\widehat{\Delta}^+_n \overset{\text{def}}{=} \tilde{f}_{\widetilde{\mathcal{S}}^+_n}(\widehat{\Delta}^+) \subset \widetilde{\mathcal{S}}^+_n \subset \mathbb{H}^3$ and $\widehat{\Delta}^-_n \overset{\text{def}}{=} \tilde{f}_{\widetilde{\mathcal{S}}^-_n}(\widehat{\Delta}^-) \subset \widetilde{\mathcal{S}}^-_n \subset \mathbb{H}^3$ are included in a hyperbolic ball of radius ρ which does not depend on n.*

Lemma 2.1 helps us to apply Arzelà–Ascoli Theorem in order to obtain:

Lemma 2.2. *There exist subsequences of functions* $\{\tilde{f}_{\widetilde{\mathcal{S}}^+_{n_k}} : \widehat{\Delta}^+ \to \mathbb{H}^3\}_{k\in\mathbb{N}}$ *and* $\{\tilde{f}_{\widetilde{\mathcal{S}}^-_{n_k}} : \widehat{\Delta}^- \to \mathbb{H}^3\}_{k\in\mathbb{N}}$ *that converge to continuous functions* $\tilde{f}_{\widetilde{\mathcal{S}}^+_\infty} : \widehat{\Delta}^+ \to \mathbb{H}^3$ *and* $\tilde{f}_{\widetilde{\mathcal{S}}^-_\infty} : \widehat{\Delta}^- \to \mathbb{H}^3$*, respectively.*

From this point on, we assume that the sequences of functions $\{\tilde{f}_{\widetilde{\mathcal{S}}^+_n} : \widehat{\Delta}^+ \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ and $\{\tilde{f}_{\widetilde{\mathcal{S}}^-_n} : \widehat{\Delta}^- \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ converge to continuous functions $\tilde{f}_{\widetilde{\mathcal{S}}^+_\infty} : \widehat{\Delta}^+ \to \mathbb{H}^3$ and $\tilde{f}_{\widetilde{\mathcal{S}}^-_\infty} : \widehat{\Delta}^- \to \mathbb{H}^3$.

Recall now that hyperbolic isometries $\{\vartheta_n \in \mathcal{I}(\mathbb{H}^3)\}_{n\in\mathbb{N}}$ *converge* to an isometry $\vartheta_\infty \in \mathcal{I}(\mathbb{H}^3)$ (we denote it by $\vartheta_n \to \vartheta_\infty$) if for any point $y \in \mathbb{H}^3$ the sequence $\{\vartheta_n.y\}_{n\in\mathbb{N}}$ converges to the point $\vartheta_\infty.y \in \mathbb{H}^3$ as $n \to \infty$.

We can prove:

Lemma 2.3. *There is a subsequence of morphisms* $\{\rho^{\mathcal{S}}_{n_k} : \pi_1(\mathcal{S}) \to \mathcal{I}(\mathbb{H}^3)\}_{k\in\mathbb{N}}$ *that converges to a morphism* $\rho^{\mathcal{S}}_\infty : \pi_1(\mathcal{S}) \to \mathcal{I}(\mathbb{H}^3)$*, i.e. for every* $\gamma \in \pi_1(\mathcal{S})$ *there exists a hyperbolic isometry that we denote by* $\rho^{\mathcal{S}}_\infty(\gamma)$ *such that* $\rho^{\mathcal{S}}_{n_k}(\gamma) \to \rho^{\mathcal{S}}_\infty(\gamma)$ *as* $k \to \infty$.

In what follows we assume that the sequence of holonomy representations $\{\rho^{\mathcal{S}}_n : \pi_1(\mathcal{S}) \to \mathcal{I}(\mathbb{H}^3)\}_{n\in\mathbb{N}}$ converges to a holonomy representation $\rho^{\mathcal{S}}_\infty : \pi_1(\mathcal{S}) \to \mathcal{I}(\mathbb{H}^3)$ as $n \to \infty$.

Now we can conclude that the following statement holds:

Lemma 2.4. *The sequences of developing maps* $\{\tilde{f}_{\widetilde{\mathcal{S}}^+_n} : \widetilde{\mathcal{S}}^+ \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ *and* $\{\tilde{f}_{\widetilde{\mathcal{S}}^-_n} : \widetilde{\mathcal{S}}^- \to \mathbb{H}^3\}_{n\in\mathbb{N}}$ *converge to continuous functions* $\tilde{f}_{\widetilde{\mathcal{S}}^+_\infty} : \widetilde{\mathcal{S}}^+ \to \mathbb{H}^3$ *and* $\tilde{f}_{\widetilde{\mathcal{S}}^-_\infty} : \widetilde{\mathcal{S}}^- \to \mathbb{H}^3$.

By construction, the surfaces $\widetilde{\mathcal{S}}^+_\infty \stackrel{\text{def}}{=} \tilde{f}_{\widetilde{\mathcal{S}}^+_\infty}(\widetilde{\mathcal{S}}^+)$ and $\widetilde{\mathcal{S}}^-_\infty \stackrel{\text{def}}{=} \tilde{f}_{\widetilde{\mathcal{S}}^-_\infty}(\widetilde{\mathcal{S}}^-)$ bound a convex domain $\widetilde{\mathcal{M}}_\infty$ in $\mathbb{H}^3$; also, they are invariant under the action of the group $\rho^{\mathcal{S}}_\infty(\pi_1(\mathcal{S}))$ of isometries of $\mathbb{H}^3$, and their boundaries at infinity coincide with the limit set of $\rho^{\mathcal{S}}_\infty(\pi_1(\mathcal{S}))$.

Another classical result due to A.D. Alexandrov [1] helps us to prove that the induced metrics of the surfaces $\widetilde{\mathcal{S}}^+_\infty$ and $\widetilde{\mathcal{S}}^-_\infty$ coincide with the pull-backs of the hyperbolic polyhedral metrics h^+_∞ and h^-_∞:

Theorem 2.5. *If a sequence of closed convex surfaces* $\mathcal{F}_n$ *in* $\mathbb{H}^3$ *converges to a closed convex surface* $\mathcal{F}$ *and if two sequences of points* X_n *and* Y_n *on* $\mathcal{F}_n$ *converge to two points* X *and* Y *of* $\mathcal{F}$*, respectively, then the distances between the points* X_n *and* Y_n *measured on the surfaces* $\mathcal{F}_n$ *converge to the distance between the points* X *and* Y *measured on* $\mathcal{F}$*, i.e.,* $\mathrm{d}_{\mathcal{F}}(X,Y) = \lim_{n\to\infty}\mathrm{d}_{\mathcal{F}_n}(X_n,Y_n)$.

Therefore, we can take the quasi-Fuchsian manifold $\mathcal{M}^\circ_\infty$ mentioned in the beginning of Section 2 to be $\mathbb{H}^3/\rho^{\mathcal{S}}_\infty(\pi_1(\mathcal{S}))$, and the convex compact domain $\mathcal{M}_\infty \subset \mathcal{M}^\circ_\infty$ to be $\widetilde{\mathcal{M}}_\infty/\rho^{\mathcal{S}}_\infty(\pi_1(\mathcal{S}))$. $\square$

3. Distance between boundary components of a convex compact domain in a quasi-Fuchsian manifold

The *distance* $d(\mathcal{K},\mathcal{L})$ *between subsets* $\mathcal{K}$ *and* $\mathcal{L}$ *of a set* $\mathcal{N}$ is defined as follows: $d(\mathcal{K},\mathcal{L}) \stackrel{\text{def}}{=} \inf\{d_{\mathcal{N}}(u,v) | u \in \mathcal{K}, v \in \mathcal{L}\}$, where $d_{\mathcal{N}}(u,v)$ stands for the distance between points u and v in $\mathcal{N}$.

The following result, which is essentially used in the demonstration of Lemma 2.1 from Section 2, is of independent interest as well:

Theorem 3.1. *Given a convex bounded domain* $\mathcal{M}$ *with the upper boundary* $\mathcal{S}^+$ *and the lower boundary* $\mathcal{S}^-$ *in a quasi-Fuchsian manifold* $\mathcal{M}^\circ$*. If the metric surface* $\mathcal{S}^+$ *possesses two homotopically different nontrivial closed simple intersecting curves* c^+_1 *and* c^+_2 *of the lengths* l^+_1 *and* l^+_2*, and* $\mathcal{S}^-$ *possesses two homotopically different nontrivial closed simple intersecting curves* c^-_1 *and* c^-_2 *of the lengths* l^-_1 *and* l^-_2 *such that* c^+_1 *and* c^-_1*, as well as* c^+_2 *and* c^-_2*, are homotopically equivalent pairs of curves in* $\mathcal{M}$*, then the distance* $d(\mathcal{S}^+,\mathcal{S}^-)$ *between* $\mathcal{S}^+$ *and* $\mathcal{S}^-$ *is bounded from above by the constant:*

$$
\begin{aligned}
d\big(\mathcal{S}^+,\mathcal{S}^-\big) < \max\Bigg\{ & \left(l^+_1 + l^-_1 + \ln\frac{2l^+_1}{l^-_1}\right), \left(l^+_1 + l^-_1 + \ln\frac{2l^-_1}{l^+_1}\right), \left(l^+_2 + l^-_2 + \ln\frac{2l^+_2}{l^-_2}\right), \left(l^+_2 + l^-_2 + \ln\frac{2l^-_2}{l^+_2}\right), \\
& 2\,\mathrm{arcosh}\left[\cosh l^+_1 \cosh\left(l^+_1 + \mathrm{arcosh}\,\frac{e^{l^+_1}(l^+_1)^2}{\varepsilon_3^2}\right)\right], 2\,\mathrm{arcosh}\left[\cosh l^-_1 \cosh\left(l^-_1 + \mathrm{arcosh}\,\frac{e^{l^-_1}(l^-_1)^2}{\varepsilon_3^2}\right)\right], \\
& 2\,\mathrm{arcosh}\left[\cosh l^+_2 \cosh\left(l^+_2 + \mathrm{arcosh}\,\frac{e^{l^+_2}(l^+_2)^2}{\varepsilon_3^2}\right)\right], 2\,\mathrm{arcosh}\left[\cosh l^-_2 \cosh\left(l^-_2 + \mathrm{arcosh}\,\frac{e^{l^-_2}(l^-_2)^2}{\varepsilon_3^2}\right)\right]\Bigg\},
\end{aligned}
$$

where the symbol ε_3 *stands for the Margulis constant of hyperbolic space* $\mathbb{H}^3$.

Note that we do not require the regularity of surface metrics in Theorem 3.1.

In order to prove Theorem 3.1, we need the following version of Margulis lemma, adapted to quasi-Fuchsian isometries of $\mathbb{H}^3$ [5]:

Lemma 3.2. *There is a universal constant $\varepsilon_3 > 0$ such that, for any properly discontinuous subgroup Γ of the group $\mathcal{I}(\mathbb{H}^3)$ of isometries of $\mathbb{H}^3$, if two closed simple intersecting curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ of the manifold $\mathbb{H}^3/\Gamma$ have lengths less than ε_3, then $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are homotopically equivalent in $\mathbb{H}^3/\Gamma$.*

The main idea of the proof of Theorem 3.1 is as follows: assuming that the distance between the surfaces $\mathcal{S}^+$ and $\mathcal{S}^-$ in the quasi-Fuchsian manifold $\mathcal{M}^\circ$ is big enough, we find a pair of closed simple intersecting curves in $\mathcal{M}^\circ$ of lengths less than ε_3, which are homotopically different. This leads us to a contradiction with Margulis lemma.

The detailed proofs of Theorems 1.3 and 3.1 are given in [7] (in English).

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