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Dimension of gradient measures

La dimension de mesures qui constituent le gradient d'une fonction

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ABSTRACT

We prove that if pure derivatives of a function on $\mathbb{R}^n$ are complex measures, then their lower Hausdorff dimension is at least $n-1$. The derivatives with respect to different coordinates may be of different order.

RÉSUMÉ

Supposons que les dérivées pures (pas nécéssairement du même ordre) d'une fonction sur $\mathbb{R}^n$ soient des mesures de Radon finies. On montre que leur dimension inférieure de Hausdorf est alors au moins $n-1$.

1. Introduction

We begin with a well-known fact: if a function f is in BV, then the lower Hausdorff dimension of ∇f is not less than $n-1$ (see [1], Lemma 3.76). By the lower Hausdorff dimension of a vector-valued complex measure μ, we mean:

$$\dim \mu = \inf\bigl\{\alpha \mid \text{there is a Borel set } F \text{ with } \mu(F) \neq 0,\ \dim F \leq \alpha\bigr\}. \tag{1}$$

In [11], this fact was treated as a manifestation of a certain more general uncertainty-type principle. We use the notation from that paper. Namely, let $\phi : S^{n-1} \to S^{n-1}$ be a mapping. Consider the class M_ϕ of vector-valued signed measures μ such that $\hat{\mu}(\xi) \parallel \phi(\frac{\xi}{|\xi|})$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$. The celebrated theorem of Uchiyama [13] shows that if $\phi(\xi)$ is not parallel to $\phi(-\xi)$ for all $\xi \in S^{n-1}$, then every μ in M_ϕ is absolutely continuous. However, can one say something if this condition does not hold? We cite a simpler version of Theorem 3 from [11].

Theorem 1.1. *Suppose that the image of ϕ contains n linearly independent points $\phi(h_1), \phi(h_2), \dots, \phi(h_n)$ and ϕ is α-Hölder in neighborhoods of h_i, $i = 1, 2, \dots, n$, $\alpha > \frac{1}{2}$. Then $\dim \mu \geq 1$ for all $\mu \in M_\phi$.*

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The relationship between BV and M_ϕ can be expressed by the formula $\{\nabla f \mid f \in \mathrm{BV}(\mathbb{R}^n)\} = M_{\mathrm{Id}}$, where Id is the identity map on the sphere. In this particular case, Theorem 1.1 is weaker (we get only dimension 1). However, it describes a much more general setting. One can make a courageous conjecture (Conjecture 1 in [11]).

Conjecture 1.2. *Suppose that the function ϕ is Lipschitz and its image contains n linearly independent points. Then $\dim\mu \geq n-1$ for all $\mu \in M_\phi$.*

Not being able to prove the conjecture, we state a result that lies towards it. In what follows, D_i means "the derivative with respect to x_i".

Theorem 1.3. *Let m be a natural number. Let f be a function such that $D_i^m f$ is a complex measure for all i. Then $\dim\mu \geq n-1$, where μ is the vector-valued complex measure whose components are $D_i^m f$.*

This theorem is a particular case of Conjecture 1.2, $\mu \in M_\phi$, where $\phi(\xi) = \frac{\xi^m}{|\xi^m|}$. When the orders of derivation differ, the homogeneity is not isotropic. However, in this case we still have the same principle.

Theorem 1.4. *Let $m_1, m_2, \dots, m_n$ be natural numbers. Let f be a function such that $D_i^{m_i} f$ is a complex measure for all i. Then $\dim\mu \geq n-1$, where μ is the vector-valued complex measure whose components are $D_i^{m_i} f$.*

The basic fact about BV-functions we started with can be proved by several methods. In [1], the proof was based on the co-area formula for BV-functions. This gives more information about those "parts" of ∇f that have dimension $n-1$: they are situated on the jumps of f. However, the applicability of the methods from [1] to Conjecture 1.2 and even to Theorem 1.3 is questionable. The proof of Theorem 1.1 is based on the application of F. and M. Rieszs' classical theorem (see [8], p. 28) on the continuity of an analytic complex measure. This gives only dimension 1 (it, however, allows one to disregard entirely the algebraic structure of ϕ).

Our strategy is, in a sense, a mixture of the two proofs indicated above. The co-area formula is replaced with the Sobolev embedding theorem with the limiting summation exponent, and Rieszs' theorem is replaced with a certain modification of the Frostman lemma.

In Section 2 we prove Theorem 1.4 (and Theorem 1.3 as a particular case), except for the modification of Frostman lemma, which we prove in Section 3. Last Section 4 contains some examples and some suggestions how to prove Conjecture 1.2.

2. Proof of the theorem

We begin with the discussion of the embedding theorem we will use. We will need some Besov spaces. The reader unfamiliar with them can either consult [2,10], or skip the details and pass to Theorem 2.2 directly.

By W_1^m, $m = (m_1, m_2, \dots, m_n)$, we denote the completion of the set $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|f\|_{W_1^m} = \sum_{i=1}^n \|D_i^{m_i} f\|_{L^1}$. Another norm on the set C_0^∞ describes the one-dimensional Besov spaces (i.e. we measure the smoothness of a function in $\mathbb{R}^n$ with respect to one coordinate),

$$\|f\|_{B^{i,\ell}_{q,\theta}} = \left(\int_0^\infty \left(h^{-\ell}\left\|\Delta_i^s(h)f\right\|_{L^q}\right)^\theta \frac{\mathrm{d}h}{h}\right)^{\frac{1}{\theta}}.$$

Here i is the number of the coordinate, $i = 1, 2, \dots, n$, $\Delta_i^s(h)$ is the operator of finite difference of order s and step h with respect to the i-th coordinate, $s > \ell$.

We cite Theorem 4 from [5] (see also Theorem B in [6] and [7]).

Theorem 2.1. *Let f be a function in W_1^m. Then, for each $i = 1, 2, \dots, n$ and any $\ell_i < m_i$, the inequality*

$$\|f\|_{B^{i,\ell_i}_{q,1}} \lesssim \sum_{j=1}^n \left\|D_j^{m_j} f\right\|_{L^1}$$

holds true if the parameters satisfy the homogeneity condition $\ell_i = m_i\left(1 - \frac{q-1}{q}\sum_{j=1}^n \frac{1}{m_j}\right)$.

Now we fix $\ell_i = m_i - 1$; therefore, $\frac{q-1}{q} = \left(\sum_{j=1}^n \frac{m_i}{m_j}\right)^{-1}$. This identity matches its individual q to each m_i, we denote it by q_i. Using the easy embedding (see [10], p. 62) for $\theta = 1$

$$\left\|D_i^{m_i-1} f\right\|_{L^{q_i}} \lesssim \|f\|_{B^{i,m_i-1}_{q_i,1}},$$

we get the following embedding theorem without Besov spaces.

Theorem 2.2. *Let f be a function in W_1^m. Then, for each $i = 1, 2, \dots, n$,*

$$\|D_i^{m_i-1} f\|_{L^{q_i}} \lesssim \sum_{j=1}^{n} \|D_j^{m_j} f\|_{L^1}$$

if the parameters satisfy the homogeneity condition $\frac{q_i-1}{q_i} = (\sum_{j=1}^{n} \frac{m_i}{m_j})^{-1}$.

Suppose now that f is a function with compact support such that $\mu_i = D_i^{m_i} f$ is a complex measure for all i. Then,

$$\|D_i^{m_i-1} f\|_{L^{q_i}} \lesssim \sum_{j=1}^{n} \mathrm{Var}\mu_j. \tag{2}$$

This can be deduced from Theorem 2.2 by a simple limiting argument. We skip the details.

Let φ be a test function in $C_0^\infty(\mathbb{R}^{n-1})$ supported in the unit ball. Let $\varphi_r(x) = \varphi(r^{-1}x)$, $r > 0$. For $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ we write $x_{[i]}$ for the $(n-1)$-dimensional vector that is obtained from x by forgetting the i-th coordinate (for example, for $n = 3$, $x_{[2]} = (x_1, x_3)$). By $B_r(z)$ we denote the $(n-1)$-dimensional ball of radius r centered at z.

Lemma 2.3. *Let the balls $B_{r_j}(y_j)$ be disjoint, and let $\psi \in C_0^\infty(\mathbb{R})$ be a test function. Suppose that f is a compactly supported function. If $\mu = (D_i^{m_i} f)_i$ is a complex measure, then, for all $i = 1, 2, \dots, n$ and any $\varphi \in C_0^\infty$ supported in the unit ball,*

$$\sum_j \left| \int_{\mathbb{R}^n} \psi(x_i)\varphi_{r_j}(x_{[i]} + y_j)\, \mathrm{d}\mu_i(x) \right| \lesssim \left(\sum_j r_j^{n-1} \right)^{\frac{1}{q_i'}} \mathrm{Var}\mu$$

for some fixed q_i' and all $y_j \in \mathbb{R}^{n-1}$ and $r_j < 1$ uniformly (the constants may depend on φ and ψ).

Proof. For simplicity, let $i = 1$. We can write:

$$\begin{aligned}
\sum_j \left| \int_{\mathbb{R}^n} \psi(x_1)\varphi_{r_j}(x_{[1]} + y_j)\, \mathrm{d}\mu_1(x) \right| &= \sum_j \left| \int_{\mathbb{R}^n} \psi'(x_1)\varphi_{r_j}(x_{[1]} + y_j) D_1^{m_1-1} f(x)\, \mathrm{d}x \right| \\
&\le \sum_j \left\| \psi'(x_1)\varphi_{r_j}(x_{[1]} + y_j) \right\|_{L^{q_1'}} \|D_1^{m_1-1} f\|_{L^{q_1}(B_{r_j}(y_j))} \\
&\lesssim \sum_j r_j^{\frac{n-1}{q_1'}} \|D_1^{m_1-1} f\|_{L^{q_1}(B_{r_j}(y_j))} \\
&\le \left(\sum_j r_j^{n-1} \right)^{\frac{1}{q_1'}} \left(\sum_j \|D_1^{m_1-1} f\|_{L^{q_1}(B_{r_j}(y_j))}^{q_1} \right)^{\frac{1}{q_1}} \le \left(\sum_j r_j^{n-1} \right)^{\frac{1}{q_1'}} \|D_1^{m-1} f\|_{L^{q_1}} \\
&\lesssim \left(\sum_j r_j^{n-1} \right)^{\frac{1}{q_1'}} \|\mu\|.
\end{aligned}$$

Here q_1 is the exponent taken from Theorem 2, and q_1' is its adjoint. The first inequality is the Hölder inequality, the second one is rescaling, the third one is Hölder again, and the fourth one is inequality (2). □

The next lemma is a generalization of Frostman's lemma (see [9], p. 112, for the original).

Lemma 2.4. *Suppose that $\varphi \in C_0^\infty(\mathbb{R}^n)$ is a radial non-negative function supported in the unit ball that decreases monotonically as the radius grows, and $\varphi(x) = 1$ when $|x| \le \frac{3}{4}$. Let μ be a complex measure such that, for every collection $B_{r_j}(x_j)$ of n-dimensional balls such that $B_{3r_j}(x_j)$ are disjoint, we have:*

$$\sum_j \left| \int_{\mathbb{R}^n} \varphi_{3r_j}(x_j + y)\, \mathrm{d}\mu(y) \right| \lesssim \left(\sum r_j^\alpha \right)^\beta$$

for some positive α and β. Then $\dim(\mu) \ge \alpha$.

We postpone its proof till Section 3.

Lemma 2.5. *Let μ be a complex Borel measure on $\mathbb{R}^{k+l}$. Suppose that $\mu(I \times A) = 0$ for every parallelepiped $I \subset \mathbb{R}^k$ and every $A \subset \mathbb{R}^l$ such that $\dim A < \alpha$. Then $\dim \mu \geq \alpha$.*

Proof of Theorem 1.3. Suppose the contrary, let F be some Borel set such that $\dim F < n-1$, but $\mu(F) \neq 0$. We may assume that $\mu_1(F) \neq 0$ (by symmetry) and F is compact (due to the regularity of the measure). Multiplying f by a test function that equals 1 on F, we make f compactly supported without loosing the condition that its higher order derivatives are signed measures. To get a contradiction, it suffices to prove that for every set $A \subset \mathbb{R}^{n-1}$ such that $\dim A < n-1$ and every function $\psi \in C_0^\infty(\mathbb{R})$, we have:

$$\int_{\mathbb{R}\times A} \psi(x_1)\,\mathrm{d}\mu_1(x) = 0. \tag{3}$$

Then, approximating the characteristic function of an interval I by smooth functions, we get the hypothesis of Lemma 2.5 with $\alpha = n-1$, which, in its turn, asserts that $\mu_1(F) = 0$.

Consider now a complex measure μ_ψ on $\mathbb{R}^{n-1}$ given by the formula $\mu_\psi(B) = \int_{\mathbb{R}\times B} \psi(x_1)\,\mathrm{d}\mu_1(x)$. By Lemma 2.3, the measure μ_ψ satisfies the hypothesis of Lemma 2.4 with $\alpha = n-1$. Therefore, $\dim \mu_\psi \geq n-1$ and (3) holds for all A with $\dim A < n-1$. □

3. Proof of Lemma 2.4

To prove Lemma 2.4, we need some preparation. First, it suffices to prove Lemma 2.4 for real-valued signed measures only.

The next lemma provides a softer substitute for the Lebesgue differentiation theorem for an arbitrary Borel measure.

Lemma 3.1. *Let μ be a signed measure, let A_+ and A_- be the sets of its Hahn decomposition. Consider the set*

$$P_+ = \bigl\{x \in A_+ \mid \exists\delta(x) \text{ such that } \forall r < \delta(x)\mu_+\bigl(B_r(x)\bigr) \leq 10\mu\bigl(B_r(x)\bigr)\bigr\}. \tag{4}$$

Then $\mu(P_+) = \mu(A_+)$.

Consider now the sets $P_+^{(N)}$ given by the formula

$$P_+^{(N)} = \left\{x \in A_+ \mid \forall r < \frac{1}{N}, \mu_+\bigl(B_r(x)\bigr) \leq 10\mu\bigl(B_r(x)\bigr)\right\}.$$

Surely, $P_+ = \bigcup_N P_+^{(N)}$. Therefore, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\mu_+(P_+^{(N)}) \geq \mu_+(A_+) - \varepsilon$. We need to change inequality (4) slightly.

Lemma 3.2. *Suppose that for some fixed x and all $r \leq 2\delta$ the inequality $\mu_+(B_r(x)) \leq 10\mu(B_r(x))$ holds true. Then*

$$\int \varphi_r(x+y)\,\mathrm{d}\mu_+(y) \leq 10 \int \varphi_r(x+y)\,\mathrm{d}\mu(y) \tag{5}$$

for all $r < \delta$ and any radial non-negative test function φ supported in $B_1(0)$ that decreases monotonically as the radius grows.

Lemma 3.3. *Let μ be a signed measure. Let μ_+ and μ_- be its Hahn decomposition. Then $\dim \mu = \min(\dim \mu_+, \dim \mu_-)$.*

Proof of Lemma 2.4. We assume the contrary; suppose that there exists some Borel set F such that $\mu(F) \neq 0$, but $\dim(F) < \alpha^- < \alpha$. By Lemma 3.3, we may assume that $F \subset A_+$; moreover, we may assume that $F \in P_+^{(N)}$ for some big N (because these sets tend to A_+ in measure) and that F is compact (by the regularity of μ). Let $\mu(F) = c_0$. By the definition of the Hausdorff dimension, there exists a covering of F with the balls $B_{r_j}(x_j)$ whose centers x_j lie in F, whose radii r_j do not exceed δ (which we take to be less than $\frac{1}{10N}$), and $\sum_j r_j^{\alpha^-} \leq c_1$ for some uniform constant c_1. We divide the set of balls into the classes of roughly equal balls: $E_k = \{j \mid r_j \in (2^{-k-1}, 2^{-k}]\}$. Surely, $|E_k| \leq 2^{k\alpha^-} c_1$. By the pigeonhole principle, there exists some $k \gtrsim \log\frac{1}{\delta}$ such that $\mu_+(F \cap \bigcup_{j\in E_k} B_{r_j}(x_j)) \geq \frac{c_0}{k^2}$. We fix δ and also fix this k for a while. Let D_k be a subset of E_k such that $\{x_j \mid j \in D_k\}$ is a maximal 2^{-k}-separated subset of $\{x_j \mid j \in E_k\}$. Then

(i) $\bigcup_{j\in D_k} B_{3r_j}(x_j) \supset F \cap \bigcup_{j\in E_k} B_j$, so $\sum_{j\in D_k} \mu_+(B_{3r_j}(x_j)) \geq \frac{c_0}{k^2}$;
(ii) the collection $B_{4r_j}(x_j)$, $j \in D_k$ covers each point only a finite number of times (uniformly).

Using these two statements and recalling that φ equals 1 on $B_{\frac{3}{4}}(0)$, we can write:

$$\frac{c_0}{k^2} \le \sum_{j\in D_k} \mu_+\big(B_{3r_j}(x_j)\big) \le \sum_{j\in D_k} \int \varphi_{4r_j}(x_j+y)\,\mathrm{d}\mu_+(y)$$
$$\le 10 \sum_{j\in D_k} \int \big|\varphi_{4r_j}(x_j+y)\,\mathrm{d}\mu(y)\big| \lesssim \Big(\sum_{j\in D_k} r_j^{\alpha}\Big)^{\beta} \le \big(|D_k|2^{-k\alpha}\big)^{\beta} \lesssim c_1^{\beta} 2^{\beta k(\alpha^- -\alpha)}.$$

We get a contradiction for δ small. □

4. Examples and conjectures

We note that Theorem 1.4 is sharp in the sense that one cannot rise the dimension. Consider a one-dimensional signed measure $\Delta_h^s = \sum_{j=0}^{s}(-1)^{s-j}C_s^j\delta_{hj}$. This measure has s vanishing moments, therefore, there exists a compactly supported function f_h^s such that $D^s f_h^s = \Delta_h^s$. Consider a function F on $\mathbb{R}^n$ given by the formula $F(x) = \prod_{i=1}^{n} f_h^{m_i}(x_i)$. Then, for each i, $D_i^{m_i}F$ is a measure supported on the $(n-1)$-dimensional hypercubes

$$\big\{x \mid x_i = hj,\ \forall k \ne i, x_k \in \big[0, (m_k+1)h\big]\big\},$$

here $j = 0, 1, 2, \ldots, m_i$. This proves that Theorem 1.4 is sharp.

Theorem 4 from [5] we have used is very strong. For the isotropic case, what we need is the inequality $\|D_i^{m-1} f\|_{L^{\frac{n}{n-1}}} \le \|f\|_{W_1^m}$, which is much easier. However, even embedding theorems from [12] are not sufficiently strong for our purposes in the general setting (they require additional assumptions on the numbers m_i).

We believe that the relationship between Conjecture 1.2 and embedding theorems are deeper. Maybe, embedding theorems for vector fields from [15] may help (there was a lot of progress in the recent years for the isotropic case, starting with [3,14]; see [4] for some examples of anisotropic theorems of such kind).

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References

[1] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford Mathematical Monographs, 2000.
[2] O.V. Besov, V.P. Il'in, S.M. Nikolski, Integral Representations of Functions and Embedding Theorems, 1975.
[3] J. Bourgain, H. Brezis, New estimates for the Laplacian, the div–curl, and related Hodge systems, C. R. Acad. Sci. Paris, Ser. I 338 (2004) 539–543.
[4] S.V. Kislyakov, D.V. Maksimov, D.M. Stolyarov, Spaces of smooth functions generated by nonhomogeneous differential expressions, Funkc. Anal. Prilozh. 47 (2) (2013) 89–92.
[5] V.I. Kolyada, On an embedding of Sobolev spaces, Mat. Zametki 54 (3) (1993) 48–71 (in Russian).
[6] V.I. Kolyada, Estimates of Fourier transform in Sobolev spaces, Stud. Math. 125 (1) (1997) 67–74.
[7] V.I. Kolyada, Rearrangements of functions and embedding of anisotropic spaces of Sobolev type, East J. Approx. 4 (2) (1998) 111–198.
[8] P. Koosis, Introduction to H^p Spaces, 2nd ed., Cambridge University Press, Cambridge, UK, 1998.
[9] P. Mattila, Geometry of Sets and Measures in Euclidean Space, Cambridge University Press, Cambridge, UK, 1995.
[10] J. Peetre, New Thoughts on Besov Spaces, Duke University Mathematical Series, vol. I, Duke University, Durham, NC, USA, 1976.
[11] M. Roginskaya, M. Wojciechowski, Singularity of vector valued measures in terms of Fourier transform, J. Fourier Anal. Appl. 12 (2) (2006) 213–223.
[12] V.A. Solonnikov, On certain inequalities for functions belonging to $\vec{W}_p(\mathbb{R}^n)$-classes, Zap. Nauč. Semin. LOMI 27 (1972) 194–210 (in Russian).
[13] A. Uchiyama, A constructive proof of the Fefferman–Stein decomposition of BMO($\mathbb{R}^n$), Acta Math. 148 (1) (1982) 215–241.
[14] J. van Schaftingen, Estimates for L^1-vector fields, C. R. Acad. Sci. Paris, Ser. I 339 (2004) 181–186.
[15] J. van Schaftingen, Limiting Sobolev inequalities for vector fields and canceling linear differential operators, J. Eur. Math. Soc. 15 (3) (2013) 877–921.