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# On the subexponentiality of the ridgelet transform

## *Sur la sous-exponentalité de la transformée en ridelettes*

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ABSTRACT

We show that we can consider the ridgelet transform for Wiener functionals as a subexponential random variable. We give an application of this result to random walks.


RÉSUMÉ

La transformée en ridelettes peut être considérée comme une variable aléatoire sous-exponentielle. On donne alors une application de ce résultat aux marches aléatoires.


### 1. Introduction, notations, results

The ridgelet transform for Wiener functionals introduced by the author [9] is considered as a real random variable defined on the measured space: $(\boldsymbol{\Gamma}, B(\boldsymbol{\Gamma}), \lambda \otimes \mu \otimes \mu)$, with $\boldsymbol{\Gamma} := ]0, +\infty[ \times \mathbf{S} \times \mathbf{H}$, $\mathbf{H}$ denoting the Cameron–Martin space and $\mathbf{S}$ its unit sphere, $B(\boldsymbol{\Gamma})$ is the Borel set of $\boldsymbol{\Gamma}$ and $\lambda$ (resp., $\mu$) is the Lebesgue measure (resp., Wiener measure), by the following: for $X \in L^2(\mu)$, $\gamma = (a, \omega_0, \omega_1) \in \boldsymbol{\Gamma} \longrightarrow R^\delta_\gamma(X) = R^\delta_{a,\omega_0,\omega_1}(X) = \langle X, \Psi_\gamma\rangle_\delta = \langle X, \Psi_{a,\omega_0,\omega_1}\rangle_\delta \in \mathbb{R}$, where $\langle .,.\rangle_\delta$ is the inner product in $L^2(\mu_\delta)$, $\mu_\delta$ being the Wiener measure of variance $\delta$ ($\delta > 0$), and $\Psi_\gamma(\omega) = \Psi_{a,\omega_0,\omega_1}(\omega) := a^{-1/2}\Psi(a^{-1}[\langle \omega, \omega_0\rangle_{\mathbf{H}}\omega_0 - \omega_1])$, $\langle .,.\rangle_{\mathbf{H}}$ being the inner product in $\mathbf{H}$ and $\Psi \in L^2(\mu)$ is fixed. As in [9], we assume $\Psi$ to be spherically symmetric, i.e., $\Psi(\omega) = \eta(\|\omega\|_{\mathbf{H}}) = \|\omega\|_{\mathbf{H}}\widetilde{\eta}(\|\omega\|_{\mathbf{H}})$, $\omega \in \mathbf{H}$, with the admissibility condition

$$C_\Psi := \int_0^\infty \mathrm{d}t\, t\big|\eta(t)\big|^2 = \int_0^\infty \mathrm{d}t\, t^3\big|\widetilde{\eta}(t)\big|^2 < \infty.$$

The concept of wavelets, ridgelets has been introduced during the last century (in the 1980s) and interest in them has grown considerably (see [2,5,6]). The author's investment in the subject [10] allowed him to give applications to the regularity of Wiener functionals, or diffusion densities, and also for solving backward stochastic differential equations.

In this paper we first show that, if $X$ is assumed to be Gaussian, then $R^\delta(X)$ is Gaussian. More generally, for all $X \in L^2(\mu)$, $R^\delta(X)$ is a subexponential random variable. We give the following application of this result. Let $\xi_1, \xi_2, \ldots$ be independent

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identically distributed random variables with a distribution $F$ such that $E(\xi_1) = -\alpha < 0$. Let $S_0 = 0$, $S_n = \xi_1 + \ldots + \xi_n$ for $n \geq 1$; let $M_n = \max(S_i, 0 \leq i \leq n)$, the maximum of the random walk to time $n$ and let $M = \sup(S_n, n \geq 0)$ be its global maximum. We denote: $\forall A > 0, \forall \varepsilon > 0, \mathbf{B}_k := \{|S_j + \alpha j| \leq j\varepsilon + A$ for all $j \leq k, \xi_{k+1} > x + k\alpha\}$. Hence we show the following main theorem.

**Theorem.** *For any fixed* $\varepsilon > 0$,

$$\lim_{A\to\infty} \lim_{x\to\infty} \inf_{n\geq 1} \mu\left\{\bigcup_{k=0}^{n-1} \mathbf{B}_k \,\middle|\, M_n > x\right\} \geq \frac{\alpha}{\alpha+\varepsilon}. \tag{1}$$

This theorem is the well-known principle of a single big jump (PSBJ) for $M_n$ (see [5,6]). This principle until now underlies the behaviour of sums of independent subexponential random variables [6]. A subexponential distribution is heavy-tailed and further examples of heavy-tailed distributions that are of use in practical applications, e.g., the modelling of insurance claim sizes, are given in [4]. These subexponential distributions are all well-behaved in a certain manner. However, mathematically there is a whole range of further possible distributions, and one of our aims is to provide a firm basis of tools that does not exclude those which are in some sense pathological.

The PSBJ not only abound in theory, but is also useful in modern probabilistic modelling, in such diverse areas as insurance risk [1], communication networks [2,3] and finance [7].

## 2. Gaussian case

Let $X \in L^2(\mu)$ be a Gaussian Wiener functional. It has the following Ito–Wiener representation: $X(\omega) = \int_0^1 \mathrm{d}B_s(\omega)h_0(s)$ with $B_s(\omega) := \omega(s)$, the coordinate application, $\omega \in \mathbf{W} := C_0(0,1)$, the classical Wiener space, $s \in [0,1]$ and $h_0 \in L^2(0,1)$ is deterministic. With the help of this representation, we calculate the ridgelet transform.

**Proposition 2.1.** *For all fixed* $\delta > 0$ *and any* $\gamma = (a, \omega_0, \omega_1) \in ]0, +\infty[ \times \mathbf{S} \times \mathbf{H}$, *we have:*

$$R_\gamma^\delta(X) = a^{-5/2}\delta^2 \int_0^1 \mathrm{d}s\, \omega_0'(s) E_\mu\big[\omega'(s) X_s(\omega)\big\langle \nabla\Psi\big(\delta A_s(\omega_0,\omega)\omega_0 - \omega_1\big), \omega_0\big\rangle_{\mathbf{H}}\big] \tag{2}$$

*assuming* $\Psi$ *to be* $C^2$*-Fréchet differentiable and where* $E_\mu[.]$ *denotes expectation with respect to Wiener measure* $\mu$, $X_s(\omega) := \int_0^s \mathrm{d}B_r(\omega)\, h_0(r)$ *(Ito–Wiener stochastic integration) and* $A_s(\omega_0, \omega) := \int_0^s \mathrm{d}r\, \omega_0'(r)\omega'(r)$, *for* $\omega_0, \omega \in \mathbf{H}$.

**Proof.** $R_\gamma^\delta(X) = \langle \int_0^1 \mathrm{d}B_s(\omega)\, h_0(s), \Psi_{a,\omega_0,\omega_1}\rangle_\delta = a^{-1/2}\delta E_\mu[X_1(\omega)\Psi(a^{-1}[\delta A_1(\omega_0,\omega)\omega_0 - \omega_1])]$.
With the help of Ito's formula, we easily get (2). □

**Theorem 2.2.** *The real random variable* $(a, \omega_0, \omega_1) \in ]0, +\infty[ \times \mathbf{S} \times \mathbf{H} \longrightarrow R_{a,\omega_0,\omega_1}^\delta(X) \in \mathbb{R}$ *is Gaussian, for any fixed* $\delta > 0$.

**Proof.** The random variable $A_s$, defined on the probability space $(\mathbf{S}, B(\mathbf{S}), \mu)$, is Gaussian. We easily deduce that the random variable $(\omega_0, \omega_1) \in \mathbf{S} \times \mathbf{H} \longrightarrow \langle \nabla\Psi(\delta A_s(\omega_0,\omega)\omega_0 - \omega_1), \omega_0\rangle_{\mathbf{H}} \in \mathbb{R}$ is Gaussian, hence from (2) the conclusion follows. □

## 3. Non-Gaussian case

Now, let $X$ be any element of $L^2(\mu)$. It has the following Ito representation: $X(\omega) = \int_0^1 \mathrm{d}B_s(\omega)h_0(s,\omega)$ with $h_0 \in L^2(\lambda \otimes \mu)$, $h_0$ being moreover an optional process with respect to the filtration on $\mathbf{W}$: $\sigma\{\omega \in \mathbf{W} \longrightarrow \boldsymbol{\omega}(s) \in \mathbb{R}; s \leq t\}$, $t \in [0,1]$.

**Theorem 3.1.** *The real random variable* $(a, \omega_0, \omega_1) \in ]0, +\infty[ \times \mathbf{S} \times \mathbf{H} \longrightarrow R_{a,\omega_0,\omega_1}^\delta(X) \in \mathbb{R}$ *is long-tailed for any fixed* $\delta > 0$.

**Proof.** The tail function associated with $R_\gamma^\delta(X)$ is:

$$\overline{F_{R_\gamma^\delta(X)}}(x) = \int\int_{\mathbf{S}\times\mathbf{H}} \mu(\mathrm{d}\omega_0)\mu(\mathrm{d}\omega_1) \int_0^\infty \mathrm{d}a\, \mathbf{1}_{\{Z^\delta(\omega_0,\omega_1) > a^{5/2}x\}}$$

where

$$Z^\delta(\omega_0,\omega_1) = a^{5/2} R_\gamma^\delta(X) = G_0(x) + G_1(x) \tag{3}$$

with $G_0(x) = \int\int_{|Z^\delta(\omega_0,\omega_1)|<x} \mu(\mathrm{d}\omega_0)\mu(\mathrm{d}\omega_1)|Z^\delta(\omega_0,\omega_1)|^{2/5}x^{-2/5}$ and $G_1(x) = \overline{F_{|Z^\delta|}}(x)$.

The long-tailedness of $G_0(x)$ follows from the equivalence: for any $y > 0$,

$$G_0(x) \sim \int \int_{|Z^\delta(\omega_0,\omega_1)|<x+y} \mu(\mathrm{d}\omega_0)\mu(\mathrm{d}\omega_1)\left|Z^\delta(\omega_0,\omega_1)\right|^{2/5} x^{-2/5} \quad \text{as } x \to \infty.$$

Moreover, as the random variable $\omega_0 \in \mathbf{S} \longrightarrow A_s(\omega_0,\omega)$ is Gaussian for any $\omega \in \mathbf{H}$, $s \in [0,1]$, we easily deduce the long-tailedness of $G_1(x)$.

Equality (3) implies then the required long-tailedness. □

Now we prove the following.

**Theorem 3.2.** *The real random variable $(a,\omega_0,\omega_1) \in ]0,+\infty[ \times \mathbf{S} \times \mathbf{H} \longrightarrow R^\delta_{a,\omega_0,\omega_1}(X) \in \mathbb{R}$ is subexponential for any fixed $\delta > 0$. Even more, this real random variable is a strong subexponential.*

**Proof.** We prove that there exists $c > 0$ such that $\overline{F_{R^\delta_\gamma(X)}}(2x) \geq c\overline{F_{R^\delta_\gamma(X)}}(x)$, for all $x$, hence with the help of Theorem 3.29 in [8] we deduce the required subexponentiality and strong subexponentiality.

Decomposing (3) of $\overline{F_{R^\delta_\gamma(X)}}(x)$ allows us to show that, $\overline{F_{R^\delta_\gamma(X)}}(2x) \geq 2^{-2/5}\overline{F_{R^\delta_\gamma(X)}}(x)$. □

## 4. Finite-time horizon asymptotics

We give here a sketch of the proof of the main theorem introduced in the first section. We shall proceed as in [8]. We show the following:

$$\mu\{M_n > x\} \geq \frac{1+o(1)}{\alpha} \int_x^{x+n\alpha} \overline{F}(y)\mathrm{d}y \quad \text{as } x \to \infty, \text{ uniformly in } n \geq 1 \tag{4}$$

where $u = o(1)$ means that $u \longrightarrow 0$ as $x \to \infty$;

$$\mu\{M_n > x\} \sim \frac{1}{\alpha} \int_x^{x+n\alpha} \overline{F}(y)\mathrm{d}y \quad \text{as } x \to \infty, \text{ uniformly in } n \geq 1 \tag{5}$$

where $u \sim v$ means that $u/v \longrightarrow 1$ as $x \to \infty$. To show (4) and (5), we put: $\widetilde{\xi_n}(\gamma) = R^1_\gamma(\xi_n)$. We moreover shall assume $\widetilde{C}_\Psi := \int_0^\infty \mathrm{d}t\, t^{3/2}\widetilde{\eta}(t) < \infty$ and $\widetilde{\eta}(t) \geq 0$ $\forall t$. $\widetilde{\xi_1}, \widetilde{\xi_2}, \ldots$ are identically distributed random variables defined on the measured space $(]0,+\infty[ \times \mathbf{S} \times \mathbf{H}, B(]0,+\infty[) \otimes B(\mathbf{S}) \otimes B(\mathbf{H}), \lambda \otimes \mu \otimes \mu)$; from Theorem 3.2 their common distribution is a strong subexponential and we apply Theorem 5.4 in [8]. □

## References

[1] S. Asmussen, Ruin Probabilities, World Scientific, 2000.
[2] Z.-E. Baarir, A. Ouafi, Étude de la transformée en ondelettes dans la compression d'images fixes, Courrier du Savoir 5 (June 2004) 69–74.
[3] F. Baccelli, S. Foss, Moments and tails in monotone–separable stochastic networks, Ann. Appl. Probab. 14 (2004) 612–650.
[4] F. Baccelli, S. Foss, M. Lelarge, Asymptotics of a maximal dater in generalized Jackson networks, J. Appl. Probab. 42 (2005) 513–530.
[5] G. Bachman, L. Narici, E. Beckenstein, Fourier and Wavelet Analysis, Springer-Verlag, New York, 2000.
[6] I. Daubechies, Ten Lectures on Wavelets, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 1992.
[7] P. Embrechts, C. Klüppelberg, T. Mikosch, Modelling Extremals Events for Insurance and Finance, Springer, Berlin, 1997.
[8] S. Foss, D. Korshunov, S. Zachary, An Introduction to Heavy-Tailed and Subexponential Distributions, Springer Series in Operations Research and Financial Engineering, Springer, New York/Dordrecht/Heidelberg/London, 2011.
[9] C. Martias, La transformée en ridelettes pour les fonctionnelles de Wiener, C. R. Acad. Sci. Paris, Ser. I 350 (5–6) (2012) 259–262.
[10] C. Martias, The Wavelet transform for Wiener functionals and some applications, Stoch. Int. J. Probab. Stoch. Process. 86 (5) (2014), http://dx.doi.org/10.1080/17442508.2013.879144.