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Algebra

A note on the controllability of pairs of matrices[☆]*Une note sur la contrôlabilité des paires de matrices*

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ABSTRACT

Let F be an arbitrary field and let n, p_1, p_2, p_3 be positive integers such that $n = p_1 + p_2 + p_3$. Let

$$(C_1, C_2) = \left(\begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix}, \begin{bmatrix} C_{1,3} \\ C_{2,3} \end{bmatrix} \right),$$

where the blocks $C_{i,j}$ are of type $p_i \times p_j$, $i \in \{1, 2\}$, $j \in \{1, 2, 3\}$. Suppose that $C_{1,1}$, $C_{1,2}$ and $C_{2,1}$ are known. In this note we identify sufficient conditions under which there exist $C_{1,3}$, $C_{2,2}$, $C_{2,3}$ such that the pair (C_1, C_2) is completely controllable when $[C_{1,1}^T | C_{2,1}^T]^T$ has full rank.

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R É S U M É

Soit F un corps arbitraire et soient n, p_1, p_2, p_3 des entiers positifs tels que $n = p_1 + p_2 + p_3$. Soit

$$(C_1, C_2) = \left(\begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix}, \begin{bmatrix} C_{1,3} \\ C_{2,3} \end{bmatrix} \right),$$

où les blocs $C_{i,j}$ sont de type $p_i \times p_j$, $i \in \{1, 2\}$, $j \in \{1, 2, 3\}$. On suppose les blocs $C_{1,1}$, $C_{1,2}$, $C_{2,1}$ connus. Nous identifions dans cette note des conditions suffisantes sous lesquelles il existe des blocs $C_{1,3}$, $C_{2,2}$, $C_{2,3}$ tels que la paire (C_1, C_2) est complètement contrôlable quand le rang $[C_{1,1}^T | C_{2,1}^T]^T = p_1$.

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1. Introduction

An important question in control theory is the Pole Assignment Problem, described as follows. Given a system

$$\dot{\chi}(t) = A\chi(t) + B\zeta(t), \tag{1}$$

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where $\chi(t) \in \mathbb{R}^{p \times 1}$ denotes the ability of a certain physical system to be controllable by the input $\zeta(t) \in \mathbb{R}^{q \times 1}$, and $A \in \mathbb{R}^{p \times p}$, $B \in \mathbb{R}^{p \times q}$, how to select the input $\zeta(t)$ in such a way that $\chi(t)$ is driven to a certain desirable state?

The aim of this note is to identify conditions under which the system (1) is completely controllable, i.e. the pair (A, B) is completely controllable.

Next we introduce some notation that is necessary for the rest of the paper. We denote by $0_{m,n}$ the zero matrix of type $m \times n$; when there is no ambiguity, we can simply denote this matrix by 0. If A is a matrix, then A^T represents its transpose.

From now on, we adopt the notation $[A|B]$ to represent a rectangular matrix where A is of type $p \times p$ and B is of type $p \times q$. Then we recall the concept of controllability.

Definition 1.1. (See [3].) Let $A \in F^{p \times p}$, $B \in F^{p \times q}$. The pair (A, B) is said to be completely controllable if all the invariant factors of the matrix pencil $[xI_p - A | -B]$ are equal to 1.

There are many results available in the literature that describe the possibility that a system of the form (1) be completely controllable, when some entries of $[A|B]$ are prescribed. In particular, in a previous paper [2] we described conditions under which there exists a completely controllable pair of the form (C_1, C_2) where $C_1 = [C_{i,j}]_{i,j \in \{1, \dots, k-1\}}$, $C_2 = [C_{i,k}]_{i \in \{1, \dots, k-1\}}$ and $C_{i,j}$ is of type $p \times p$ ($i \in \{1, \dots, k-1\}$, $j \in \{1, \dots, k\}$), when $k-1$ of its blocks are fixed and the others vary. When the blocks are not necessarily of the same type, this problem becomes more difficult, and it remains open. To give some insight into this question, we start by considering the case $k=3$. In this note, we identify sufficient conditions under which there exists a completely controllable pair of the form

$$(C_1, C_2) = \left(\left[\begin{array}{cc} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{array} \right], \left[\begin{array}{c} C_{1,3} \\ C_{2,3} \end{array} \right] \right), \tag{2}$$

when the blocks $C_{1,1}$, $C_{1,2}$, $C_{2,1}$ are prescribed and $[C_{1,1}^T | C_{2,1}^T]^T$ has full rank.

2. Main results

Our main result is the following, where we provide sufficient conditions under which there exists a completely controllable pair of the form (2), when $[C_{1,1}^T | C_{2,1}^T]^T$ has full rank and $C_{2,1} \neq 0$.

Theorem 2.1. (See [1].) Let F be an arbitrary field and let n, p_1, p_2, p_3 be positive integers such that $n = p_1 + p_2 + p_3$. Let $C_{1,1} \in F^{p_1 \times p_1}$, $C_{1,2} \in F^{p_1 \times p_2}$, $C_{2,1} \in F^{p_2 \times p_1}$ be given. Let $r = \text{rank } C_{2,1}$. If

$$\text{rank}[C_{1,1}^T | C_{2,1}^T]^T = p_1, \tag{3}$$

with $r > 0$, and one of the following conditions holds, then there exist $C_{1,3} \in F^{p_1 \times p_3}$, $C_{2,2} \in F^{p_2 \times p_2}$, $C_{2,3} \in F^{p_2 \times p_3}$ such that the pair of the form (2) is completely controllable. The conditions are the following:

- (a_{2.1}) $p_1 \leq \min\{p_2, p_3\}$;
- (b_{2.1}) $p_2 < p_1 \leq p_3$;
- (c_{2.1}) $r \leq p_3 < p_1 \leq p_2$;
- (d_{2.1}) $\max\{p_2, p_3\} < p_1$, $p_1 = p_2 + s$ and $p_1 - p_3 \leq s$.

Our approach led us to consider the description of the characteristic polynomial of a matrix of the form

$$C = \left[\begin{array}{ccc} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{array} \right] \in F^{n \times n}, \tag{4}$$

when the blocks $C_{1,1}$, $C_{1,2}$, $C_{2,1}$ are prescribed, $C_{i,j}$ is of type $p_i \times p_j$, $i, j \in \{1, 2, 3\}$ and $n = p_1 + p_2 + p_3$.

Corollary 2.2. (See [1].) Let F be an arbitrary field and let n, p_1, p_2, p_3 be positive integers such that $n = p_1 + p_2 + p_3$. Let $f(x) \in F[x]$ be a monic polynomial of degree n . Let $C_{1,1} \in F^{p_1 \times p_1}$, $C_{1,2} \in F^{p_1 \times p_2}$, $C_{2,1} \in F^{p_2 \times p_1}$ be given. Let $r = \text{rank } C_{2,1}$. If (3) is satisfied with $r > 0$, and one of the conditions (a_{2.1}), (b_{2.1}), (c_{2.1}), (d_{2.1}) holds, then there exist $C_{1,3} \in F^{p_1 \times p_3}$, $C_{2,2} \in F^{p_2 \times p_2}$, $C_{2,3} \in F^{p_2 \times p_3}$, $C_{3,1} \in F^{p_3 \times p_1}$, $C_{3,2} \in F^{p_3 \times p_2}$, $C_{3,3} \in F^{p_3 \times p_3}$ such that the matrix of the form (4) has the characteristic polynomial $f(x)$.

Remark 1. (See [1].) Let F be an arbitrary field and let n, p_1, p_2, p_3 be positive integers such that $n = p_1 + p_2 + p_3$. Let $C_{1,1} \in F^{p_1 \times p_1}$, $C_{1,2} \in F^{p_1 \times p_2}$, $C_{2,1} \in F^{p_2 \times p_1}$ be given. Let $r = \text{rank } C_{2,1}$. If (3) holds with $r > 0$, $p_1 \leq p_2$ and $p_1 > p_3$, there is no guarantee of the existence of a completely controllable pair of the form (2) with the given blocks.

For example, let $p_1 = p_2 = 3, p_3 = 1$, and suppose that

$$C_{1,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad C_{1,2} = 0_{3,3}, \quad C_{2,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Obviously, $p_1 \leq p_2$ and $p_1 > p_3$. For any choice of $C_{1,3}, C_{2,2}$ and $C_{2,3}$ it is not hard to check that the matrix pencil of the form

$$\left[\begin{array}{cc|c} xI_{p_1} - C_{1,1} & -C_{1,2} & -C_{1,3} \\ -C_{2,1} & xI_{p_2} - C_{2,2} & -C_{2,3} \end{array} \right] \tag{5}$$

has always invariant factors different from 1. Thus, the pair of the form (2) is not completely controllable.

Remark 2. (See [1].) Let F be an arbitrary field and let n, p_1, p_2, p_3 be positive integers such that $n = p_1 + p_2 + p_3$. Let $C_{1,1} \in F^{p_1 \times p_1}, C_{1,2} \in F^{p_1 \times p_2}, C_{2,1} \in F^{p_2 \times p_1}$ be given. Let $r = \text{rank } C_{2,1}$. If (3) holds with $r > 0, \max\{p_2, p_3\} < p_1, p_1 = p_2 + s$ and $p_1 - p_3 > s$, there is no guarantee of the existence of a completely controllable pair of the form (2) with the given blocks. For example, let $p_1 = 4, p_2 = 2, p_3 = 1$ and suppose that

$$C_{1,1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad C_{1,2} = 0_{4,2}, \quad C_{2,1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Clearly, $p_1 - p_3 = 3 > 2 = p_1 - p_2 = s$. For any choice of $C_{1,3}, C_{2,2}$ and $C_{2,3}$, the matrix pencil of the form (5) has always invariant factors different from 1. Thus the pair of the form (2) is not completely controllable.

Proposition 2.3. Let F be an arbitrary field and let n, p_1, p_2, p_3 be positive integers such that $n = p_1 + p_2 + p_3$. Let $C_{1,1} \in F^{p_1 \times p_1}, C_{1,2} \in F^{p_1 \times p_2}, C_{2,1} \in F^{p_2 \times p_1}$ be given. Let $r = \text{rank } C_{2,1}$. If condition (3) holds with $0 < r \leq p_3$, then there exist $C_{1,3} \in F^{p_1 \times p_3}, C_{2,2} \in F^{p_2 \times p_2}, C_{2,3} \in F^{p_2 \times p_3}$ such that the pair of the form (2) is completely controllable.

Corollary 2.4. Let F be an arbitrary field and let n, p_1, p_2, p_3 be positive integers such that $n = p_1 + p_2 + p_3$. Let $f(x) \in F[x]$ be a monic polynomial of degree n . Let $C_{1,1} \in F^{p_1 \times p_1}, C_{1,2} \in F^{p_1 \times p_2}, C_{2,1} \in F^{p_2 \times p_1}$ be given. Let $r = \text{rank } C_{2,1}$. If condition (3) holds with $0 < r \leq p_3$, then there exist $C_{1,3} \in F^{p_1 \times p_3}, C_{2,2} \in F^{p_2 \times p_2}, C_{2,3} \in F^{p_2 \times p_3}, C_{3,1} \in F^{p_3 \times p_1}, C_{3,2} \in F^{p_3 \times p_2}, C_{3,3} \in F^{p_3 \times p_3}$ such that the matrix of the form (4) has the characteristic polynomial $f(x)$.

A natural question posed is the following: does the rank of matrix $C_{1,2}$ play a crucial role in such way this problem has a solution? In particular, if $C_{1,2}$ has full rank, does the problem have always a solution? The next remark shows that this question has a negative answer.

Remark 3. (See [1].) Let F be an arbitrary field and let n, p_1, p_2, p_3 be positive integers such that $n = p_1 + p_2 + p_3$. Let $C_{1,1} \in F^{p_1 \times p_1}, C_{1,2} \in F^{p_1 \times p_2}, C_{2,1} \in F^{p_2 \times p_1}$ be given. Let $r = \text{rank } C_{2,1}$. If (3) holds with $r > 0, \max\{p_2, p_3\} < p_1, p_1 = p_2 + s, p_1 - p_3 > s$, and $C_{1,2}$ has full rank, there is no guarantee of the existence of a completely controllable pair of the form (2) with the given blocks.

For example, let $p_1 = 4, p_2 = 2, p_3 = 1$ and suppose that

$$C_{1,1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad C_{1,2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_{2,1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Clearly, $p_1 - p_3 = 3 > 2 = p_1 - p_2 = s$ and $C_{1,2}$ has full rank. Nevertheless, for any choice of $C_{1,3}, C_{2,2}$ and $C_{2,3}$, the matrix pencil of the form (5) has invariant factors different from 1. Consequently, the pair of the form (2) is not completely controllable.

In a final reflection, other questions can naturally arise. For example: does the rank of the matrix $[C_{1,1}|C_{1,2}]$ also play an important role in this problem? At first sight it seems that if this matrix has full rank, then the problem has a solution. As previously illustrated on the above example, this question also has a negative answer. Clearly, the matrix pencil $[C_{1,1}|C_{1,2}]$ has full rank; nevertheless, for any arbitrary choice of $C_{1,3}, C_{2,2}$ and $C_{2,3}$, the matrix pencil of the form (5) has invariant factors different from 1, which means that the pair of the form (2) is not completely controllable.

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