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Complex analysis

Weak solutions to complex Monge–Ampère equations on compact Kähler manifolds

Solutions faibles des équations de Monge–Ampère complexes sur des variétés de Kähler compactes

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ABSTRACT

We show a general existence theorem of solutions to the complex Monge–Ampère type equation on compact Kähler manifolds.

RÉSUMÉ

Nous montrons un théorème général d'existence et d'unicité de solution d'une équation de type Monge–Ampère complexe sur des variétés de Kähler compactes.

1. Introduction

Let (X,ω) be a compact Kähler manifold of dimension n. Throughout this note, θ denotes a smooth closed form of bidegree $(1,1)$ which is nonnegative and big, i.e. such that $\int_X \theta^n > 0$. Recall that a θ-*plurisubharmonic* (θ-psh for short) function is an upper semi-continuous function φ such that $\theta + dd^c\varphi$ is nonnegative in the sense of currents. The set of all θ-psh functions φ on X will be denoted by $PSH(X,\theta)$ and endowed with the weak topology, which coincides with the $L^p(X)$-topology. We shall consider the existence and uniqueness of the weak solution to the following complex Monge–Ampère equations

$$\left(\theta + dd^c\varphi\right)^n = F(\varphi,\cdot)d\mu \tag{1}$$

where φ is a θ-psh function, $F(t,x) \ge 0$ is a measurable function on $\mathbb{R}\times X$ and μ is a positive measure. It is well known that we cannot always make sense to the left-hand side of (1) as a nonnegative measure. But according to [4] (see also [6,7,12]), we can define the non-pluripolar product $(\theta + dd^c u)^n$ as the limit of $\mathbf{1}_{(u>-j)}(\theta + dd^c(\max(u,-j)))^n$. It was shown in [7] that its trivial extension is a nonnegative closed current and

$$\int_X \left(\theta + dd^c u\right)^n \le \int_X \theta^n.$$

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Denote by $\mathcal{E}(X,\theta)$ the set of all θ-psh with full non-pluripolar Monge–Ampère measure, i.e. the θ-psh functions for which the last inequality becomes an equality.

For F smooth and $\mu = \mathrm{d}V$ is a smooth positive volume form, the equation has been studied extensively by various authors, see for example [1,2,7,15,13,14,16], etc., and references therein. Recently, Kołodziej treated the case F bounded by a function independent of the first variable and $\mu = \omega^n$, where ω is a Kähler form on X. In this paper, we consider a more general case. Our main purpose is to prove the following theorem.

Main Theorem. *Assume that $F : \mathbb{R} \times X \to [0,+\infty)$ is a measurable function such that:*

1) *for all $x \in X$ the function $t \mapsto F(t,x)$ is continuous and nondecreasing;*
2) *$F(t,\cdot) \in L^1(X,\mathrm{d}\mu)$ for all $t \in \mathbb{R}$;*
3)

$$\lim_{t\to-\infty} \int_X F(t,x)\,\mathrm{d}\mu \le \int_X \theta^n < \lim_{t\to+\infty} \int_X F(t,x)\,\mathrm{d}\mu.$$

Then there exists a unique (up to additive constant) θ-psh function $\varphi \in \mathcal{E}(X,\theta)$ solution to

$$\left(\theta + \mathrm{dd}^c\varphi\right)^n = F(\varphi,\cdot)\,\mathrm{d}\mu.$$

2. Proof

Lemma 2.1. *Let μ be a positive measure on X vanishing on all pluripolar subsets of X and $u_j \in \mathcal{E}(X,\theta)$ such that $u_j \ge u_0$ for some $u_0 \in \mathcal{E}(X,\theta) \cap L^1(\mathrm{d}\mu)$.*

If $u_j \to u$ in $L^1(X)$, then

$$\lim_{j\to+\infty} \int_X u_j\,\mathrm{d}\mu = \int_X u\,\mathrm{d}\mu.$$

Proof. Since $u_0 \in L^1(\mathrm{d}\mu)$ and the measure μ puts no mass on pluripolar subsets of X, then

$$\int_\alpha^{+\infty} \int_{(u_j<-t)} \mathrm{d}\mu\,\mathrm{d}t \le \int_\alpha^{+\infty} \int_{(u_0<-t)} \mathrm{d}\mu\,\mathrm{d}t \to 0 \quad \text{as } \alpha \to +\infty.$$

Hence, by the Dunford–Petit theorem (see for example [10] p. 274), we have that the sequence (u_j) is weakly relatively compact in $L^1(\mathrm{d}\mu)$. Let $\hat{u} \in L^1(\mathrm{d}\mu)$ be a cluster point of (u_j). After extracting a subsequence, we may assume that (u_j) converges to $\hat{u}$ weakly in $L^1(\mathrm{d}\mu)$. On the other hand, we have $u_j \to u$ in $L^1(X)$. So, choosing a subsequence if necessary, we can assume that $u_j \to u$ point-wise on $X \setminus A$, where $A = \{\limsup_{j\to\infty} u_j < u\}$. But A is negligible, hence, by [3] A is pluripolar subset of X, thus $\mu(A) = 0$. It follows from Lebesgue's dominated convergence theorem that $u_j \to u$ weakly in $L^1(\mathrm{d}\mu)$. Therefore $\hat{u} = u$ μ-a.e. Hence u is the unique cluster point of (u_j), which means that (u_j) converges to u weakly in $L^1(\mathrm{d}\mu)$ and the proof is complete. □

The following corollary is the global version of Corollary 1.4 in [8].

Corollary 2.2. *Let μ be a nonnegative measure that puts no mass on pluripolar sets of X. Then for any sequence $u_j \in \mathcal{E}(X,\theta)$ converging weakly, one can extract a subsequence that converges pointwise $\mathrm{d}\mu$-almost everywhere.*

Proof of the Main Theorem. The set of $\varphi \in PSH(X,\theta)$ normalized by $\sup_X \varphi = 0$ is compact (cf. [11,12]). Then there exists a positive constant $C_0 > 0$ such that

$$\int_X -u\theta^n \le C_0, \quad \forall u \in PSH(X,\theta); \qquad \sup_X u = 0.$$

Consider the set

$$\mathcal{H} := \left\{\varphi \in PSH(X,\theta);\, \varphi \le 0 \text{ and } \int_X -\varphi\theta^n \le C_0\right\}$$

It is obvious that $\mathcal{H}$ is a compact convex subset of $L^1(X)$.

From the conditions of the main theorem, there exists a real number c_0 such that

$$\int_X F(c_0,\cdot)\mathrm{d}\mu = \int_X \theta^n.$$

Fix a function $\varphi \in \mathcal{H}$. Then there exists a real number $c_\varphi \ge c_0$ such that

$$\int_X F(\varphi + c_\varphi,\cdot)\mathrm{d}\mu = \int_X \theta^n.$$

Since $F(\varphi + c_\varphi,\cdot) \in L^1(X,\mathrm{d}\mu)$ and μ vanishes on pluripolar sets, it follows by [7,5] that there exists a function $\tilde{\varphi} \in \mathcal{E}(X,\theta)$ such that $\sup_X \tilde{\varphi} = 0$ and

$$\left(\theta + \mathrm{dd}^c\tilde{\varphi}\right)^n = F(\varphi + c_\varphi,\cdot)\mathrm{d}\mu.$$

The function $\tilde{\varphi}$ does not depend on the constant c_φ. Indeed, assume that there exist two constant c_φ and c'_φ such that

$$\int_X F(\varphi + c_\varphi,\cdot)\mathrm{d}\mu = \int_X F\left(\varphi + c'_\varphi,\cdot\right)\mathrm{d}\mu = \int_X \theta^n.$$

If $c_\varphi \le c'_\varphi$ then $F(\varphi + c_\varphi,\cdot)\mathrm{d}\mu \le F(\varphi + c'_\varphi,\cdot)\mathrm{d}\mu$. Thence

$$F(\varphi + c_\varphi,\cdot)\mathrm{d}\mu = F\left(\varphi + c'_\varphi,\cdot\right)\mathrm{d}\mu.$$

By the uniqueness result in [7] and [9], we get that $\tilde{\varphi}$ is unique and therefore independent of the constant c_φ.

From the definition of $\mathcal{H}$ we have $\tilde{\varphi} \in \mathcal{H}$. Consider the map $\Phi : \mathcal{H} \to \mathcal{H}$ defined by $\varphi \mapsto \tilde{\varphi}$. In fact, the range of Φ is equal to $\mathcal{H} \cap \mathcal{E}(X,\theta)$.

We claim that Φ is continuous. Indeed, let $\varphi_j \in \mathcal{H}$ be a converging sequence with limit $\varphi \in \mathcal{H}$ in $L^1(X)$-topology. Let ψ be any cluster point of the sequence $\tilde{\varphi}_j := \phi(\varphi_j)$. We may assume, up to extracting, that $\tilde{\varphi}_j$ converges towards ψ in $L^1(X)$. Since the measure μ vanishes on pluripolar subsets, then by Corollary 2.2 above, we can extract a subsequence, which is still denoted by φ_j, so that $\varphi_j \to \varphi$ μ-a.e. We claim that the sequence c_{φ_j} is bounded. Indeed, by construction we have $c_{\varphi_j} \ge c_0$. Now if $c_{\varphi_j} \to +\infty$ then

$$\int_X \theta^n = \liminf_{j\to\infty} \int_X F(\varphi_j + c_{\varphi_j},\cdot)\mathrm{d}\mu > \int_X \theta^n,$$

which is impossible.

So by passing to a subsequence, we may assume that $c_{\varphi_j} \to c$. Therefore $F(\varphi_j + c_{\varphi_j},\cdot) \to F(\varphi + c,\cdot)$ in $L^1(\mathrm{d}\mu)$. Since $\tilde{\varphi}_j \to \psi$ in $L^1(X)$, then $\psi = (\limsup_{j\to+\infty} \tilde{\varphi}_j)^*$ and therefore by Hartogs' lemma $\sup_X \psi = 0$. Let denote $\psi_j := (\sup_{k\ge j} \tilde{\varphi}_k)^* = (\lim_{l\to+\infty} \max_{l\ge k\ge j} \tilde{\varphi}_k)^*$. Since the set $(\sup_{k\ge j} \tilde{\varphi}_k < (\sup_{k\ge j} \tilde{\varphi}_k)^*)$ is pluripolar, then by the continuity of the complex Monge–Ampère operator along monotonic sequences, we have:

$$\begin{aligned}
\left(\theta + \mathrm{dd}^c\psi\right)^n &= \lim_{j\to+\infty} \left(\theta + \mathrm{dd}^c\psi_j\right)^n \\
&= \lim_{j\to+\infty} \lim_{l\to+\infty} \left(\theta + \mathrm{dd}^c \max_{l\ge k\ge j} \tilde{\varphi}_k\right)^n \\
&\ge \lim_{j\to+\infty} \lim_{l\to+\infty} \min_{l\ge k\ge j} F(\varphi_k + c_{\varphi_k},\cdot)\mathrm{d}\mu \\
&= \liminf_{j\to+\infty} F(\varphi_j + c_{\varphi_j},\cdot)\mathrm{d}\mu \\
&= F(\varphi + c_\varphi,\cdot)\mathrm{d}\mu.
\end{aligned}$$

Thence $(\theta + \mathrm{dd}^c\psi)^n = (\theta + \mathrm{dd}^c\tilde{\varphi})^n$. By uniqueness (shown in [7]), we get $\tilde{\varphi} = \psi$ and therefore Φ is continuous. Now, Shauder's fixed point theorem implies that there exists a function $u \in \mathcal{H}$ such that $\Phi(u) = u$. Since $\Phi(\mathcal{H}) \subset \mathcal{E}(X,\theta)$ we have $u \in \mathcal{E}(X,\theta)$ and

$$\left(\theta + \mathrm{dd}^c u\right)^n = F(u + c_u,\cdot)\mathrm{d}\mu.$$

The function $\varphi := u + c_u$ is the required solution.

Uniqueness follows in a classical way from the comparison principle [3] and its generalization [7]. Indeed assume that there exist tow solutions φ_1 and φ_2 in $\mathcal{E}(X,\theta)$ such that

$$\left(\theta + \mathrm{dd}^c\varphi_i\right)^n = F(\varphi_i,.), \quad i = 1,2. \tag{2}$$

Then, since F is non-decreasing with respect to the first variable, we have

$$F(\varphi_1,.)\mathrm{d}\mu \le F(\varphi_2,.)\mathrm{d}\mu \quad \text{on } (\varphi_1<\varphi_2). \tag{3}$$

On the other hand, by the comparison principle we have

$$\int_{(\varphi_1<\varphi_2)} \left(\theta+\mathrm{dd}^c\varphi_2\right)^n \le \int_{(\varphi_1<\varphi_2)} \left(\theta+\mathrm{dd}^c\varphi_1\right)^n. \tag{4}$$

Combining (2), (3) and (4), we get

$$\int_{(\varphi_1<\varphi_2)} F(\varphi_1,.)\mathrm{d}\mu \le \int_{(\varphi_1<\varphi_2)} F(\varphi_2,.)\mathrm{d}\mu \le \int_{(\varphi_1<\varphi_2)} F(\varphi_1,.)\mathrm{d}\mu.$$

Hence

$$F(\varphi_1,.)=F(\varphi_2,.)\mu\text{-almost everywhere} \quad \text{on } (\varphi_1<\varphi_2).$$

In the same way, we get the equality on $(\varphi_1>\varphi_2)$ and then on X. Hence

$$\left(\theta+\mathrm{dd}^c\varphi_1\right)^n=\left(\theta+\mathrm{dd}^c\varphi_2\right)^n.$$

Once more, the uniqueness result of [7] and [9] implies that $\varphi_1-\varphi_2=Cst$. □

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