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Lagrange interpolation problem for quaternion polynomials

*Le problème d'interpolation de Lagrange pour les polynômes de quaternions*

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ABSTRACT

Interpolation theory for complex polynomials is well understood. In the non-commutative quaternionic setting, the polynomials can be evaluated “from the left” and “from the right”. If the interpolation problem involves interpolation conditions of the same (left or right) type, the results are very much similar to the complex case: a consistent problem has a unique solution of a low degree (less than the number of interpolation conditions imposed), and the solution set of the homogeneous problem is an ideal in the ring $\mathbb{H}[z]$. The problem containing both “left” and “right” interpolation conditions is quite different: there may exist infinitely many low-degree solutions and the solution set of the homogeneous problem is a quasi-ideal in $\mathbb{H}[z]$.

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R É S U M É

La théorie de l'interpolation pour les polynômes complexes est bien comprise. Dans le cadre des quaternions non commutatifs, les polynômes peuvent être évalués « de la gauche » et « de la droite ». Si le problème d'interpolation implique des conditions d'interpolation du même type (gauche ou droite), les résultats sont très similaires au cas complexe : un problème constant a une solution unique d'un faible degré (moins que le nombre de conditions d'interpolation qui sont imposées), et l'ensemble du problème homogène est un idéal de l'anneau $\mathbb{H}[z]$. Le problème qui contient des conditions d'interpolation à la fois « gauches » et « droites » est tout à fait différent : il peut exister une infinité de solutions d'un faible degré, et l'ensemble des solutions du problème homogène est un quasi-idéal de $\mathbb{H}[z]$.

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1. Introduction

Given distinct points $z_1, \dots, z_n \in \mathbb{C}$ and target values $c_1, \dots, c_n \in \mathbb{C}$, the Lagrange interpolation problem consists of finding a complex polynomial $f \in \mathbb{C}[z]$ such that $f(z_i) = c_i$ for $i = 1, \dots, n$. It turns out that the Lagrange polynomial

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$$\tilde{f}(z) = \sum_{k=1}^n \frac{c_k p_k(z)}{p_k(z_k)} \left(\text{where } p_k(z) = \prod_{\substack{j=1 \\ j \neq k}}^n (z - z_j) \right) \tag{1.1}$$

is a unique solution of the problem of degree less than n and that all solutions are parameterized by the formula $f(z) = \tilde{f}(z) + p(z)h(z)$, where $p(z) = \prod_{i=1}^n (z - z_i)$ and h is the free parameter in $\mathbb{C}[z]$. Over the years, Lagrange interpolation has been playing a prominent role in approximation theory and numerical analysis; more recent applications include image processing and control theory. The problem can be settled exactly as in the complex case for polynomials over any field (including finite fields, which has important applications in cryptography). However, interpolation problems in non-commutative polynomial rings have not attracted much attention so far. The objective of this paper is to consider the Lagrange interpolation problem for polynomials over the skew field \mathbb{H} of real quaternions

$$\alpha = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3 \quad (x_0, x_1, x_2, x_3 \in \mathbb{R}), \tag{1.2}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are imaginary units commuting with the reals and such that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. We denote by $\mathbb{H}[z]$ the ring of polynomials in one formal variable z that commutes with quaternionic coefficients. The ring operations in $\mathbb{H}[z]$ are defined as in the commutative case, but as multiplication in \mathbb{H} is not commutative, multiplication in $\mathbb{H}[z]$ is not commutative as well. For $\alpha \in \mathbb{H}$ and $f \in \mathbb{H}[z]$, we define $f^{e\ell}(\alpha)$ and $f^{er}(\alpha)$ (left and right evaluations of f at α) by

$$f^{e\ell}(\alpha) = \sum_{k=0}^n \alpha^k f_k, \quad f^{er}(\alpha) = \sum_{k=0}^n f_k \alpha^k \quad \text{if } f(z) = \sum_{k=0}^n z^k f_k = \sum_{k=0}^n f_k z^k, \quad f_k \in \mathbb{H}. \tag{1.3}$$

Since \mathbb{R} is the center of \mathbb{H} , the ring $\mathbb{R}[z]$ of polynomials with real coefficients is the center of $\mathbb{H}[z]$. Therefore, $f^{e\ell}(x) = f^{er}(x)$ for every $f \in \mathbb{H}[z]$ if $x \in \mathbb{R}$ and, on the other hand, if $f \in \mathbb{R}[z]$, then $f^{e\ell}(\alpha) = f^{er}(\alpha)$ for every $\alpha \in \mathbb{H}$. In general, interpolation conditions produced by the left and right evaluations should be distinguished. We will consider the interpolation problem whose data set consists of two (not necessarily disjoint) finite sets

$$\Lambda = \{\alpha_1, \dots, \alpha_n\} \quad \text{and} \quad \Omega = \{\beta_1, \dots, \beta_m\} \tag{1.4}$$

of distinct elements in \mathbb{H} along with the respective target values c_1, \dots, c_n and d_1, \dots, d_m in \mathbb{H} . The *two-sided Lagrange problem* consists in finding a polynomial $f \in \mathbb{H}[z]$ such that

$$f^{e\ell}(\alpha_i) = c_i \quad \text{for } i = 1, \dots, n, \tag{1.5}$$

$$f^{er}(\beta_j) = d_j \quad \text{for } j = 1, \dots, m. \tag{1.6}$$

Since right and left evaluations coincide at real points, we may assign all real interpolation nodes to the left set Λ , assuming therefore that $\Omega \cap \mathbb{R} = \emptyset$. We emphasize that the sets (1.4) do not have to be disjoint, so that we may have left and right interpolation conditions at the same interpolation node $\alpha_i = \beta_j$.

2. The solvability criterion and description of all solutions

For $\alpha \in \mathbb{H}$ of the form (1.2), its real and imaginary parts, the quaternion conjugate and the absolute value are defined as $\text{Re}(\alpha) = x_0$, $\text{Im}(\alpha) = \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$, $\bar{\alpha} = \text{Re}(\alpha) - \text{Im}(\alpha)$, and $|\alpha| = \sqrt{\alpha\bar{\alpha}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$, respectively. Two quaternions α and β are called *equivalent* (conjugate to each other) if $\alpha = h^{-1}\beta h$ for some nonzero $h \in \mathbb{H}$. It follows (see e.g., [9]) that

$$\alpha \sim \beta \quad \text{if and only if} \quad \text{Re}(\alpha) = \text{Re}(\beta) \quad \text{and} \quad |\text{Im}(\alpha)| = |\text{Im}(\beta)|. \tag{1.1}$$

The conjugacy class of an $\alpha \in \mathbb{H}$ will be denoted by $[\alpha]$. Observe that $[\alpha] = \{\alpha\}$ if and only if α is real.

2.1. Consistency

In the complex setting, the Lagrange problem with distinct interpolation nodes is always consistent. In the quaternionic case, inconsistency may occur if the set $\Lambda \cup \Omega$ contains more than two points from the same conjugacy class; the one-sided version of this phenomenon was observed in [2].

Lemma 2.1. *For $f \in \mathbb{H}[z]$ and three distinct equivalent quaternions $\alpha \sim \beta \sim \gamma$,*

$$f^{e\ell}(\gamma) = (\gamma - \beta)(\alpha - \beta)^{-1} f^{e\ell}(\alpha) + (\alpha - \gamma)(\alpha - \beta)^{-1} f^{e\ell}(\beta), \tag{2.2}$$

$$f^{er}(\gamma) = (\alpha - \beta)^{-1} f^{er}(\alpha)\gamma - \beta(\alpha - \beta)^{-1} f^{er}(\alpha) + \alpha(\alpha - \beta)^{-1} f^{er}(\beta) - (\alpha - \beta)^{-1} f^{er}(\beta)\gamma, \tag{2.3}$$

$$f^{er}(\gamma) = f^{er}(\alpha)(\alpha - \beta)^{-1}(\gamma - \beta) + f^{er}(\beta)(\alpha - \beta)^{-1}(\alpha - \gamma), \tag{2.4}$$

$$f^{e\ell}(\gamma) = \gamma f^{er}(\alpha)(\alpha - \beta)^{-1} - f^{er}(\alpha)(\alpha - \beta)^{-1}\beta + f^{er}(\beta)(\alpha - \beta)^{-1}\alpha - \gamma f^{er}(\beta)(\alpha - \beta)^{-1}. \tag{2.5}$$

Lemma 2.1 shows that left (or right) evaluations of $f \in \mathbb{H}[z]$ at any two points from the same conjugacy class uniquely determine left and right evaluations of f at any point in this conjugacy class. Thus, if the set $\Lambda \cup \Omega$ contains more than two points from the same conjugacy class, the corresponding target values must satisfy certain conditions for the Lagrange problem to have a solution. Let V be a conjugacy class such that $V \cap \Lambda = \{\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}\}$ contains at least two elements and let $V \cap \Omega = \{\beta_{j_1}, \dots, \beta_{j_s}\}$. For the assigned target values c_{i_ℓ} and d_{j_r} , we verify equalities (cf. (2.2) and (2.3))

$$c_{i_\ell} = (\alpha_{i_\ell} - \alpha_{i_2})(\alpha_{i_1} - \alpha_{i_2})^{-1}c_{i_1} + (\alpha_{i_1} - \alpha_{i_\ell})(\alpha_{i_1} - \alpha_{i_2})^{-1}c_{i_2},$$

$$d_{j_r} = (\alpha_{i_1} - \alpha_{i_2})^{-1}c_{i_1}\beta_{j_r} - \alpha_{i_2}(\alpha_{i_1} - \alpha_{i_2})^{-1}c_{i_1} + \alpha_{i_1}(\alpha_{i_1} - \alpha_{i_2})^{-1}c_{i_2} - (\alpha_{i_1} - \alpha_{i_2})^{-1}c_{i_2}\beta_{j_r}$$

for $\ell = 3, \dots, k$ and $r = 1, \dots, s$. If at least one of them fails, the Lagrange problem (1.5), (1.6) does not have solutions, by Lemma 2.1. Otherwise, any polynomial $f \in \mathbb{H}[z]$ satisfying interpolation conditions (1.5) at α_{i_1} and α_{i_2} will satisfy interpolation conditions at α_{i_ℓ} (for $\ell = 3, \dots, k$) and right conditions at β_{j_r} (for $r = 1, \dots, s$) automatically, again by Lemma 2.1. Hence, removing interpolation conditions at these points, we get a reduced interpolation problem with the same solution set as the original one. Alternatively, if $V \cap \Omega$ contains at least two elements β_{j_1}, β_{j_2} , we may use relations (2.4) and (2.5) to check if other interpolation conditions on V are compatible with those two at β_{j_1} and β_{j_2} and, if this is the case, all other conditions can be removed without affecting the solution set of the problem. After completing consistency verifications in all conjugacy classes having more than two common elements with $\Lambda \cup \Omega$, we either conclude that the original problem is inconsistent or reduce it to a problem for which *none three of the interpolation nodes belong to the same conjugacy class*.

2.2. Minimal polynomials

Since the left and the right division algorithms hold in $\mathbb{H}[z]$, any ideal (left or right) in $\mathbb{H}[z]$ is principal. Given $\Lambda = \{\alpha_1, \dots, \alpha_n\} \subset \mathbb{H}$, the set $\mathbf{I}_\Lambda = \{f \in \mathbb{H}[z] : f^{\mathbf{e}_\ell}|_\Lambda = 0\}$ is a right ideal in $\mathbb{H}[z]$; hence there exists a unique monic polynomial generating \mathbf{I}_Λ . We will call this polynomial the *left minimal polynomial* of Λ and will denote it by $P_{\Lambda, \ell}$. This polynomial can be alternatively defined as a unique monic polynomial with the set of left zeros equal to Λ . If *none three elements of Λ are equivalent*, the left minimal polynomial $P_{\Lambda, \ell}$ can be constructed recursively as follows:

$$p_0(z) \equiv 1, \quad p_{j+1}(z) = p_j(z) \cdot (z - p_j^{\mathbf{e}_\ell}(\alpha_{j+1})^{-1} \cdot \alpha_{j+1} \cdot p_j^{\mathbf{e}_\ell}(\alpha_{j+1})), \quad P_{\Lambda, \ell}(z) := p_n(z). \tag{2.6}$$

The assumption on Λ guarantees that $p_j^{\mathbf{e}_\ell}(\alpha_{j+1}) \neq 0$ for all $j = 1, \dots, n - 1$. It is also clear from (2.6) that $\deg(P_{\Lambda, \ell}) = n$. Recursion (2.6) produces the left minimal polynomial as a product of linear factors. Although the outcome $P_{\Lambda, \ell}$ of the recursion does not depend on the order in which the elements of Λ are arranged, different permutations of Λ produce via recursion (2.6) different factorizations of $P_{\Lambda, \ell}$.

The *right minimal polynomial* $P_{\Omega, \mathbf{r}}$ of the set $\Omega = \{\beta_1, \dots, \beta_m\}$ is defined as a unique monic polynomial with the set of right zeros equal to Ω . In case *none three of the elements of Ω belong to the same conjugacy class*, this polynomial is of degree m and can be constructed recursively as follows:

$$q_0(z) \equiv 1, \quad q_{j+1}(z) = (z - q_j^{\mathbf{e}_\mathbf{r}}(\beta_{j+1}) \cdot \beta_{j+1} \cdot q_j^{\mathbf{e}_\mathbf{r}}(\beta_{j+1})^{-1}) \cdot q_j(z), \quad P_{\Omega, \mathbf{r}}(z) := q_m(z). \tag{2.7}$$

It makes sense to define minimal polynomials of the empty set by letting $P_{\emptyset, \ell}(z) = P_{\emptyset, \mathbf{r}}(z) \equiv 1$.

2.3. Sylvester equation

It is not hard to show that for $\alpha \approx \beta$ and any $\Delta \in \mathbb{H}$, the Sylvester equation $\alpha q - q\beta = \Delta$ has a unique solution given by the formula $q = (\bar{\alpha}\Delta - \Delta\beta)(\beta^2 - \beta(\alpha + \bar{\alpha}) + |\alpha|^2)^{-1}$. If α and β are equivalent, we may write them, due to characterization (2.1) as

$$\alpha = x + yI, \quad \beta = x + y\tilde{I} \quad (x \in \mathbb{R}, y > 0, I^2 = \tilde{I}^2 = -1). \tag{2.8}$$

The elements I and \tilde{I} are purely imaginary unit quaternions. Since \mathbb{H} is a (four-dimensional) vector space over \mathbb{R} , we may define orthogonal complements with respect to the usual Euclidean metric in \mathbb{R}^4 . For α and β as in (2.8), we define the plane (the two-dimensional subspace of $\mathbb{H} \cong \mathbb{R}^4$) $\Pi_{\alpha, \beta}$ via the formula

$$\Pi_{\alpha, \beta} = \begin{cases} \text{span}\{1, I\} = \{u + vI : u, v \in \mathbb{R}\}, & \text{if } \beta = \alpha, \\ (\text{span}\{1, I\})^\perp, & \text{if } \beta = \bar{\alpha}, \\ \text{span}\{I + \tilde{I}, 1 - \tilde{I}\}, & \text{if } \beta \neq \alpha, \bar{\alpha}. \end{cases} \tag{2.9}$$

Lemma 2.2. *Let $\alpha \sim \beta$ be of the form (2.8). Then the Sylvester equation $\alpha q - q\beta = \Delta$ has a solution if and only if $\bar{\alpha}\Delta = \Delta\beta$, in which case the solution set for the equation is the affine plane $(2\text{Im}(\alpha))^{-1} \cdot \Delta + \Pi_{\alpha, \beta}$.*

2.4. Main results

We assume that none three of the interpolation nodes in $\Lambda \cup \Omega$ belong to the same conjugacy class. If there are pairs (α_i, β_j) of equivalent left and right nodes, we will rearrange the sets (1.4) so that

$$\alpha_i \sim \beta_i \quad (i = 1, \dots, k); \quad [\alpha_i] \cap \Omega = \emptyset \quad (i = k + 1, \dots, n); \quad [\beta_j] \cap \Lambda = \emptyset \quad (j = k + 1, \dots, m). \tag{2.10}$$

If $[\alpha_i] \cap [\beta_j] = \emptyset$ for all $(\alpha_i, \beta_j) \in \Lambda \times \Omega$, we let $k = 0$ in (2.10).

Theorem 2.3. *Let us assume that interpolation nodes are arranged as in (2.10). There is an $f \in \mathbb{H}[z]$ satisfying conditions (1.5), (1.6) if and only if*

$$\bar{\alpha}_i(c_i - d_i) = (c_i - d_i)\beta_i \quad (i = 1, \dots, k). \tag{2.11}$$

For any polynomial $f(z) = \sum_{j=0}^N z^j f_j$, we define its conjugate $f^\sharp(z) = \sum_{j=0}^N z^j \bar{f}_j$. The antilinear involution $f \mapsto f^\sharp$ on $\mathbb{H}[z]$ can be viewed as an extension of the quaternionic conjugation $\alpha \mapsto \bar{\alpha}$ from \mathbb{H} to $\mathbb{H}[z]$. The polynomial ff^\sharp is real (since $(ff^\sharp)^\sharp = f^\sharp f = ff^\sharp$), and therefore $(ff^\sharp)^{e\ell}(\alpha) = (ff^\sharp)^{er}(\alpha)$ for all $\alpha \in \mathbb{H}$. If $f^{e\ell}(\alpha) = 0$ or if $f^{er}(\alpha) = 0$, then any element from the conjugacy class $[\alpha]$ is a zero of ff^\sharp (see [6]). Therefore, if the interpolation nodes are arranged as in (2.10), then $(P_{\Lambda, \ell}^\sharp P_{\Lambda, \ell})(\beta_j) \neq 0$ and $(P_{\Omega, r}^\sharp P_{\Omega, r})(\alpha_i) \neq 0$ for $i, j > k$.

For the sets (1.4), we let $P_{\Lambda_i, \ell}$ be the left minimal polynomial of the set $\Lambda_i := \Lambda \setminus \{\alpha_i\}$ for $i = 1, \dots, n$, and we let $P_{\Omega_j, r}$ be the right minimal polynomial of the set $\Omega_j := \Omega \setminus \{\beta_j\}$ for $j = 1, \dots, m$. If the interpolation nodes are arranged as in (2.10), then $(P_{\Lambda_i, \ell}^\sharp P_{\Lambda_i, \ell})^{e\ell}(\beta_i) \neq 0$ and $(P_{\Omega_i, r}^\sharp P_{\Omega_i, r})(\alpha_i) \neq 0$ for all $i = 1, \dots, k$. The next theorem is the main result that we announce in this note.

Theorem 2.4. *Let the interpolation nodes be arranged as in (2.10) and let us assume that conditions (2.11) are met. Define the elements*

$$\rho_i = \begin{cases} P_{\Lambda_i, \ell}^{e\ell}(\alpha_i)^{-1} [(P_{\Omega_i, r}^\sharp P_{\Omega_i, r})(\alpha_i)]^{-1} \cdot c_i \cdot P_{\Omega, r}^{e\ell}(c_i^{-1} \alpha_i c_i), & \text{if } c_i \neq 0 \text{ \& } i = 1, \dots, k, \\ P_{\Lambda_i, \ell}^{e\ell}(\alpha_i)^{-1} [(P_{\Omega, r}^\sharp P_{\Omega, r})(\alpha_i)]^{-1} \cdot c_i \cdot P_{\Omega, r}^{e\ell}(c_i^{-1} \alpha_i c_i), & \text{if } c_i \neq 0 \text{ \& } i = k + 1, \dots, n, \\ 0, & \text{if } c_i = 0, \end{cases} \tag{2.12}$$

$$\gamma_j = \begin{cases} P_{\Lambda, \ell}^{er}(d_j \beta_j d_j^{-1}) \cdot d_j \cdot [(P_{\Lambda_j, \ell}^\sharp P_{\Lambda_j, \ell})(\beta_j)]^{-1} P_{\Omega_j, r}^{er}(\beta_j)^{-1}, & \text{if } d_j \neq 0 \text{ \& } j = 1, \dots, k, \\ P_{\Lambda, \ell}^{er}(d_j \beta_j d_j^{-1}) \cdot d_j \cdot [(P_{\Lambda, \ell}^\sharp P_{\Lambda, \ell})(\beta_j)]^{-1} P_{\Omega_j, r}^{er}(\beta_j)^{-1}, & \text{if } d_j \neq 0 \text{ \& } j = k + 1, \dots, m, \\ 0, & \text{if } d_j = 0, \end{cases} \tag{2.13}$$

$$\tilde{\alpha}_i = P_{\Lambda_i, \ell}^{e\ell}(\alpha_i)^{-1} \cdot \alpha_i \cdot P_{\Lambda_i, \ell}^{e\ell}(\alpha_i), \quad \tilde{\beta}_i = P_{\Omega_i, r}^{er}(\beta_i) \cdot \beta_i \cdot P_{\Omega_i, r}^{er}(\beta_i)^{-1} \quad \text{for } i = 1, \dots, k, \tag{2.14}$$

and let $\Pi_{\tilde{\alpha}_i, \tilde{\beta}_i}$ be the plane defined via formula (2.9) (since $\alpha_i \sim \beta_i$, it follows from (2.14) that $\tilde{\alpha}_i \sim \tilde{\beta}_i$). All polynomials $f \in \mathbb{H}[z]$ satisfying conditions (1.5), (1.6) are given by the formula

$$f(z) = \sum_{i=k+1}^n P_{\Lambda_i, \ell}(z) \cdot \rho_i \cdot P_{\Omega, r}(z) + \sum_{j=k+1}^m P_{\Lambda, \ell}(z) \cdot \gamma_j \cdot P_{\Omega_j, r}(z) + P_{\Lambda, \ell}(z) \cdot h(z) \cdot P_{\Omega, r}(z) + \sum_{i=1}^k P_{\Lambda_i, \ell}(z) \cdot (\rho_i + (z - \tilde{\alpha}_i)(2\text{Im}(\tilde{\alpha}_i))^{-1}(\rho_i - \gamma_i)) \cdot P_{\Omega_i, r}(z) + \sum_{i=1}^k P_{\Lambda, \ell}(z) \cdot q_i \cdot P_{\Omega_i, r}(z) \tag{2.15}$$

with free parameters $h \in \mathbb{H}[z]$ and $q_i \in \Pi_{\tilde{\alpha}_i, \tilde{\beta}_i}$ for $i = 1, \dots, k$.

We point out several consequences of the parameterization formula (2.15).

1. Since $\deg(P_{\Lambda, \ell}) = n$, $\deg(P_{\Lambda_i, \ell}) = n - 1$, $\deg(P_{\Omega_j, r}) = m$, and $\deg(P_{\Omega_i, r}) = m - 1$, it follows that formula (2.15) with the parameter $h \equiv 0$ produces all low-degree solutions to the problem (1.5), (1.6). The formula still contains k parameters $q_i \in \Pi_{\tilde{\alpha}_i, \tilde{\beta}_i}$.
2. If $[\alpha_i] \cap [\beta_j] = \emptyset$ for all $(\alpha_i, \beta_j) \in \Lambda \times \Omega$, there is a unique polynomial \tilde{f} of degree less than $n + m$ satisfying conditions (1.5), (1.6). Letting $k = 0$ in (2.15), we get all solutions for this special case:

$$f = \tilde{f} + P_{\Lambda, \ell} \cdot h \cdot P_{\Omega, r}, \tag{2.16}$$

$$\tilde{f}(z) = \sum_{i=k+1}^n P_{\Lambda_i, \ell}(z) \cdot \rho_i \cdot P_{\Omega, r}(z) + \sum_{j=k+1}^m P_{\Lambda, \ell}(z) \cdot \gamma_j \cdot P_{\Omega_j, r}(z), \quad h \in \mathbb{H}[z].$$

3. Specializing the formula (2.16) further to the case where $\Omega = \emptyset$ and therefore, $P_{\Omega, \mathbf{r}} \equiv 1$ and, according to (2.12), $\rho_i = P_{\Lambda_i, \ell}^{e_\ell}(\alpha_i)^{-1}$, we recover a known result, which is very much parallel to (1.1): if none of the three α_i 's is equivalent, then all solutions to the left Lagrange problem (1.5) are given by

$$f = \tilde{f}_\ell + P_{\Lambda, \ell} h, \quad \tilde{f}_\ell(z) = \sum_{i=1}^n P_{\Lambda_i, \ell}(z) \cdot P_{\Lambda_i, \ell}^{e_\ell}(\alpha_i)^{-1} \cdot c_i, \quad h \in \mathbb{H}[z].$$

Similarly, letting $\Lambda = \emptyset$, so that $P_{\Lambda, \ell} \equiv 1$ and $\gamma_i = P_{\Omega_j, \mathbf{r}}^{e_r}(\beta_j)^{-1}$, we get all solutions to right Lagrange problem (1.6), under the assumption that none of the three β_j 's is equivalent. We recall that one-sided Lagrange interpolation problems have been considered earlier in the context of quaternionic Vandermonde matrices; see e.g., [4,5,7]. For the Lagrange problem in related settings, we refer to [1,3].

4. The sets $\{f \in \mathbb{H}[z] : f^{e_\ell}|_\Lambda = 0\}$ and $\{f \in \mathbb{H}[z] : f^{e_r}|_\Omega = 0\}$ are respectively, a right and a left ideal in $\mathbb{H}[z]$. Their intersection (a quasi-ideal in $\mathbb{H}[z]$; see [8]) is the solution set of the homogeneous version of problem (1.5), (1.6). Letting $\alpha_i = \beta_j = 0$ for all i, j , we conclude from (2.15) that the elements of this quasi-ideal are parameterized by the formula:

$$f(z) = \sum_{i=1}^k P_{\Lambda, \ell}(z) \cdot q_i \cdot P_{\Omega_i, \mathbf{r}}(z) + P_{\Lambda, \ell}(z) \cdot h(z) \cdot P_{\Omega, \mathbf{r}}(z), \quad h \in \mathbb{H}[z], \quad q_i \in \Pi_{\tilde{\alpha}_i, \tilde{\beta}_i}. \tag{2.17}$$

It is clear from (2.17) that any such f is a right multiple of $P_{\Lambda, \ell}$. However, since q_i is in $\Pi_{\tilde{\alpha}_i, \tilde{\beta}_i}$, and therefore satisfies the Sylvester equation $\tilde{\alpha}_i q_i = q_i \tilde{\beta}_i$, it follows from (2.14) that

$$P_{\Lambda, \ell}(z) \cdot q_i \cdot P_{\Omega_i, \mathbf{r}}(z) = P_{\Lambda_i, \ell}(z) \cdot (z - \tilde{\alpha}_i) \cdot q_i \cdot P_{\Omega_i, \mathbf{r}}(z) = P_{\Lambda_i, \ell}(z) \cdot q_i \cdot (z - \tilde{\beta}_i) \cdot P_{\Omega_i, \mathbf{r}}(z) = P_{\Lambda_i, \ell}(z) \cdot q_i \cdot P_{\Omega, \mathbf{r}}(z)$$

for $i = 1, \dots, k$, and hence f of the form (2.17) is a left multiple of $P_{\Omega, \mathbf{r}}$ as well.

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