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Vertex-reinforced random walk on \mathbb{Z} with sub-square-root weights is recurrent



Récurrence d'une marche aléatoire renforcée par sommets sur \mathbb{Z} avec poids inférieur à racine carrée

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ABSTRACT

We prove that vertex-reinforced random walk on \mathbb{Z} with weight of order k^α , for $\alpha \in [0, 1/2)$, is recurrent. This confirms a conjecture of Volkov for $\alpha < 1/2$. The conjecture for $\alpha \in [1/2, 1)$ remains open.

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R É S U M É

On démontre que toute marche aléatoire renforcée par sommets sur \mathbb{Z} avec poids de l'ordre de k^α , pour $\alpha \in [0, 1/2)$, est récurrente. Ce résultat confirme une conjecture de Volkov pour $\alpha < 1/2$. La conjecture reste ouverte pour $\alpha \in [1/2, 1)$.

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1. Introduction

Linearly vertex-reinforced random walk (VRRW for short), introduced by Pemantle [3], was studied on \mathbb{Z} by Pemantle and Volkov [4]. A striking phenomenon was proved for this model [4,7]: the random walk will eventually visit just 5 sites on \mathbb{Z} almost surely.

In contrast, Volkov later in [8] studied non-linear vertex reinforced random walk on \mathbb{Z} with some weight function $w : \{0, 1, 2, \dots\} \rightarrow (0, \infty)$. This process, denoted by $(X_n, n \geq 0)$, is defined as follows. Fix $X_0 = 0$. Then for all $n \geq 0$,

$$\mathbb{P}(X_{n+1} = X_n \pm 1 \mid X_1, \dots, X_n) = \frac{w(Z_n(X_n \pm 1))}{w(Z_n(X_n - 1)) + w(Z_n(X_n + 1))}, \quad (1)$$

where $Z_n(y) = \#\{m \leq n : X_m = y\}$ is the local time in $y \in \mathbb{Z}$ at time n . For $w(k) = k^\alpha (c + o(1))$, $\alpha \geq 0$, Volkov proved the existence of a phase transition for this model. That is, there is a large time T_0 such that after T_0 , the walk visits 2, 5 or ∞ sites when $\alpha > 1$, $\alpha = 1$ and $\alpha < 1$ respectively. In the case of $\alpha < 1$, though it was proved that the random walk will visit

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infinitely many sites, it is not clear whether it will visit every site of \mathbb{Z} infinitely many times with probability 1. Namely, the question whether the random walk is recurrent¹ was left open.

Recently, Schapira was able to move one step further towards a positive answer to this question, and in [5] proved a 0–1 law for VRRW on \mathbb{Z} with weight of order k^α , for $\alpha \in [0, 1/2)$. In this paper, we show that in this regime the walk is in fact recurrent.

Theorem. *Vertex-reinforced random walk on \mathbb{Z} with weight $w(k) \approx k^\alpha$, $\alpha \in [0, 1/2)$, is recurrent.*

The notation $w(k) \approx k^\alpha$ means that the ratio between the two quantities is bounded between two constants independent of k (except for $w(0)$, on which we make no requirements).

The proof of the theorem consists of a martingale argument, which is a modification of a similar martingale argument used in [1] for edge-reinforced random walk on \mathbb{Z} . Another ingredient of the proof is the fact that for small α , the random walk will not visit the nearby sites too many times before moving to a new site (see Lemma 2). This fact was basically proved by Schapira via a kind of domino principle. This is the part of the proof that only works for $0 \leq \alpha < 1/2$.

Let us remark that Arvind Singh [6] arrived at a similar result simultaneously, using a martingale argument similar in spirit but different in some technical details.

2. Proof

Recall the definition of $Z_i(j)$. For all $i \in \mathbb{N}$, we define a sequence of random variables $F_i : \mathbb{Z} \setminus \{0\} \rightarrow \mathbb{R}^+$

$$F_i(v) = \begin{cases} \sum_{j=0}^{v-1} \frac{1}{w(Z_i(j)) \cdot w(Z_i(j+1))} & \text{if } v > 0; \\ \sum_{j=v}^{-1} \frac{1}{w(Z_i(j)) \cdot w(Z_i(j+1))} & \text{if } v < 0. \end{cases} \quad (2)$$

Note that $F_i(\cdot)$ depends on the history of the random walk up to time i and is \mathcal{F}_i -measurable where \mathcal{F}_i is the σ -field spanned by X_1, \dots, X_i . Then we have the following lemma.

Lemma 1. *Let $X_0 = 0$. Let $T = \min\{i > 0 : X_i = 0\}$ i.e. the first time the process returns to the origin. Then $\{F_{\min(T,i)}(X_{\min(T,i)}) : i = 1, 2, \dots\}$ is a supermartingale.*

Proof. We think about moving from $F_i(X_i)$ to $F_{i+1}(X_{i+1})$ as being composed of two steps: moving X and updating the weights. We will prove that F_i satisfies the following two properties:

- (i) *harmonicity:* for all $i \in \mathbb{N}$, with respect to the random walk's transition probability at time i , $F_i(v)$ is harmonic (in v) on $\mathbb{Z} \setminus \{0\}$. In other words, the first step is a martingale;
- (ii) *monotonicity:* for any fixed $v \in \mathbb{Z} \setminus \{0\}$, $F_i(v)$ is monotone decreasing in i .

Let us prove (i). We condition on \mathcal{F}_i , and denote $v = X_i$ for brevity, and assume $v > 0$ (the other case is similar). We get:

$$\begin{aligned} \mathbb{E}(F_i(X_{i+1}) \mid \mathcal{F}_i) &= \mathbb{P}(X_{i+1} = v + 1)F_i(v + 1) + \mathbb{P}(X_{i+1} = v - 1)F_i(v - 1) \\ &= \frac{w(Z_i(v + 1))}{w(Z_i(v - 1)) + w(Z_i(v + 1))} \sum_{j=0}^v \frac{1}{w(Z_i(j)) \cdot w(Z_i(j + 1))} \\ &\quad + \frac{w(Z_i(v - 1))}{w(Z_i(v - 1)) + w(Z_i(v + 1))} \sum_{j=0}^{v-2} \frac{1}{w(Z_i(j)) \cdot w(Z_i(j + 1))} \\ &= \sum_{j=0}^{v-2} \frac{1}{w(Z_i(j)) \cdot w(Z_i(j + 1))} + \frac{w(Z_i(v + 1))}{w(Z_i(v - 1)) + w(Z_i(v + 1))} \\ &\quad \cdot \left(\frac{1}{w(Z_i(v - 1)) \cdot w(Z_i(v))} + \frac{1}{w(Z_i(v)) \cdot w(Z_i(v + 1))} \right) \\ &= \sum_{j=0}^{v-1} \frac{1}{w(Z_i(j)) \cdot w(Z_i(j + 1))} = F_i(v). \end{aligned}$$

¹ The definition of recurrence we use here is that the random walk visits every vertex of \mathbb{Z} infinitely many times almost surely and the definition of transience is that the random walk visits every vertex of \mathbb{Z} finitely many times almost surely.

Hence, we proved (i); (ii) follows from the fact that for fixed j , $Z_i(j)$, the random walk's local time is monotone increasing in time i .

Now by harmonicity and monotonicity of $F_i(v)$, one has:

$$\mathbb{E}(F_{i+1}(X_{i+1}) \mid \mathcal{F}_i) \leq \mathbb{E}(F_i(X_{i+1}) \mid \mathcal{F}_i) = F_i(X_i),$$

so $F_i(X_i)$ is a supermartingale. \square

Remark. Lemma 1 holds more generally for any vertex-reinforced random walk on \mathbb{Z} with increasing weight sequence. In fact, it holds for any self-interacting process where the vertex weights are increasing, and \mathbb{Z} may be replaced with any tree (also remarked in [2]).

To prove the theorem, we need a second lemma. Let T_n denote the hitting time of a vertex $n \in \mathbb{Z}$. Then,

Lemma 2. *Almost surely, $I := \liminf_{n \rightarrow \infty} Z_{T_n}(n - 1) < \infty$.*

Proof. The claim is equivalent to showing

$$\lim_{k \rightarrow \infty} \mathbb{P}\left(\liminf_{n \rightarrow \infty} Z_{T_n}(n - 1) > k\right) = 0. \tag{3}$$

Note that for any fixed k

$$\begin{aligned} \mathbb{P}\left(\liminf_{n \rightarrow \infty} Z_{T_n}(n - 1) > k\right) &= \mathbb{P}\left(\bigcup_{N \geq 0} \bigcap_{n \geq N} \{Z_{T_n}(n - 1) > k\}\right) \\ &= \sup_{N \geq 0} \mathbb{P}\left(\bigcap_{n \geq N} \{Z_{T_n}(n - 1) > k\}\right) \\ &\leq \sup_{N \geq 0} \mathbb{P}(Z_{T_N}(N - 1) > k). \end{aligned}$$

We now apply formula (4.3) in [5], which claims that

$$\sup_{N \geq 0} \mathbb{P}(Z_{T_N}(N - 1) > k) \leq Ce^{-ck^c}, \tag{4}$$

where c and C are some positive constants (possibly depending on the weight w). Hence, (3) follows from (4). This concludes the proof of the lemma. \square

By the same argument as the proof of Lemma 2, one can prove the same behavior in the negative direction i.e. $\liminf_{n \rightarrow -\infty} Z_{T_n}(n + 1) < \infty$.

Finally, we also use the 0–1 law proved by Schapira, which is stated as follows.

Lemma 3. (See [5], Theorem 1.1.) *Vertex-reinforced random walk on \mathbb{Z} with weight $w(k) \approx k^\alpha$, $k \geq 1$ for some $\alpha \in [0, 1/2)$, is either recurrent or transient.*

Proof of the theorem. By Lemma 3, we know that X_n is either recurrent or transient. Now suppose that X_n is transient, then X_n will visit the origin just finitely many times almost surely. By Lemma 1, $F_i(X_i)$ will be a supermartingale eventually. Since it is positive, it converges to a finite random variable almost surely. On the other hand, by Lemma 2 there will be infinitely many vertices N , such that the increment of $F_i(X_i)$ at time T_N is bounded from below by a positive random variable. Indeed, the only update to Z that happens at time T_N is the increase of $Z(N)$ to 1, but $Z(N)$ does not appear in the sum defining F_{T_n-1} . Hence

$$F_{T_n}(X_{T_n}) - F_{T_n-1}(X_{T_n-1}) = \frac{1}{w(Z_{T_n}(N - 1))w(1)} \geq \frac{1}{w(I)w(1)} > 0$$

(where I is still $\liminf_{n \rightarrow \infty} Z_{T_n}(n - 1) < \infty$). This contradicts the convergence of $F_i(X_i)$. Therefore, we can conclude the theorem. \square

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