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Harmonic analysis

Weak-L^p bounds for the Carleson and Walsh–Carleson operators

Estimation $L^{p,\infty}$ pour les opérateurs de Carleson et de Walsh–Carleson

Francesco Di Plinio [a,b]

[a] *INdAM – Cofund Marie Curie Fellow at Dipartimento di Matematica, Università degli Studi di Roma "Tor Vergata", Via della Ricerca Scientifica, 00133 Roma, Italy*
[b] *The Institute for Scientific Computing and Applied Mathematics, Indiana University, 831 East Third Street, Bloomington, IN 47405, USA*

A R T I C L E I N F O

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A B S T R A C T

We prove a weak-L^p bound for the Walsh–Carleson operator for p near 1, improving on a theorem of Sjölin. We relate our result to the conjectures that the Walsh–Fourier and Fourier series of a function $f \in L\log L(\mathbb{T})$ converge for almost every $x \in \mathbb{T}$.

R É S U M É

Nous prouvons une estimation $L^{p,\infty}$ pour l'opérateur de Walsh–Carleson, pour p proche de 1, qui constitue une amélioration d'un théorème de Sjölin. Nous interprétons nos résultats par rapport à la conjecture selon laquelle la série de Fourier d'une fonction $f \in L\log L(\mathbb{T})$ est convergente presque partout.

1. Motivation and main result

The $L^p(\mathbb{T})$, $1 < p < \infty$ boundedness of the Carleson maximal operator

$$\mathrm{C}f(x) = \sup_{n\in\mathbb{N}} \left| \mathrm{p.v.} \int_{\mathbb{T}} f(x-t)\mathrm{e}^{2\pi i n t} \frac{\mathrm{d}t}{t} \right|, \quad x \in \mathbb{T},$$

first proved in [3,10], entails as a consequence the almost everywhere convergence of the sequence $S_n f$ of partial Fourier sums for each $f \in L^p(\mathbb{T})$. A natural question, posed for instance by Konyagin in [11], is whether, given an Orlicz function $\Phi(t)$ such that $L^1(\mathbb{T}) \subsetneq L^\Phi(\mathbb{T}) \subsetneq L^p(\mathbb{T})$ for all $p > 1$, it is true that

$$\|\mathrm{C}f\|_{1,\infty} \leqslant c\|f\|_{L^\Phi(\mathbb{T})}, \tag{1}$$

so that, equivalently, $S_n f$ converges almost everywhere to f whenever $f \in L^\Phi(\mathbb{T})$. It is a result of Antonov [1] that (1) holds true for $\Phi(t) = t\log(\mathrm{e}+t)\log\log\log(\mathrm{e}^{\mathrm{e}^{\mathrm{e}}}+t)$. Antonov's proof makes use of an approximation technique, relying on the smoothness of the Dirichlet kernels, to upgrade the restricted weak-type estimate of Hunt [10]:

E-mail address: diplinio@mat.uniroma2.it.

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$$\|\mathrm{C}\mathbf{1}_E\|_{p,\infty} \leqslant c\frac{p^2}{p-1}|E|^{\frac{1}{p}} \quad \forall E \subset \mathbb{T}, \quad \forall 1 < p < \infty, \tag{2}$$

to the mixed bound:

$$\|\mathrm{C}f\|_{1,\infty} \leqslant c\|f\|_1 \log\left(\mathrm{e} + \frac{\|f\|_\infty}{\|f\|_1}\right), \tag{3}$$

which, in turn, yields that $\mathrm{C}: L\log L\log\log\log L(\mathbb{T}) \to L^{1,\infty}(\mathbb{T})$, in view of the log-convexity of the latter space. A larger quasi-Banach rearrangement invariant space QA such that $\mathrm{C}: QA \to L^{1,\infty}(\mathbb{T})$ was later found in [2]. In [4] it is shown that, however, Antonov's space is the largest, in a suitable sense, Orlicz space $L^\Phi(\mathbb{T})$ such that the embedding $L^\Phi(\mathbb{T}) \hookrightarrow QA$ holds. The results of [1,2] have been reproved by Lie [14], where (3) is obtained directly, without the use of approximation techniques.

We note that estimate (2) is in fact equivalent to (1) with $\Phi(t) = t\log(\mathrm{e}+t)$, restricted to indicator functions; see [19, Remark 1]. This leads to the conjecture that (1) holds for the space $L\log L(\mathbb{T})$, a consequence of which would be the unrestricted version of Hunt's estimate (2):

$$\|\mathrm{C}f\|_{p,\infty} \leqslant \frac{c}{p-1}\|f\|_p, \quad \forall 1 < p \leqslant 2. \tag{4}$$

On the other hand, a suitable choice of $p \in (1,2)$ in (4) yields (3) directly, and in turn, recovers (1) for Antonov's Φ; thus, the weak-L^p estimate (4) arises naturally as an intermediate result between the conjectured $L\log L(\mathbb{T})$ bound in (1) and the presently known best Orlicz space bound. That the "$L\log L$ conjecture" implies (4) is a particular case of the following observation, due to Andrei Lerner (personal communication). Assuming (1) holds for a given Φ, one has the pointwise inequality $M^\#(|\mathrm{C}f|^{\frac{1}{2}}) \leqslant (M_\Phi f)^{\frac{1}{2}}$, the latter being the local Orlicz maximal function associated with Φ [9, Proposition 5.2]. It follows that

$$\|\mathrm{C}f\|_{p,\infty} \leqslant c\left\|\left(M^\#\left(|\mathrm{C}f|^{\frac{1}{2}}\right)\right)^2\right\|_{p,\infty} \leqslant c\|M_\Phi f\|_{p,\infty} \leqslant c\left(\sup_{t\geqslant 1}\frac{\Phi(t)}{t^p}\right)^{\frac{1}{p}}\|f\|_p, \quad \forall 1 < p \leqslant 2. \tag{5}$$

Using Antonov's $\Phi(t) = t\log(\mathrm{e}+t)\log\log\log(\mathrm{e}^{\mathrm{e}^{\mathrm{e}}}+t)$ in (5) leads to

$$\|\mathrm{C}f\|_{p,\infty} \leqslant \frac{c}{p-1}\log\log\left(\mathrm{e}^{\mathrm{e}} + \frac{1}{p-1}\right)\|f\|_p, \quad \forall 1 < p \leqslant 2; \tag{6}$$

to the best of the author's knowledge, there seems to be no better weak-L^p bound than (6) in the current literature, and in particular the validity of (4), which can be thought of as a weakening of the $L\log L$ conjecture, is open.

The main new result of this article is that the analogue of (4) holds for the Walsh–Fourier version of the Carleson operator, which is often thought of as a discrete model of the Fourier case: see [21, Chapter 8] for the relevant definitions.

Theorem 1.1. *Denote by $\mathrm{W}_n f(x)$ the n-th partial Walsh–Fourier sum of $f \in L^1(\mathbb{T})$. There exists an absolute constant $c > 0$ such that the Walsh–Carleson maximal operator $\mathrm{W}f(x) := \sup_{n\in\mathbb{N}}|\mathrm{W}_n f(x)|$ satisfies the operator norm bound*:

$$\|\mathrm{W}\|_{L^p(\mathbb{T})\to L^{p,\infty}(\mathbb{T})} \leqslant \frac{c}{p-1}, \quad \forall 1 < p \leqslant 2. \tag{7}$$

Theorem 1.1 is a strengthening of the Walsh analogue of (2), obtained by Sjölin in [17], and recovers the correspondent version of (3), first established in [18], without the need for approximation techniques developed therein. The bound $\mathrm{W}: L\log L\log\log\log L(\mathbb{T}) \to L^{1,\infty}(\mathbb{T})$, which is the Walsh case of Antonov's result, follows as a further consequence. Furthermore, if we assume that the Walsh case of the $L\log L$ conjecture is sharp, in the sense that there exists no Young function Φ with $\mathrm{W}: L^\Phi(\mathbb{T}) \to L^{1,\infty}(\mathbb{T})$ and such that $\limsup_{t\to\infty}(t\log(\mathrm{e}+t))^{-1}\Phi(t) = 0$, then the bound (7) is sharp, up to a doubly logarithmic term in $(p-1)^{-1}$; see [6, Section 2] for details.

The proof is given in the upcoming Section 2. In the final Section 3, we discuss analogous results for the lacunary versions of C and W.

2. Proof of Theorem 1.1

We will prove (7) by relying on the (Walsh) phase plane model sums (see for instance [20,21]). The main technical tool not present in the classical works mentioned above is a discrete variant of the multi-frequency Calderón–Zygmund decomposition of [15] (Lemma 2.2 below). Similar arguments have already found ample use in the treatment of discrete modulation-invariant singular integrals [16,7,5,6].

Let $\mathcal{D}$ be the standard dyadic grid on $\mathbb{R}_+$; below, we indicate with $\mathbf{S}$ an arbitrary finite collection of *bitiles*, that is rectangles $s = I_s \times \omega_s \subset \mathcal{D}\times\mathcal{D}$ with $|\omega_s| = 2|I_s|^{-1}$. Denoting by ω_{s_1}, ω_{s_2}, respectively, the left and right dyadic child of ω_s, each bitile s is thought of as the union of the two *tiles* (dyadic rectangles of area 1) $s_1 = I_s \times \omega_{s_1}$, $s_2 = I_s \times \omega_{s_2}$. Writing W_n for the n-th Walsh character on $\mathbb{T}$, the Walsh wave packet time-frequency adapted to a tile $t = I_t \times \omega_t$ is then defined as

$$w_t(x) = \mathrm{Dil}^2_{|I_t|} \mathrm{Tr}_{\inf I_t} W_{n_t}(x) = |I_t|^{-1/2} W_{n_t}\left(\frac{x - \inf I_t}{|I_t|}\right), \quad n_t := |I_t| \inf \omega_t.$$

The model sums for the Walsh–Carleson maximal operator W are then given by

$$\mathrm{W}_{\mathbf{S}} f(x) = \sum_{s\in\mathbf{S}} \varepsilon_s \langle f, w_{s_1}\rangle w_{s_1}(x) \mathbf{1}_{\omega_{s_2}}\big(N(x)\big),$$

where $N : \mathbb{R}^+ \to \mathbb{R}^+$ is an (arbitrary) measurable choice function, and $\{\varepsilon_s\} \in \{-1, 0, 1\}^{\mathbf{S}}$. By the reduction given in, e.g., [20,21], Theorem 1.1 is a consequence of the bound (p' is the Hölder dual of p):

$$\|\mathrm{W}_{\mathbf{S}} f\|_{p,\infty} \lesssim p' \|f\|_p, \quad \forall 1 < p \leqslant 2; \tag{8}$$

in (8) and in what follows, the constants implied by the almost inequality signs are meant to be absolute (in particular, independent of f, $1 < p \leqslant 2$, $\mathbf{S}$, N and $\{\varepsilon_s\}$) and may vary at each occurrence. Observe that (8) is recovered by taking $G = \{|\mathrm{W}_{\mathbf{S}} f| > \lambda\}$, $g(x) = \mathbf{1}_{G'}(x) \exp(-\mathrm{i} \arg(\mathrm{W}_{\mathbf{S}} f(x)))$ in the bound:

$$\big|\langle \mathrm{W}_{\mathbf{S}} f, g\rangle\big| \lesssim p' \|f\|_p |G|^{\frac{1}{p'}}, \quad \forall |g| \leqslant \mathbf{1}_{G'}, \tag{9}$$

where $G' \subset G$ is a suitably chosen, possibly depending on f, major subset of G: that is, $|G| \leqslant 4|G'|$. By dyadic scale-invariance of the family of operators $\{\mathrm{W}_{\mathbf{S}}\}$ over all choices of $\mathbf{S} \subset \mathcal{D} \times \mathcal{D}$ and measurable functions N, and by linearity in f, it suffices to prove (9) in the case $\|f\|_p = 1$, $1 \leqslant |G| < 4$, to which we turn in Subsection 2.2. In the upcoming Subsection 2.1, we recall some tools of discrete time-frequency analysis.

2.1. Trees, size and density

We will use the Fefferman order relation on either tiles or bitiles: $s \ll s'$ if $I_s \subset I_{s'}$ and $\omega_s \supset \omega_{s'}$. We say that $\mathbf{S}$ is a *convex* collection of bitiles if $s, s' \in \mathbf{S}$, $s \ll s'' \ll s'$ implies $s'' \in \mathbf{S}$. There is no restriction to prove (8) under the further assumption that $\mathbf{S}$ is convex, and we do so. A convex collection of bitiles $\mathbf{T} \subset \mathbf{S}$ is called *tree* with top bitile $s_{\mathbf{T}}$ if $s \ll s_{\mathbf{T}}$ for all $s \in \mathbf{T}$. We simplify the notation and write $I_{\mathbf{T}} := I_{s_{\mathbf{T}}}$, $\omega_{\mathbf{T}} = \omega_{s_{\mathbf{T}}}$. We will call *forest* a collection of convex trees $\mathbf{T} \in \mathcal{F}$, and will make use of the quantity

$$\mathrm{tops}(\mathcal{F}) := \sum_{\mathbf{T}\in\mathcal{F}} |I_{\mathbf{T}}|.$$

The above definitions make their first appearance in the proof of boundedness of the Carleson operator by C. Fefferman [8], and have since then been recast in several works, the first of which is [12].

Given a measurable function $N : \mathbb{R} \to \mathbb{R}$ and $G \subset \mathbb{R}$, define

$$\mathrm{dense}_G(\mathbf{S}) = \sup_{s\in\mathbf{S}} \sup_{s'\in\mathbf{S}: s\ll s'} \frac{|G \cap I_{s'} \cap N^{-1}(\omega_{s'_2})|}{|I_{s'}|}.$$

Furthermore, for $f \in L^2(\mathbb{T})$, we set

$$\mathrm{size}_f(\mathbf{S}) = \sup_{s\in\mathbf{S}} \max_{j=1,2} \frac{|\langle f, w_{s_j}\rangle|}{|I_s|^{\frac{1}{2}}}.$$

We observe that size, dense are monotone increasing with respect to set inclusion. One has $\mathrm{dense}_G(\mathbf{S}) \leqslant 1$ for each $G \subset \mathbb{R}$, and it is immediate to see that

$$\mathrm{size}_f(\mathbf{S}) \leqslant \sup_{s\in\mathbf{S}} \inf_{x\in I_s} \mathrm{M}_1 f(x), \tag{10}$$

where M_p, $1 \leqslant p < \infty$, denotes the dyadic p-th Hardy–Littlewood maximal function. Finally, we recall *verbatim* a result from [7] (Lemma 2.13 therein).

Lemma 2.1. *Let $h \in L^2(\mathbb{R})$ and $\mathcal{F}$ be a forest with* $\mathrm{dense}_G(\mathcal{F}) \leqslant \delta$, $\mathrm{tops}(\mathcal{F}_\delta) \lesssim \delta^{-1}|G|$. *Then for all* $g : \mathbb{R} \to \mathbb{C}$, $|g| \leqslant \mathbf{1}_G$,

$$\big|\langle \mathrm{W}_{\mathcal{F}} h, g\rangle\big| \lesssim \min\big\{\mathrm{size}_h(\mathcal{F})|G|, \delta^{\frac{1}{2}} \sqrt{|G|} \|h\|_2\big\}.$$

2.2. Proof of (9)

Recall that we are assuming $\|f\|_p = 1$, $1 \leqslant |G| < 4$. For an appropriate (absolute) choice of $c > 0$,

$$\left|E := \{\mathsf{M}_p f \geqslant c\}\right| \lesssim c^{-p}\|\mathsf{M}_p f\|_p^p \leqslant \tfrac{1}{4}. \tag{11}$$

Set $G' := G \backslash E$; by the above, $|G'| \geqslant \frac{1}{2}$, so that G' is a major subset of G. Since $w_{s_1}(x)\mathbf{1}_{\omega_{s_2}}(N(x))$ is supported inside I_s, we have that $\langle w_{s_1}, g\rangle = 0$ when $|g| \leqslant \mathbf{1}_{G'}$ and $I_s \cap G' = \emptyset$. This means that

$$\langle \mathsf{W}_{\mathbf{S}} f, g\rangle = \langle \mathsf{W}_{\mathbf{S}_{\mathrm{good}}} f, g\rangle, \quad \mathbf{S}_{\mathrm{good}} := \left\{s \in \mathbf{S}\colon\ I_s \cap E^c \neq \emptyset\right\}. \tag{12}$$

Therefore, from now on, we will just replace $\mathbf{S}$ by $\mathbf{S}_{\mathrm{good}}$ in (9). Note that, as a consequence of (10) and of the definition of $\mathbf{S}_{\mathrm{good}}$, we have $\mathrm{size}_f(\mathbf{S}_{\mathrm{good}}) \lesssim 1$.

The next step is an application of the density decomposition lemma (for instance, [7, Lemma 2.6]) to $\mathbf{S}_{\mathrm{good}}$, writing:

$$\mathbf{S}_{\mathrm{good}} = \bigcup_{\delta \in 2^{-\mathbb{N}}} \mathcal{F}_\delta, \qquad \mathrm{size}_f(\mathcal{F}_\delta) \lesssim 1, \qquad \mathrm{dense}_G(\mathcal{F}_\delta) \leqslant \delta, \qquad \mathrm{tops}(\mathcal{F}_\delta) \lesssim \delta^{-1}|G|. \tag{13}$$

We claim the single forest estimate

$$\left|\langle \mathsf{W}_{\mathcal{F}_\delta} f, g\rangle\right| \lesssim \delta^{\frac{1}{p'}}. \tag{14}$$

Assuming that (14) holds true,

$$\left|\langle \mathsf{W}_{\mathbf{S}_{\mathrm{good}}} f, g\rangle\right| \leqslant \sum_{\delta \in 2^{-\mathbb{N}}} \left|\langle \mathsf{W}_{\mathcal{F}_\delta} f, g\rangle\right| \lesssim \sum_{\delta \in 2^{-\mathbb{N}}} \delta^{\frac{1}{p'}} \lesssim p',$$

that is, we have proved (9). The remainder of the section is then devoted to the proof of the single forest estimate (14). The key tool is provided by the lemma below.

Lemma 2.2. *For each $\delta \in 2^{-\mathbb{N}}$, there is a function h_δ such that*

$$\|h_\delta\|_2 \lesssim \delta^{-\frac{1}{2}+\frac{1}{p'}}, \qquad \langle f, w_{s_1}\rangle = \langle h_\delta, w_{s_1}\rangle \quad \forall s \in \mathcal{F}_\delta.$$

In particular, we see from Lemma 2.2 that $\langle \mathsf{W}_{\mathcal{F}_\delta} f, g\rangle = \langle \mathsf{W}_{\mathcal{F}_\delta} h_\delta, g\rangle$ and that $\mathrm{size}_{h_\delta}(\mathcal{F}_\delta) = \mathrm{size}_f(\mathcal{F}_\delta) \lesssim 1$; therefore, we may use Lemma 2.1 to bound

$$\left|\langle \mathsf{W}_{\mathcal{F}_\delta} f, g\rangle\right| = \left|\langle \mathsf{W}_{\mathcal{F}_\delta} h_\delta, g\rangle\right| \lesssim \delta^{\frac{1}{2}}|G|^{\frac{1}{2}}\|h_\delta\|_2 \lesssim \delta^{\frac{1}{p'}},$$

which is (14). We have thus completed the proof of Theorem 1.1, up to showing Lemma 2.2 holds true.

2.3. Proof of Lemma 2.2

This argument is analogous to [5, Lemma 5.1]. We argue under the additional assumption that f is supported on $E = \{\mathsf{M}_p f \geqslant c\}$; the general case requires only trivial modifications. Let $I \in \mathbf{I}$ be the maximal dyadic intervals of E; for each $I \in \mathbf{I}$, let $t \in T_I$ be the collection of all tiles having $I_t = I$ and which are comparable under $\ll$ to some tile $s_1 \in \mathcal{F}_\delta$. The tiles of T_I are obviously pairwise disjoint.

The definition of $\mathbf{S}_{\mathrm{good}}$ ensures that, whenever $I_s \cap I \neq \emptyset$ for some $s \in \mathbf{S}_{\mathrm{good}}$ and $I \in \mathbf{I}$, the inclusion $I \subsetneq I_s$ must hold. It follows that if $t \in T_I$, $s_1 \in \{s_1\colon\ s \in \mathbf{T} \in \mathcal{F}_\delta\}$ are related under $\ll$, then $t \ll s_1$. By standard properties of the Walsh wave packets, w_{s_1} is a scalar multiple of w_t on I; in particular, $w_{s_1}\mathbf{1}_I$ belongs to H_I, the subspace of $L^2(I)$ spanned by $\{w_t\colon\ t \in T_I\}$. A further consequence is that, if N_I is the number of trees $\mathbf{T} \in \mathcal{F}_\delta$ with $I \subset I_{\mathbf{T}}$, we have $\#T_I \leqslant N_I$. For $v \in H_I$, we have the inequality:

$$\|v\|_{L^{p'}(I)} \lesssim N_I^{\frac{1}{2}-\frac{1}{p'}}\|v\|_{L^2(I)}.$$

Since $\|f\|_{L^p(I)} \lesssim 1$ by maximality of I in E, it then follows that

$$\left|(f, v)_{L^2(I)}\right| \leqslant \|f\|_{L^p(I)}\|v\|_{L^{p'}(I)} \lesssim N_I^{\frac{1}{2}-\frac{1}{p'}}\|v\|_{L^2(I)} \quad \forall v \in H_I,$$

and consequently h_I, the projection of $f\mathbf{1}_I$ on H_I, satisfies $\|h_I\|_{L^2(I)} \lesssim N_I^{\frac{1}{2}-\frac{1}{p'}}$. Setting $h_\delta := \sum_{I \in \mathbf{I}} h_I$, we see that

$$\|h_\delta\|_2^2 = \sum_{I \in \mathbf{I}} |I|\|h_I\|_{L^2(I)}^2 \lesssim \sum_{I \in \mathbf{I}} |I| N_I^{1-\frac{2}{p'}} \lesssim \left\|\sum_{\mathbf{T} \in \mathcal{F}_\delta} \mathbf{1}_{I_{\mathbf{T}}}\right\|_1^{1-\frac{2}{p'}} \left(\sum_{I \in \mathbf{I}} |I|\right)^{\frac{2}{p'}} \lesssim \delta^{-1+\frac{2}{p'}};$$

in the last step, we made use of the bound on tops from (13), and of (11) to estimate the sum over I. Finally, in view of the above discussion, if $s \in \mathbf{T} \in \mathcal{F}_\delta$:

$$\langle f, w_{s_1}\rangle = \sum_{I\in\mathbf{I}} \langle f\mathbf{1}_I, w_{s_1}\rangle = \sum_{I\in\mathbf{I}} \langle f\mathbf{1}_I, c w_{t(s_1;I)}\rangle = \sum_{I\in\mathbf{I}} \langle h_I, w_{s_1}\rangle = \langle h_\delta, w_{s_1}\rangle$$

where $t(s_1; I)$ is the unique (if any) element t of T_I with $t \ll s_1$. This finishes the proof of the lemma.

3. The $L \log\log L$ conjecture and weak-L^p bounds for the lacunary Carleson operator

It is conjectured in [11, Conjecture 3.2] that the subsequence $S_{n_j} f$ of the partial Fourier sums of $f \in L\log\log L(\mathbb{T})$ converges almost everywhere whenever n_j is a lacunary sequence of integers, in the sense that $n_{j+1} \geqslant \theta n_j$ for all j and for some $\theta > 1$; if true, this result would be sharp. This is equivalent to the conjecture that the *lacunary* Carleson maximal operator:

$$\mathrm{C}_{\{n_j\}} f(x) = \sup_{j\in\mathbb{N}} \left| \mathrm{p.v.} \int_{\mathbb{T}} f(x-t) \mathrm{e}^{2\pi \mathrm{i} n_j t} \frac{\mathrm{d}t}{t} \right|, \quad x \in \mathbb{T},$$

satisfies

$$\|\mathrm{C}_{\{n_j\}} f\|_{1,\infty} \leqslant c\|f\|_{L^\Phi(\mathbb{T})}, \tag{15}$$

for $\Phi(t) = t\log\log(\mathrm{e}^{\mathrm{e}} + t)$, with constant $c > 0$ depending only on the lacunarity constant θ of the sequence $\{n_j\}$. By (5), if the above conjectured bound held true, the weak-L^p estimate

$$\|\mathrm{C}_{\{n_j\}} f\|_{p,\infty} \leqslant c\log\bigl(\mathrm{e} + (p-1)^{-1}\bigr)\|f\|_p, \quad \forall 1 < p \leqslant 2 \tag{16}$$

would follow. The current best result [6,13] is that (15) holds with $\Phi(t) = t\log\log(\mathrm{e}^{\mathrm{e}} + t)\log\log\log\log(\mathrm{e}^{\cdots^{\mathrm{e}}} + t)$. However, we remark that the argument for the main theorem in [13] can be suitably reformulated to prove the stronger (16) in place of the main result therein, which is an estimate of the same type as (3), with a $\log\log$ in place of the log. Therefore, the weaker form of Konyagin's $L\log\log L$ conjecture given by (16) holds true. Finally, we mention that the Walsh analogue of (16) is explicitly proved in [6].

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